ANALYSIS OF STRUCTURES WITH RANDOM AXIAL STIFFNESS DESCRIBED BY IMPRECISE PROBABILITY DENSITY FUNCTIONS

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Abstract

This paper addresses the analysis of structures with random axial stiffness described by *imprecise probability density functions* (*PDF*s). Uncertainties are modelled as random variables whose *PDF* is assumed to depend on interval basic parameters (mean-value, variance etc.). The main purpose of the analysis is to propagate the *imprecise PDF* of the random axial stiffness by establishing approximate bounds on the mean-value and variance of the response. To this aim, an efficient method is proposed which relies on the combination of standard probabilistic analysis with the so-called *improved interval analysis* via *extra unitary interval* and the *Rational Series Expansion*, recently introduced by the authors. The accuracy of the proposed bounds of response statistics is demonstrated by appropriate comparisons with the results obtained performing standard *Monte Carlo Simulation* in conjunction with a combinatorial procedure.

Keywords: Imprecise probability; Interval basic parameters; Improved interval analysis; Rational Series Expansion; Explicit expressions; Upper bound and lower bound.

1. INTRODUCTION

Uncertainties affecting both structural parameters (e.g. material and/or geometric properties, fabrication details, etc.) and external loads play a crucial role in the prediction of structural behavior. In the last decades, several methods have been developed to analyze the effects of uncertain properties on structural response. Such methods require some mathematical description of uncertainties based on available empirical information. The most common description is by no means the statistical one which characterizes the uncertain parameters by defining an appropriate *probability density function* (*PDF*). However, available data are often quite limited and of poor quality as well as imprecise, diffuse, fluctuating, incomplete, fragmentary, vague or ambiguous. It follows that available data are often insufficient to empirically determine the *PDF* of an uncertain variable. As a consequence, the "basic" parameters (e.g. mean-value, variance, etc.) of the *PDF* are affected by uncertainties. These uncertainties can sometimes be substantial and in many applications "precise probabilities" cannot be considered as adequate and credible models of real states. This issue has been the subject of considerable debate in the last decades and a new family of non-probabilistic or "possibilistic" assessment methods has been derived [1,2].

When the information relating to an uncertain quantity of interest is expressed only as a set of possible values that the quantity might take, this information is usually referred to as "imprecise". This is distinct from the conventional probabilistic treatment of uncertainty where a probability measure is assigned to possible values of the uncertain quantity. The extension of probabilistic analysis to include imprecise information is now well established in the theory of "imprecise probabilities" which may be viewed as a generalization of the traditional probability theory (see e.g., [3-6]). An imprecise probability arises when one's lower probability for an event is strictly smaller than one’s upper probability for the same event [5]. A key feature of imprecise probabilities is the identification of bounds on probabilities for events of interest; the uncertainty of an event is characterized by two measure values—a lower probability and an upper probability. The distance between the probability bounds reflects the indeterminacy in model specifications expressed as imprecision of the models. This imprecision is the concession for not introducing artificial model assumptions [2,7].

Different representations of imprecise probabilities have been proposed in the literature. For example, Dempster [3] and Shafer [4] formulated a theory, sometimes called *evidence theory*, which can be considered as a variant of probability theory, in which the elements of the sample space are not single points but sets of values. Walley [5] coined the term *imprecise probability*; his theory is based on the subjective behavioral interpretation of the probability with the lower and the upper previsions. Weichselberger [6] introduced the *interval probability* as a generalization of Kolmogorov's classical probability; the resulting theory does not depend upon interpretations of the probability concept. The generalization is performed through the use of lower and upper probabilities, denoted by $\underline{P}(A)$ and $\overline{P}(A)$, respectively, with $0 \le \underline{P}(A) \le \overline{P}(A) \le 1$. The special case with $\underline{P}(A) = \overline{P}(A)$ for all events A provides precise probability, whilst $\underline{P}(A) = 0$ and $\overline{P}(A) = 1$ represents complete lack of knowledge about *A*. In order to unify the standard interval analysis [8-11] with the traditional probability theory, the *probability bounds analysis* was introduced [12,13]. In this approach, also known as *P-box*, an imprecise random variable is represented by upper and lower bounds of its *cumulative density function* (*CDF*), rather than upper and lower bounds of its *PDF*. The *fuzzy probability* has been formulated [14] considering probability distributions with fuzzy parameters. In the framework of imprecise probability, the structural reliability bounds have been afterwards determined by considering imprecise parameters of the *PDF* associated with the stress and strength [15,16]. Although being very general, the application of the previously described theories is often limited to simple models, mainly because of the computational burden associated to the propagation of the imprecise probability description [17,18].

In the framework of imprecise probability, the interval analysis is certainly a very effective tool for the evaluation of the bounds of response statistical moments. However, the application of approaches based on the *classical interval analysis* (*CIA*) to engineering problems is hindered by the so-called *dependency phenomenon* [2,19] which often leads to an overestimation of the interval result unacceptable for design purposes.

In this paper, a method for the analysis of structural systems with random axial stiffness is presented. Uncertainties are modelled as random variables with *imprecise PDF* so as to take into account that the *PDF* itself is subject to doubt. Specifically, the *imprecise PDF* is assumed to depend on interval "basic" parameters (e.g. means, variances, etc.) which possess bounded descriptions. The aim of the analysis is to propagate the *imprecise PDF*s of the random axial stiffness by establishing approximate bounds on the mean-value and variance of the response. The proposed approach relies on a combination of probabilistic and non-probabilistic tools. Specifically, the random character of uncertainty is handled by performing a standard probabilistic analysis while imprecision is processed by applying the *improved interval analysis* via *extra unitary interval* (*IIA* via *EUI*) [20] in conjunction with the so-called *Rational Series Expansion* (*RSE*) (see e.g., [21-23]). The main steps of the proposed procedure may be summarized as follows: *i*) the derivation of approximate analytical expressions of the interval mean-value and variance of structural response by applying the *RSE* which enables to determine the inverse of the random stiffness matrix in approximate explicit form; *ii*) the evaluation of explicit bounds of the interval mean-value and variance of structural response by adopting the *IIA* via *EUI* as an effective remedy to the overestimation due to the *dependency phenomenon*.

To demonstrate the effectiveness of the presented procedure, a braced shear-type frame and a 3D truss structure with random axial stiffness of braces and bars, respectively, characterized by *imprecise PDF* are analyzed.

The paper is organized as follows: in Section 2, preliminary concepts and definitions concerning the *imprecise PDF* model assumed in the paper are introduced; in Section 3, approximate explicit expressions of the mean-value and variance of displacements of structures with random axial stiffness are derived by means of the *RSE*; in Section 4, under the assumption of *imprecise PDF* of the random axial stiffness, approximate explicit expressions of the bounds of the interval mean-value and variance of displacements are derived; finally, in Section 5, numerical results are presented to demonstrate the accuracy and efficiency of the proposed method.

2. IMPRECISE PROBABILITY: PRELIMINARY CONCEPTS AND DEFINITIONS

Information on an uncertain quantity of interest is usually referred to as "imprecise" when it is expressed only as a set of possible values that the quantity might take. Here, in the framework of imprecise probability analysis, it is assumed that a random variable possesses a family of *probability density functions* (*PDF*s). In particular, let us introduce the function $p_X(x;\mathbf{a})$ which represents the family of *PDF*s of the random variable X (with $x \in \mathbb{R}$). This function, herein referred to as *imprecise PDF*, depends on the set of epistemic "basic" parameters $a_1, a_2, \ldots, a_s$, collected into the vector $\mathbf{a} = \left[a_1, a_2, \ldots, a_s\right]^{\mathrm{T}}$, that lies within the admissible closed region Q. Hereafter, it is assumed that the epistemic parameters define a bounded set of interval variables. This means that the vector $\mathbf{a}$ is constrained by an *s*-dimensional box *Q*. According to the *interval analysis* formalism, the set-interval vector of epistemic parameters $\mathbf{a}$ is represented by $\mathbf{a}^I \triangleq [\underline{\mathbf{a}}, \overline{\mathbf{a}}] \in \mathbb{IR}^s$, such that $\underline{\mathbf{a}} \leq \mathbf{a} \leq \overline{\mathbf{a}}$, where $\mathbb{IR}$ is the set of all closed real interval numbers. The symbols $\underline{\mathbf{a}}$ and $\overline{\mathbf{a}}$ denote the lower bound (LB) and upper bound (UB) vectors, while the apex *I* characterizes interval variables; the *i*-th element of the interval vector $\mathbf{a}^I$ can be defined as $a_i^I \triangleq [\underline{a}_i, \overline{a}_i]$, where $a_i^I \in \mathbb{IR}$, $\underline{a}_i$ and $\overline{a}_i$ are the LB and UB of the *i*-th epistemic basic parameter a_i^I, respectively.

As can be readily inferred, the statistical moment of order k of the random variable X with *imprecise PDF* $p_X(x;\mathbf{a}^I)$ is defined by an interval. Indeed, the set of *PDFs* describing the random variable X yields a set of statistical moments. This concept is formally expressed by introducing the so-called *interval stochastic average operator* $\mathrm{E}^I\langle\bullet\rangle$, i.e.:

$$\mathrm{E}^I\left\langle X^k\right\rangle = \left[\underline{\mathrm{E}}\left\langle X^k\right\rangle, \overline{\mathrm{E}}\left\langle X^k\right\rangle\right]. \tag{1}$$

In the previous expression, $\underline{\mathrm{E}}\left\langle X^k\right\rangle$ and $\overline{\mathrm{E}}\left\langle X^k\right\rangle$ denote the LB and UB, respectively, of the k-th order statistical moment of the random variable X, characterized by the *imprecise PDF* $p_X(x;\mathbf{a}^I)$. Based on standard probability theory and *classical interval analysis* (*CIA*), such bounds can be evaluated as:

$$\begin{aligned} \underline{\mathrm{E}}\left\langle X^k\right\rangle &= \min_{\mathbf{a}^I \in Q}\left\{\int_{-\infty}^{\infty} x^k\, p_X(x;\mathbf{a}^I)\,\mathrm{d}x\right\}; \\ \overline{\mathrm{E}}\left\langle X^k\right\rangle &= \max_{\mathbf{a}^I \in Q}\left\{\int_{-\infty}^{\infty} x^k\, p_X(x;\mathbf{a}^I)\,\mathrm{d}x\right\} \end{aligned} \tag{2a,b}$$

where the symbols $\min\{\bullet\}$ and $\max\{\bullet\}$ mean minimum and maximum value of the quantity into parentheses under the condition that $\mathbf{a}^I \in Q$, respectively.

Taking into account the definitions (2a,b), the LB and UB of the interval expectation (or mean-value) of the random variable X, $\mu_X^I = \mathrm{E}^I\langle X\rangle = [\underline{\mu}_X, \overline{\mu}_X]$, are given, respectively, as:

$$\begin{aligned} \underline{\mu}_X &= \underline{\mathrm{E}}\langle X\rangle = \min_{\mathbf{a}^I \in Q}\left\{\int_{-\infty}^{\infty} x\, p_X(x;\mathbf{a}^I)\,\mathrm{d}x\right\}; \\ \overline{\mu}_X &= \overline{\mathrm{E}}\langle X\rangle = \max_{\mathbf{a}^I \in Q}\left\{\int_{-\infty}^{\infty} x\, p_X(x;\mathbf{a}^I)\,\mathrm{d}x\right\}. \end{aligned} \qquad (3a,b)$$

Similarly, the LB and UB of the interval variance of the random variable X, $\sigma_X^{2\,I} = [\underline{\sigma}_X^2, \overline{\sigma}_X^2] = \mathrm{E}^I\langle X^2\rangle - (\mu_X^I)^2$, are defined as:

$$\begin{aligned} \underline{\sigma}_X^2 &= \min_{\mathbf{a}^I \in Q}\left\{\int_{-\infty}^{\infty} \left(x-\mu_X^I\right)^2 p_X(x;\mathbf{a}^I)\,\mathrm{d}x\right\} = \underline{\mathrm{E}}\langle X^2\rangle - (\overline{\mu}_X)^2; \\ \overline{\sigma}_X^2 &= \max_{\mathbf{a}^I \in Q}\left\{\int_{-\infty}^{\infty} \left(x-\mu_X^I\right)^2 p_X(x;\mathbf{a}^I)\,\mathrm{d}x\right\} = \overline{\mathrm{E}}\langle X^2\rangle - (\underline{\mu}_X)^2. \end{aligned} \qquad (4a,b)$$

The definitions in Eqs. (2)-(4) can be extended to cover the case of a joint *imprecise PDF*, $p_{\mathbf{X}}(\mathbf{x};\mathbf{a}_{\mathbf{X}}^I)$, with $\mathbf{X} = [\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_r]^{\mathrm{T}}$ and $\mathbf{a}_{\mathbf{X}}^I = [\mathbf{a}_{X_1}^{\mathrm{T}}, \mathbf{a}_{X_2}^{\mathrm{T}}, \ldots, \mathbf{a}_{X_r}^{\mathrm{T}}]^{\mathrm{T}}$ being the vectors collecting, respectively, the random uncertain parameters and the corresponding epistemic interval variables; the *k*-th element of $\mathbf{a}_{\mathbf{X}}^I$ is constrained by an s_k-dimensional box Q_k, that is $\mathbf{a}_{X_k} \in \mathbb{IR}^{s_k}$.

3. LINEAR STRUCTURES WITH RANDOM AXIAL STIFFNESS

3.1 Explicit inverse of the random stiffness matrix by the Rational Series Expansion

By applying the *unimodal formulation*, discretized structures (like frames and truss structures) can be decomposed into their constituent unimodal components (one for a truss element, two for a beam element of Euler-Bernoulli type, etc.) [24]. It follows that the equations governing the response of a structure with *n* degrees of freedom and *m* unimodal components, subjected to static loads, can be written as follows:

$$\begin{aligned} &\mathbf{C}^{\mathrm{T}}\mathbf{Q} = \mathbf{f}\,; && \text{equilibrium equations;} \\ &\mathbf{Q} = \mathbf{E}\,\mathbf{q}\,; && \text{constitutive equations;} \\ &\mathbf{C}\,\mathbf{U} = \mathbf{q}. && \text{compatibility equations} \end{aligned} \qquad (5a\text{-}c)$$

where $\mathbf{U}$ is the *n*-vector of nodal displacements; $\mathbf{Q}$ and $\mathbf{q}$ are the *m*-vectors of internal forces and deformations, respectively; $\mathbf{C}^{\mathrm{T}}$ is the $n \times m$ equilibrium matrix and $\mathbf{E}$ is the $m \times m$ diagonal internal stiffness matrix. Let $\rho_j = E_j A_j / L_j$ be the axial stiffness of the *j*-th element, where E_j, A_j and L_j are the Young's modulus, cross-sectional area and length of the element, respectively. Let us assume now that $r \le m$ elements possess uncertain elastic modulus. Denoting by X_j the zero-mean random fluctuation of the uncertain elastic modulus around the nominal value, $E_{0,j}$, of the *j*-th element, such that $E_j = E_{0,j}\left(1 + X_j\right)$, one gets:

$$\rho_j = \frac{E_{0,j}\left(1 + X_j\right)A_j}{L_j} = \rho_{0,j}(1 + X_j) \qquad (6)$$

where $\rho_{0,j} = E_{0,j} A_j / L_j$ is the nominal value of the axial stiffness of the *j*-th element with $j = 1, 2, \ldots, r \le m$; the conditions $\left| X_j \right| < 1$ must be satisfied in order to guarantee always positive values of the axial stiffness. Then, the internal stiffness matrix $\mathbf{E}(\mathbf{X})$ can be written as:

$$\mathbf{E}(\mathbf{X}) = \mathbf{E}_0 + \sum_{j=1}^{r} X_j \, \mathbf{l}_{E,j} \mathbf{l}_{E,j}^{\mathrm{T}}, \tag{7}$$

where $\mathbf{E}_0$ is the nominal internal stiffness matrix and $\mathbf{l}_{E,j}$ is a vector of order *n* having zero entries except the *j*-th which is equal to $\sqrt{\rho_{0,j}}$. Notice that the dyadic product $\mathbf{l}_{E,j} \mathbf{l}_{E,j}^{\mathrm{T}}$ gives a change of rank one to the nominal internal stiffness matrix.

After simple manipulations, the equilibrium equations, in the framework of the displacement method, can be written as:

$$\mathbf{K}(\mathbf{X}) \, \mathbf{U}(\mathbf{X}) = \mathbf{f} \tag{8}$$

where

$$\mathbf{K}(\mathbf{X}) = \mathbf{C}^{\mathrm{T}} \mathbf{E}(\mathbf{X}) \; \mathbf{C} \tag{9}$$

is the $n \times n$ random stiffness matrix which depends on the r dimensionless random variables, X_j, collected into the vector $\mathbf{X}$. Furthermore, in Eq. (8) $\mathbf{f}$ is the n-vector listing the external nodal forces which, without loss of generality, are assumed to be deterministic; $\mathbf{U}(\mathbf{X})$ is the n-vector of the unknown nodal random displacements.

Taking into account Eq.(7), the random stiffness matrix $\mathbf{K}(\mathbf{X})$ in Eq.(9) can be recast as sum of its nominal value, $\mathbf{K}_0$, plus r rank-one random modifications, i.e.:

$$\mathbf{K}(\mathbf{X}) = \mathbf{K}_0 + \sum_{j=1}^{r} X_j \, \mathbf{K}_j \; = \mathbf{K}_0 + \sum_{j=1}^{r} X_j \; \mathbf{v}_j \, \mathbf{v}_j^{\mathrm{T}} \tag{10}$$

where

$$\mathbf{K}_0 = \mathbf{C}^{\mathrm{T}} \, \mathbf{E}_0 \, \mathbf{C}\,; \quad \mathbf{v}_j = \mathbf{C}^{\mathrm{T}} \, \mathbf{l}_{E,j}. \tag{11a,b}$$

and $\mathbf{K}_j = \mathbf{v}_j \, \mathbf{v}_j^{\mathrm{T}}$ is a rank-one matrix.

The solution of Eq.(8) can be formally written as:

$$\mathbf{U}(\mathbf{X}) = \mathbf{K}(\mathbf{X})^{-1} \mathbf{f}. \tag{12}$$

The probabilistic characterization of the random displacement vector $\mathbf{U}(\mathbf{X})$ can be performed straightforwardly by applying classical *Monte Carlo simulation* (*MCS*) method. Unfortunately, this method is very expensive from a computational point of view, especially for large-size structures. In this context, the knowledge of the explicit inverse of the random stiffness matrix $\mathbf{K}(\mathbf{X})$, providing the explicit relationship between the displacement vector $\mathbf{U}(\mathbf{X})$ and the random variables X_j, is very useful in order to evaluate the statistics of the stochastic response. Recently, by properly modifying the Neumann series expansion [21-23], the authors derived the so-called *Rational Series Expansion* (*RSE*) which provides an approximate explicit expression of the inverse of an invertible matrix with rank-*r* modifications.

An essential step for the application of the *RSE* is the decomposition of the matrix to be inverted as sum of the nominal value plus a deviation given by a superposition of rank-one matrices, as performed in Eq. (10) for the random stiffness matrix $\mathbf{K}(\mathbf{X})$. Then, the *RSE* leads to the following approximate explicit expression of the inverse of the random stiffness matrix [22,23]:

$$\mathbf{K}(\mathbf{X})^{-1} = \left[\mathbf{K}_0 + \sum_{j=1}^{r} X_j \mathbf{v}_j \, \mathbf{v}_j^{\mathrm{T}} \right]^{-1} = \mathbf{K}_0^{-1} - \sum_{j=1}^{r} \frac{X_j}{1+X_j \, d_j} \mathbf{D}_j + \sum_{j=1}^{r} \sum_{\substack{k=1 \\ k \neq j}}^{r} \frac{X_j \, X_k}{1+X_k \, d_k} d_{jk} \, \mathbf{D}_{jk} - \sum_{j=1}^{r} \sum_{k=1}^{r} \sum_{\substack{\ell=1 \\ \ell \neq j}}^{r} \frac{X_j \, X_k \, X_\ell}{1+X_\ell \, d_\ell} d_{jk} \, d_{k\ell} \, \mathbf{D}_{j\ell} + \cdots \tag{13}$$

where

$$\begin{aligned} d_i &= \mathbf{v}_i^{\mathrm{T}} \mathbf{K}_0^{-1} \mathbf{v}_i; \quad \mathbf{D}_i = \mathbf{K}_0^{-1} \mathbf{v}_i \, \mathbf{v}_i^{\mathrm{T}} \mathbf{K}_0^{-1}; \\ d_{is} &= \mathbf{v}_i^{\mathrm{T}} \mathbf{K}_0^{-1} \mathbf{v}_s; \quad \mathbf{D}_{is} = \mathbf{K}_0^{-1} \mathbf{v}_i \, \mathbf{v}_s^{\mathrm{T}} \mathbf{K}_0^{-1}; \quad (i, s = j, k, \ell, \cdots). \end{aligned} \tag{14a-d}$$

Equation (13) holds provided that the conditions $|X_i d_i| < 1$ are satisfied. These conditions guarantee the convergence of the Taylor series expansion $\sum_{s=0}^{\infty} (-1)^s (X_i \, d_i)^s$ to the function $1/(1 + X_i \, d_i)$ [22]. Obviously, the accuracy of Eq.(13) depends on the magnitude of the fluctuations X_j of the uncertain parameters.

For small degrees of uncertainties, i.e. $|X_j| \ll 1$, an accurate approximation of the inverse of the random stiffness matrix can be obtained by retaining only first-order terms of the *RSE* in Eq.(13) [22,25], thus obtaining the following handy formula i.e.:

$$\mathbf{K}(\mathbf{X})^{-1} = \left[\mathbf{K}_0 + \sum_{j=1}^{r} X_j \mathbf{v}_j \, \mathbf{v}_j^{\mathrm{T}} \right]^{-1} \approx \mathbf{K}_0^{-1} - \sum_{j=1}^{r} \frac{X_j}{1+X_j \, d_j} \mathbf{D}_j . \tag{15}$$

Notice that, if only one uncertain parameter is present, then the *RSE* (15) provides the exact inverse of the stiffness matrix and reduces to the well-known Sherman-Morison formula [26,27].

3.2 Mean-value vector and covariance of the response

In the previous section, an approximate explicit expression of the inverse of the random stiffness matrix of a structure with uncertain axial stiffness has been derived by applying the *RSE*. The goal is now to find the mean-value vector and the covariance matrix of the structural response. For the sake of simplicity, attention is focused on problems of practical interest which involve small fluctuations of the uncertain axial stiffness, i.e. $\left|X_j\right| \ll 1$.

Upon replacing the *RSE* of the inverse of the stiffness matrix (15) truncated to first-order terms into Eq.(12), the following approximate explicit relationship between the displacement vector $\mathbf{U}(\mathbf{X})$ and the random variables X_j is derived:

$$\mathbf{U}(\mathbf{X}) = \mathbf{K}(\mathbf{X})^{-1}\mathbf{f} \approx \mathbf{K}_0^{-1}\mathbf{f} - \sum_{i=1}^{r} \frac{X_i}{1+X_i\, d_i} \mathbf{D}_i\, \mathbf{f}. \tag{16}$$

Based on Eq. (16) and applying standard probability theory, the mean-value vector, $\boldsymbol{\mu}_{\mathbf{U}}$, and the covariance matrix, $\boldsymbol{\Sigma}_{\mathbf{U}}$, of the stochastic response can be evaluated as follows:

$$\begin{aligned}
\boldsymbol{\mu}_{\mathbf{U}} &= \mathrm{E}\left\langle \mathbf{U}(\mathbf{X}) \right\rangle = \mathbf{K}_0^{-1}\mathbf{f} - \sum_{i=1}^{r} \mathrm{E}\left\langle \chi_i \right\rangle \mathbf{D}_i\, \mathbf{f}\ ; \\
\boldsymbol{\Sigma}_{\mathbf{U}} &= \mathrm{E}\left\langle \mathbf{U}(\mathbf{X})\mathbf{U}^{\mathrm{T}}(\mathbf{X}) \right\rangle - \boldsymbol{\mu}_{\mathbf{U}}\boldsymbol{\mu}_{\mathbf{U}}^{\mathrm{T}} = \sum_{i=1}^{r}\sum_{j=1}^{r} \left[\mathrm{E}\left\langle \chi_i\, \chi_j \right\rangle - \mathrm{E}\left\langle \chi_i \right\rangle \mathrm{E}\left\langle \chi_j \right\rangle \right] \mathbf{D}_i\, \mathbf{f}\, \mathbf{f}^{\mathrm{T}}\, \mathbf{D}_j^{\mathrm{T}}
\end{aligned} \tag{17a,b}$$

where $\mathrm{E}\langle\bullet\rangle$ denotes the stochastic average operator and χ_i is an auxiliary random variable defined as:

$$\chi_i = \frac{X_i}{1 + X_i\, d_i} \; . \tag{18}$$

Without loss of generality, it can be assumed that the random variables X_i are independent, so that Eq. (17b) takes the following simplified expression:

$$\mathbf{\Sigma}_{\mathbf{U}} = \mathrm{E}\left\langle \mathbf{U}(\mathbf{X})\mathbf{U}^{\mathrm{T}}(\mathbf{X})\right\rangle - \boldsymbol{\mu}_{\mathbf{U}}\boldsymbol{\mu}_{\mathbf{U}}^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_{\chi_i}^2 \mathbf{D}_i\, \mathbf{f}\, \mathbf{f}^{\mathrm{T}} \mathbf{D}_i^{\mathrm{T}} \tag{19}$$

where

$$\sigma_{\chi_i}^2 = \mathrm{E}\left\langle \chi_i^2 \right\rangle - \left(\mathrm{E}\langle \chi_i \rangle\right)^2 . \tag{20}$$

The previous equations provide substantial computational savings over classical *MCS* method since they just involve the evaluation of the statistics of the auxiliary random variables χ_i without requiring the repeated inversion of the global random stiffness matrix.

4. INTERVAL MEAN-VALUE VECTOR AND INTERVAL COVARIANCE MATRIX OF THE RESPONSE

Let us assume now that the information on the uncertain axial stiffness (6) is imprecise, that is only a set of possible values that the quantity might take is known. Under this assumption, the zero-mean random variables X_j collected into the vector $\mathbf{X}$ are more appropriately described by a family of joint *PDFs*. Such a family is represented by the function $p_{\mathbf{X}}(\mathbf{x};\mathbf{a}_{\mathbf{X}}^{I})$, herein referred to as joint *imprecise PDF*, which depends on the set of epistemic “basic” parameters $\mathbf{a}_{X_1},\mathbf{a}_{X_2},\ldots,\mathbf{a}_{X_r}$, collected into the vector $\mathbf{a}_{\mathbf{X}}^{I}=\left[\mathbf{a}_{X_1}^{\mathrm{T}},\mathbf{a}_{X_2}^{\mathrm{T}},\ldots,\mathbf{a}_{X_r}^{\mathrm{T}}\right]^{\mathrm{T}}$ and constrained to belong to an $r\times s$-dimensional box. This means that the interval vector $\mathbf{a}_{\mathbf{X}}^{I}$ defines a bounded set of interval variables $\mathbf{a}_{\mathbf{X}}^{I}\triangleq[\underline{\mathbf{a}}_{\mathbf{X}},\overline{\mathbf{a}}_{\mathbf{X}}]\in\mathbb{IR}^{r\times s}$.

In this context, both the mean-value vector, $\boldsymbol{\mu}_{\mathbf{U}}$, and the covariance matrix, $\boldsymbol{\Sigma}_{\mathbf{U}}$, of the random displacements $\mathbf{U}(\mathbf{X})$ have an interval nature. Therefore, the aim of the analysis is the evaluation of the LB and UB of response statistics by performing suitable interval computations. From an engineering point of view, the r random variables X_i can be assumed to be independent so that the multidimensional joint *imprecise PDF*, $p_{\mathbf{X}}(\mathbf{x};\mathbf{a}_{X_i}^{I})$, can be written as:

$$p_{\mathbf{X}}(\mathbf{x};\mathbf{a}_{\mathbf{X}}^{I})=\prod_{k=1}^{r}p_{X_k}(x_k;\mathbf{a}_{X_k}^{I}) \tag{21}$$

where $p_{X_k}(x_k;\mathbf{a}_{X_k}^{I})$ is the marginal *imprecise PDF* of the random variable X_k with the interval vector $\mathbf{a}_{X_k}\in\mathbb{IR}^{s_k}$ of basic parameters belonging to the admissible closed region Q_k.

As outlined in Section 2, the interval nature of statistics related to the random variables X_i with *imprecise PDF* can be formally expressed by introducing the *interval stochastic average operator* $\mathrm{E}^{I}\langle\bullet\rangle$. In particular, upon replacing the stochastic average operator by the interval one, Eqs.(17a) and (19) yield the interval mean-value vector and covariance matrix of the random displacements as:

$$\begin{aligned}\boldsymbol{\mu}_{\mathbf{U}}^{I}&=\mathrm{E}^{I}\left\langle\mathbf{U}(\mathbf{X})\right\rangle=\mathbf{K}_0^{-1}\mathbf{f}-\sum_{i=1}^{r}\mu_{\chi_i}^{I}\mathbf{D}_i\,\mathbf{f};\\ \boldsymbol{\Sigma}_{\mathbf{U}}^{I}&=\mathrm{E}^{I}\left\langle\mathbf{U}(\mathbf{X})\mathbf{U}^{\mathrm{T}}(\mathbf{X})\right\rangle-\boldsymbol{\mu}_{\mathbf{U}}^{I}\boldsymbol{\mu}_{\mathbf{U}}^{I\,\mathrm{T}}=\sum_{i=1}^{r}\sigma_{\chi_i}^{2\,I}\mathbf{D}_i\,\mathbf{f}\,\mathbf{f}^{\mathrm{T}}\mathbf{D}_i^{\mathrm{T}}\end{aligned} \tag{22a,b}$$

where

$$\mu_{\chi_i}^I \equiv \mathrm{E}^I\langle\chi_i\rangle = \left[\underline{\mu}_{\chi_i}, \overline{\mu}_{\chi_i}\right]; \\ \sigma_{\chi_i}^{2\;I} \equiv \mathrm{E}^I\langle\chi_i^2\rangle - \left(\mathrm{E}^I\langle\chi_i\rangle\right)^2 = \left[\underline{\sigma}_{\chi_i}^2, \overline{\sigma}_{\chi_i}^2\right] \tag{23a,b}$$

are the interval mean-value and variance of the auxiliary random variable χ_i introduced in Eq. (18). According to standard probability theory, such quantities are defined as:

$$\mu_{\chi_i}^I = \int_{-\infty}^{+\infty} \frac{x_i}{1+d_i\, x_i}\, p_{X_i}(x_i;\mathbf{a}_{X_i}^I)\mathrm{d}x_i; \\ \sigma_{\chi_i}^{2\;I} = \int_{-\infty}^{+\infty} \left(\frac{x_i}{1+d_i\, x_i} - \mu_{\chi_i}^I\right)^2 p_{X_i}(x_i;\mathbf{a}_{X_i}^I)\mathrm{d}x_i \tag{24a,b}$$

where $p_{X_i}(x_i;\mathbf{a}_{X_i}^I)$ is the marginal *imprecise PDF* of the random variable X_i. In the context of the *CIA*, the LB and UB of $\mu_{\chi_i}^I$ and $\sigma_{\chi_i}^{2\;I}$ are given by definition as

$$\underline{\mu}_{\chi_i} = \min_{\mathbf{a}_{X_i}^I \in Q_i} \left\{ \int_{-\infty}^{\infty} \frac{x_i}{1+d_i\, x_i}\, p_{X_i}(x_i;\mathbf{a}_{X_i}^I)\,\mathrm{d}x_i \right\}; \\ \overline{\mu}_{\chi_i} = \max_{\mathbf{a}_{X_i}^I \in Q_i} \left\{ \int_{-\infty}^{\infty} \frac{x_i}{1+d_i\, x_i}\, p_{X_i}(x_i;\mathbf{a}_{X_i}^I)\,\mathrm{d}x_i \right\} \tag{25a,b}$$

and

$$\overline{\sigma}_{\chi_i}^2 = \max_{\mathbf{a}_{X_i}^I \in Q_i} \left\{ \int_{-\infty}^{\infty} \left(\frac{x_i}{1+d_i\, x_i} - \mu_{\chi_i}^I\right)^2 p_{X_i}(x_i;\mathbf{a}_{X_i}^I)\,\mathrm{d}x_i \right\}; \\ \underline{\sigma}_{\chi_i}^2 = \min_{\mathbf{a}_{X_i}^I \in Q_i} \left\{ \int_{-\infty}^{\infty} \left(\frac{x_i}{1+d_i\, x_i} - \mu_{\chi_i}^I\right)^2 p_{X_i}(x_i;\mathbf{a}_{X_i}^I)\,\mathrm{d}x_i \right\}. \tag{26a,b}$$

By applying the *improved interval analysis* via *extra unitary interval* (*IIA* via *EUI*) [20], the interval mean-value of the random variable χ_i can be rewritten in *affine form* as:

$$\mu_{\chi_i}^I = \mu_{0,\chi_i} + \Delta\mu_{\chi_i}\, \hat{e}_i^I \tag{27}$$

where $\hat{e}_i^I$ is the *EUI* associated to the interval mean-value of the i-th random variable χ_i; μ_{0,χ_i} and $\Delta\mu_{\chi_i}$ denote the midpoint value and deviation amplitude (or radius) of $\mu_{\chi_i}^I$ defined, respectively, as:

$$\mu_{0,\chi_i} = \frac{\underline{\mu}_{\chi_i} + \overline{\mu}_{\chi_i}}{2}; \qquad \Delta\mu_{\chi_i} = \frac{\overline{\mu}_{\chi_i} - \underline{\mu}_{\chi_i}}{2}. \tag{28a,b}$$

Similarly, the interval variance of the random variable χ_i can be expressed in the following *affine form*:

$$\sigma_{\chi_i}^{2\ I} = \sigma_{0,\chi_i}^2 + \Delta\sigma_{\chi_i}^2\, \hat{e}_i^I \tag{29}$$

where σ_{0,χ_i}^2 and $\Delta\sigma_{\chi_i}^2$ denote the midpoint value and deviation amplitude (or radius) of $\sigma_{\chi_i}^{2\ I}$ defined, respectively, as:

$$\sigma_{0,\chi_i}^2 = \frac{\underline{\sigma}_{\chi_i}^2 + \overline{\sigma}_{\chi_i}^2}{2}; \qquad \Delta\sigma_{0,\chi_i}^2 = \frac{\overline{\sigma}_{\chi_i}^2 - \underline{\sigma}_{\chi_i}^2}{2}. \tag{30a,b}$$

Substituting Eqs. (27) and (29) into Eqs. (22a,b), the interval mean-value vector and covariance matrix of the random displacements can be rewritten as sum of the midpoint value plus an interval deviation:

$$\boldsymbol{\mu}_{\mathbf{U}}^{I} = \mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{U}}^{I}\right\} + \mathrm{dev}\left\{\boldsymbol{\mu}_{\mathbf{U}}^{I}\right\};$$
$$\boldsymbol{\Sigma}_{\mathbf{U}}^{I} = \mathrm{mid}\left\{\boldsymbol{\Sigma}_{\mathbf{U}}^{I}\right\} + \mathrm{dev}\left\{\boldsymbol{\Sigma}_{\mathbf{U}}^{I}\right\} \tag{31a,b}$$

where

$$\mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{U}}^{I}\right\} = \mathbf{K}_0^{-1}\mathbf{f} - \sum_{i=1}^{r} \mu_{0,\chi_i} \mathbf{D}_i \, \mathbf{f} \;;$$
$$\mathrm{mid}\left\{\boldsymbol{\Sigma}_{\mathbf{U}}^{I}\right\} = \sum_{i=1}^{r} \sigma_{0,\chi_i}^2 \; \mathbf{D}_i \, \mathbf{f} \, \mathbf{f}^{\mathrm{T}} \mathbf{D}_i^{\mathrm{T}} \tag{32a,b}$$

and

$$\mathrm{dev}\left\{\boldsymbol{\mu}_{\mathbf{U}}^{I}\right\} = -\sum_{i=1}^{r} \Delta\mu_{\chi_i} \; \mathbf{D}_i \; \mathbf{f} \; \hat{e}_i^{I} \;;$$
$$\mathrm{dev}\left\{\boldsymbol{\Sigma}_{\mathbf{U}}^{I}\right\} = \sum_{i=1}^{r} \Delta\sigma_{\chi_i}^2 \; \mathbf{D}_i \, \mathbf{f} \, \mathbf{f}^{\mathrm{T}} \mathbf{D}_i^{\mathrm{T}} \hat{e}_i^{I} . \tag{33a,b}$$

In the previous equations, $\mathrm{mid}\{\bullet\}$ and $\mathrm{dev}\{\bullet\}$ denote the midpoint and interval deviation of the quantity between curly brackets.

Based on Eqs. (31a) and following the *IIA* via *EUI* [20,22], the LB and UB of the interval mean-value vector of displacements can be evaluated as:

$$\underline{\boldsymbol{\mu}}_{\mathbf{U}} = \mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{U}}^{I}\right\} - \Delta\boldsymbol{\mu}_{\mathbf{U}};$$
$$\overline{\boldsymbol{\mu}}_{\mathbf{U}} = \mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{U}}^{I}\right\} + \Delta\boldsymbol{\mu}_{\mathbf{U}} \tag{34a,b}$$

where

$$\Delta\boldsymbol{\mu}_{\mathbf{U}} = \sum_{i=1}^{r_K} \Delta\mu_{\chi_i} \left|\mathbf{D}_i \mathbf{f}\right| \tag{35a,b}$$

is the deviation amplitude vector and $|\bullet|$ denotes the component wise absolute value.

Similarly, Eq. (31b) leads to the following expressions of the LB and UB of the covariance matrix of displacements:

$$\underline{\boldsymbol{\Sigma}}_{\mathbf{U}} = \boldsymbol{\Sigma}_{0,\mathbf{U}} - \Delta\boldsymbol{\Sigma}_{\mathbf{U}};$$
$$\overline{\boldsymbol{\Sigma}}_{\mathbf{U}} = \boldsymbol{\Sigma}_{0,\mathbf{U}} + \Delta\boldsymbol{\Sigma}_{\mathbf{U}} \qquad (36a,b)$$

where:

$$\Delta\boldsymbol{\Sigma}_{\mathbf{U}} = \sum_{i=1}^{r} \Delta\sigma_{\chi_i}^2 \left|\mathbf{D}_i\,\mathbf{f}\,\mathbf{f}^{\mathrm{T}}\mathbf{D}_i^{\mathrm{T}}\right| \qquad (37a,b)$$

is the deviation amplitude matrix.

5. NUMERICAL APPLICATIONS

To assess the accuracy of the proposed procedure, two numerical applications concerning a braced shear-type frame and a 3D truss with uncertain axial stiffness of braces and bars, respectively, are presented. The fluctuations of the uncertain parameters, $\rho_i = \rho_{0,i}\left(1+X_i\right)$, around the nominal value are modeled as zero-mean independent random variables with Gaussian *imprecise PDF*, i.e.:

$$p_{X_i}(x_i;\sigma_{X_i}^I) = \frac{1}{\sigma_{X_i}^I\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x_i}{\sigma_{X_i}^I}\right)^2\right] \qquad (38)$$

where $\sigma_{X_i}^I = \left[\underline{\sigma}_{X_i}, \overline{\sigma}_{X_i}\right]$ is the interval standard deviation. In order to ensure positive values of the axial stiffness, the midpoint value of $\sigma_{X_i}^I$ is assumed to be sufficiently small, say it is set to $\sigma_{0,X_i} = 0.11$.

Figure 1 shows the realizations of the Gaussian *imprecise PDF* (38) pertaining to the LB and UB, $\underline{\sigma}_{X_i}$ and $\overline{\sigma}_{X_i}$, of the interval standard deviation, $\sigma_{X_i}^I$, for two different deviation amplitudes, say $\Delta\sigma_{X_i} = 0.011$ and $\Delta\sigma_{X_i} = 0.022$. As expected, the larger the degree of uncertainty affecting the standard deviation the greater is the deviation of the *PDF* from the nominal one pertaining to the midpoint value $\sigma_{0,X_i} = 0.11$.

The accuracy of the proposed procedure is demonstrated by performing appropriate comparisons with the results obtained by applying classical *Monte Carlo Simulation* (*MCS*) in conjunction with the *Vertex Method* (*VM*). The latter is a combinatorial procedure, first introduced by Dong and Shah (1987) [28], which may be viewed as the non-probabilistic counterpart of *MCS*. For a problem involving r interval parameters, basically it consists of performing 2^r deterministic analyses as many as are the combinations of the endpoints of the parameters. Then, the LB and UB of the response quantity of interest are evaluated as the minimum and maximum among the responses pertaining to the explored combinations. The main steps of the procedure resulting from the joint application of *MCS* and the *VM* (*MCS*-*VM*), to handle the probabilistic and interval character of uncertainties, respectively, are summarized by the following flow-chart:

- input: geometry; material properties; boundary and loading conditions; *imprecise PDF* of the uncertain parameters.
- start combination loop, say for $k = 1, 2, \ldots, N_\mathrm{c} = 2^r$
    - o set $\sigma_{X_i} = \sigma_{X_i}^{(k)},\ i = 1, 2, \ldots, r$
    - start sample loop, say for $j = 1, 2, \ldots, N_\mathrm{g}$

- generate a sample of r independent Gaussian random variables $X_i^{(k,j)}$, $i = 1, 2, \ldots, r$, with $\sigma_{X_i} = \sigma_{X_i}^{(k)}$
- set $\rho_i \equiv \rho_i^{(k,j)} = \rho_{0,i}\left(1 + X_i^{(k,j)}\right)$, $i = 1, 2, \ldots, r$
- evaluate $\mathbf{U}^{(k,j)}\left(\mathbf{X}\right) = \mathbf{K}^{(k,j)}(\mathbf{X})^{-1}\mathbf{f}$

- end sample loop
- evaluate response statistics

$$\mu_{U_s}^{(k)} = \mathrm{E}\left\langle U_s^{(k)}\left(\mathbf{X}\right)\right\rangle = \sum_{j=1}^{N_\mathrm{g}} \frac{U_s^{(k,j)}}{N_\mathrm{g}};\ \mathrm{E}\left\langle U_s^{(k)^2}\left(\mathbf{X}\right)\right\rangle = \sum_{j=1}^{N_\mathrm{g}} \frac{{U_s^{(k,j)}}^2}{N_\mathrm{g}};$$

$$\sigma_{U_s}^{2(k)} = \mathrm{E}\left\langle U_s^{(k)^2}\left(\mathbf{X}\right)\right\rangle - \left(\mu_{U_s}^{(k)}\right)^2,\ s = 1, 2, \ldots, n$$

- end combination loop
- compute the LB and UB of response statistics as

$$\underline{\mu}_{U_s} = \min\left\{\mu_{U_s}^{(1)}, \mu_{U_s}^{(2)}, \ldots, \mu_{U_s}^{(N_\mathrm{c})}\right\};\quad \overline{\mu}_{U_s} = \max\left\{\mu_{U_s}^{(1)}, \mu_{U_s}^{(2)}, \ldots, \mu_{U_s}^{(N_\mathrm{c})}\right\}$$

$$\underline{\sigma}_{U_s}^2 = \min\left\{\sigma_{U_s}^{2(1)}, \sigma_{U_s}^{2(2)}, \ldots, \sigma_{U_s}^{2(N_\mathrm{c})}\right\};\quad \overline{\sigma}_{U_s}^2 = \max\left\{\sigma_{U_s}^{2(1)}, \sigma_{U_s}^{2(2)}, \ldots, \sigma_{U_s}^{2(N_\mathrm{c})}\right\},\ s = 1, 2, \ldots, n.$$

The *MCS-VM* is computationally intensive since it requires $N_\mathrm{c} \times N_\mathrm{g}$ deterministic analyses. Unlike the proposed procedure, it becomes unfeasible as the number of uncertain parameters increases.

5.1 Braced shear-type frame with uncertain axial stiffness

As first application, the braced shear-type frame depicted in Figure 2 is analyzed. The geometry can be deduced from Figure 2, where $H_1 = 2.55$ m, $H_2 = 2.75$ m, $H_3 = 4.55$ m, $L_1 = L_6 = 3.42$ m and $L_2 = L_3 = L_4 = L_5 = 3.40$ m. The beams have rectangular cross-section with $b = 0.50$ m and $h = 0.23$ m at the first and second floor, $b = 0.46$ m and $h = 0.87$ m at the third floor. All the columns have square cross-section with $b = h = 0.46$ m. Young's modulus of beams and columns material is $E = E_0 = 28.50$ GPa. The axial stiffnesses of the diagonal braces are modeled as independent random variables $\rho_i = \rho_{0,i}(1 + X_i)$, $i = 1, 2, \ldots, 6$, with fluctuations around the nominal value, $\rho_{0,i} = 1.20 \times 10^8$ N/m, described by the Gaussian *imprecise PDF* (38). The frame is subjected at each floor to deterministic static loads of intensity: $f_1 = 200$ kN, $f_2 = 400$ kN and $f_3 = 600$ kN (see Figure 2). As response quantities of interest, the floor displacements U_i, $i = 1, 2, 3$, are selected (see Figure 2).

First, the accuracy of the proposed approximate explicit expressions of response statistics (Eqs. (17a) and (19)) derived by using the *RSE* in the context of classical probability theory is assessed. To this aim, the uncertain parameters are assumed to be described by a Gaussian *PDF* with standard deviation equal to the midpoint value of $\sigma^I_{X_i}$, say $\sigma_{0,X_i} = 0.11$ (continuous black curve in Figure 1). Table 1 shows a very good match between the estimates of the mean-value and variance of the floor displacements provided by the proposed method and *MCS* with $N_{\mathrm{g}} = 5 \times 10^4$ samples.

Then, the bounds of response statistics of the structure with uncertain parameters characterized by *imprecise PDF* are evaluated. Tables 2 and 3 list the LB and UB of the interval mean-value and variance of the floor displacements, respectively, obtained by applying the proposed procedure and

MCS-VM for a deviation amplitude of the standard deviation of the random variables X_i equal to $\Delta\sigma_{X_i} = 0.022$. A very good agreement between the proposed approximate explicit expressions of the bounds of response statistics (Eqs.(34a,b) and (36a,b)) and the reference ones is observed. By inspection of Tables 2 and 3, it is inferred that the LB and UB of the interval mean-value of displacements are very close to each other as well as to the mean-values obtained setting $\sigma_{0,X_i} = 0.11$ (see Table 1). This implies that the deviation amplitude of the relevant interval is very small, that is the mean-value is slightly affected by the imprecision of the *PDF*. It is worth remarking that the presented procedure is much less expensive than the *MCS-VM*. Indeed, in the present case, the latter requires $N_{\mathrm{c}} \times N_{\mathrm{g}} = 2^6 \times 5 \times 10^4$ deterministic analyses, $N_{\mathrm{g}} = 5 \times 10^4$ being the number of considered samples.

In order to estimate the influence of the *imprecise PDF* on response statistics, the so-called *coefficient of interval uncertainty* (c.i.u) of the interval mean-value and variance of displacements is evaluated, i.e.:

$$\text{c.i.u. }[\mu_{U_j}^{I}] = \frac{\Delta\mu_{U_j}}{\left|\mathrm{mid}\left\{\mu_{U_j}^{I}\right\}\right|};$$
$$\text{c.i.u. }[\sigma_{U_j}^{2I}] = \frac{\Delta\sigma_{U_j}^{2I}}{\mathrm{mid}\left\{\sigma_{U_j}^{2I}\right\}}. \qquad (39a,b)$$

The c.i.u. provides a measure of the dispersion of interval statistics around their midpoint value. In Table 4, the proposed c.i.u. of the interval mean-value and variance of floor displacements for two different values of the deviation amplitude of the standard deviation of the uncertain parameters, say $\Delta\sigma_{X_i} = 0.011$ and $\Delta\sigma_{X_i} = 0.022$, is reported. In agreement with the small influence of imprecision deduced from Tables 2 and 3, the interval mean-value of the response exhibits a very small dispersion around the midpoint value. Conversely, imprecision proves to have a significant influence on the interval variance of floor displacements which is characterized by a large dispersion around the midpoint value. In particular, it can be seen that the dispersion of response statistics around the midpoint value increases with the deviation amplitude of the interval standard deviation, $\Delta\sigma_{X_i}$, of the uncertain parameters.

5.2 3D truss with uncertain Young's moduli

The second application concerns the 3D 26-bar truss structure under deterministic static loads shown in Figure 3. The following geometrical and mechanical properties are assumed: $A_0 = A_{0,i} = 4.27\times10^{-4}$ m^2, $E_0 = E_{0,i} = 2.1\times10^{8}$ kN/m^2, $i = 1,2,\ldots,26$, and $f = 200$ kN. Young's moduli of $r = 12$ bars are modeled as independent random variables, $E_i = E_0(1+X_i)$, $i = 1,2,\ldots,12$, (see bar numbering in Figure 3) with fluctuations around the nominal value characterized by a Gaussian *imprecise PDF* (see Eq. (38)).

First, the accuracy of response statistics provided by the *RSE* is assessed under the assumption that the uncertain parameters are described by a Gaussian *PDF* with deterministic standard deviation $\sigma_{0,X_i} = 0.11$ (continuous black curve in Figure 1). Figure 4 displays the comparison between the proposed estimates of the mean-value and variance of the nodal displacements (Eqs. (17a) and (19)) and those provided by standard *MCS* with $N_{\mathrm{g}} = 5\times10^{4}$ samples. Notice that the proposed explicit expressions of response statistics are in very good agreement with *MCS* data.

Then, the effectiveness of the presented method for the evaluation of the bounds of response statistics under the assumption of Gaussian *imprecise PDF* of the axial stiffness is scrutinized. In

Figure 5, the LB and UB of the interval mean-value of nodal displacements for a deviation amplitude of the interval standard deviation of the random variables X_i equal to $\Delta\sigma_{X_i} = 0.011$ are plotted. A very good agreement between the proposed estimates and those provided by the *MCS-VM* is observed. Notice that, considering $N_\mathrm{g} = 5\times10^4$ samples, the *MCS-VM* requires $N_\mathrm{c} \times N_\mathrm{g} = 2^{12} \times 5\times10^4$ deterministic analyses of the truss. It is worth observing that the LB and UB of the interval mean-value of nodal displacements are very close to each other as well as to the mean-values obtained setting $\sigma_{0,X_i} = 0.11$ (see Figure 4), that is the deviation amplitude of the relevant interval is very small. Numerical results, omitted for conciseness, show that this circumstance holds even for larger deviation amplitudes $\Delta\sigma_{X_i}$ of the interval standard deviation of the random variables X_i. This implies that the mean-value of the response is slightly affected by both the probabilistic and non-probabilistic character of the uncertain parameters.

Figure 6 demonstrates the accuracy of the proposed bounds of the interval variance of nodal displacements by appropriate comparisons with those obtained by applying the *MCS-VM* for two different values of $\Delta\sigma_{X_i}$. As expected, the region of the interval variance of the response becomes wider as larger deviation amplitudes $\Delta\sigma_{X_i}$ are considered.

Table 5 lists the c.i.u. of the mean-value and variance of the nodal displacements in the load direction for two different deviation amplitudes of the interval standard deviation of the random variables X_i, say $\Delta\sigma_{X_i} = 0.011$ and $\Delta\sigma_{X_i} = 0.022$. Notice that the interval mean-value of the response exhibits a very small dispersion around the midpoint value. Indeed, as already mentioned, it is slightly affected by both randomness and imprecision. Conversely, the interval variance of the nodal displacements in the load direction is strongly influenced by imprecision which produces a large dispersion around the midpoint value. In particular, it is observed that the larger the deviation amplitude of the interval standard deviation of X_i, $\Delta\sigma_{X_i}$, the greater is the dispersion of response statistics around the midpoint value.

The influence of the number of uncertain parameters is also investigated. Figure 7 displays the proposed bounds of the interval variance of nodal displacements of the 3D truss with $r = 12$ and $r = 26$ bars exhibiting uncertain Young's moduli. Obviously, the *MCS-VM* procedure is unfeasible when $r = 26$ random variables are involved. As expected, both the LB and UB of response variance increase when a larger number of uncertain parameters is considered. Table 6 lists the c.i.u. of the interval mean-value and variance of the nodal displacements of the truss in the load direction when Young's moduli of all the $r = 26$ bars are assumed to be uncertain. The comparison with the c.i.u. of displacements pertaining to the truss with $r = 12$ uncertain parameters, reported in Table 5, shows that the dispersion of response statistics around the midpoint value slightly increases when a larger number of uncertainties is considered.

6. CONCLUSIONS

The analysis of structures with uncertain axial stiffness described by *imprecise PDF* with interval basic parameters (mean-value, variance, etc.) has been addressed. The challenging task of processing simultaneously the random and interval character of uncertainties has been faced by applying standard probabilistic analysis in conjunction with the so-called *improved interval analysis* via *extra unitary interval* and the *Rational Series Expansion*, recently introduced by the authors. The main feature of the proposed procedure is the capability of providing approximate explicit expressions of the bounds of the interval mean-value and variance of the response. Other remarkable advantages of the presented method are: *i*) the computational efficiency even when a large number of uncertain parameters is involved; *ii*) the ability to limit the conservatism affecting the *classical interval analysis*; *iii*) the high accuracy even when the basic parameters of the *PDF* are affected by relatively high degrees of uncertainty.

The effectiveness of the proposed procedure has been demonstrated through appropriate comparisons with the bounds of response statistics obtained by applying classical *Monte Carlo Simulation* in conjunction with a combinatorial procedure so as to handle simultaneously the random and interval nature of uncertainties, respectively.

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Figure Captions

Figure 1. Realizations of the Gaussian *imprecise PDF* of the dimensionless fluctuation of the uncertain axial stiffness: a) $\Delta\sigma_{X_i} = 0.011$; b) $\Delta\sigma_{X_i} = 0.022$.

Figure 2. Braced shear-type frame with uncertain axial stiffness of braces.

Figure 3. 3D truss structure with uncertain Young's moduli.

Figure 4. a) Mean-value and b) variance of the nodal displacements of the 3D truss with random Young's moduli described by Gaussian *PDF* ($\sigma_{0,X_i} = 0.11$): comparison between the proposed estimates and *MCS* data.

Figure 5. a) Lower bound and b) upper bound of the interval mean-value of the nodal displacements of the 3D truss with random Young's moduli described by Gaussian *imprecise PDF* ($\Delta\sigma_{X_i} = 0.022$): comparison between the proposed estimates and those obtained by the joint application of *MCS* and the *VM*.

Figure 6. Lower bound and upper bound of the interval variance of the nodal displacements of the 3D truss with random Young's moduli described by Gaussian *imprecise PDF*: comparison between the proposed estimates and those obtained by the joint application of *MCS* and the *VM* for: a) $\Delta\sigma_{X_i} = 0.011$, b) $\Delta\sigma_{X_i} = 0.022$.

Figure 7. Influence of the number r of bars with random Young's moduli on the proposed LB and UB of the interval variance of the nodal displacements of the 3D truss.

Table Captions

Table 1. Mean-value and variance of floor displacements of the braced shear-type frame with random axial stiffness of braces described by a Gaussian *PDF* ($\sigma_{0,X_i} = 0.11$): comparison between the proposed estimates and *MCS* data.

Table 2. LB and UB of the interval mean-value of floor displacements of the braced shear-type frame with random axial stiffness of braces described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11$, $\Delta\sigma_{X_i} = 0.022$): comparison between the proposed estimates and the reference ones provided by the *MCS-VM*.

Table 3. LB and UB of the interval variance of floor displacements of the braced shear-type frame with random axial stiffness of braces described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11$, $\Delta\sigma_{X_i} = 0.022$): comparison between the proposed estimates and the reference ones provided by the *MCS-VM*.

Table 4. Proposed *coefficient of interval uncertainty* of the mean-value and variance of the floor displacements of the braced shear-type frame for two different deviation amplitudes, $\Delta\sigma_{X_i}$, of the standard deviation of the random variables X_i ($i = 1, 2, \ldots, r = 6$) described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11$).

Table 5. Proposed *coefficient of interval uncertainty* of the mean-value and variance of the nodal displacements in the load direction of the 3D truss for two different deviation amplitudes $\Delta\sigma_{X_i}$ of the standard deviation of the random variables X_i ($i = 1, 2, \ldots, r = 12$) described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11$).

Table 6. Proposed *coefficient of interval uncertainty* of the mean-value and variance of the nodal displacements in the load direction of the 3D truss with uncertain Young's moduli of all the bars described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11$; $\Delta\sigma_{X_i} = 0.022$, $i = 1, 2, \ldots, r = 26$).

Figure 1

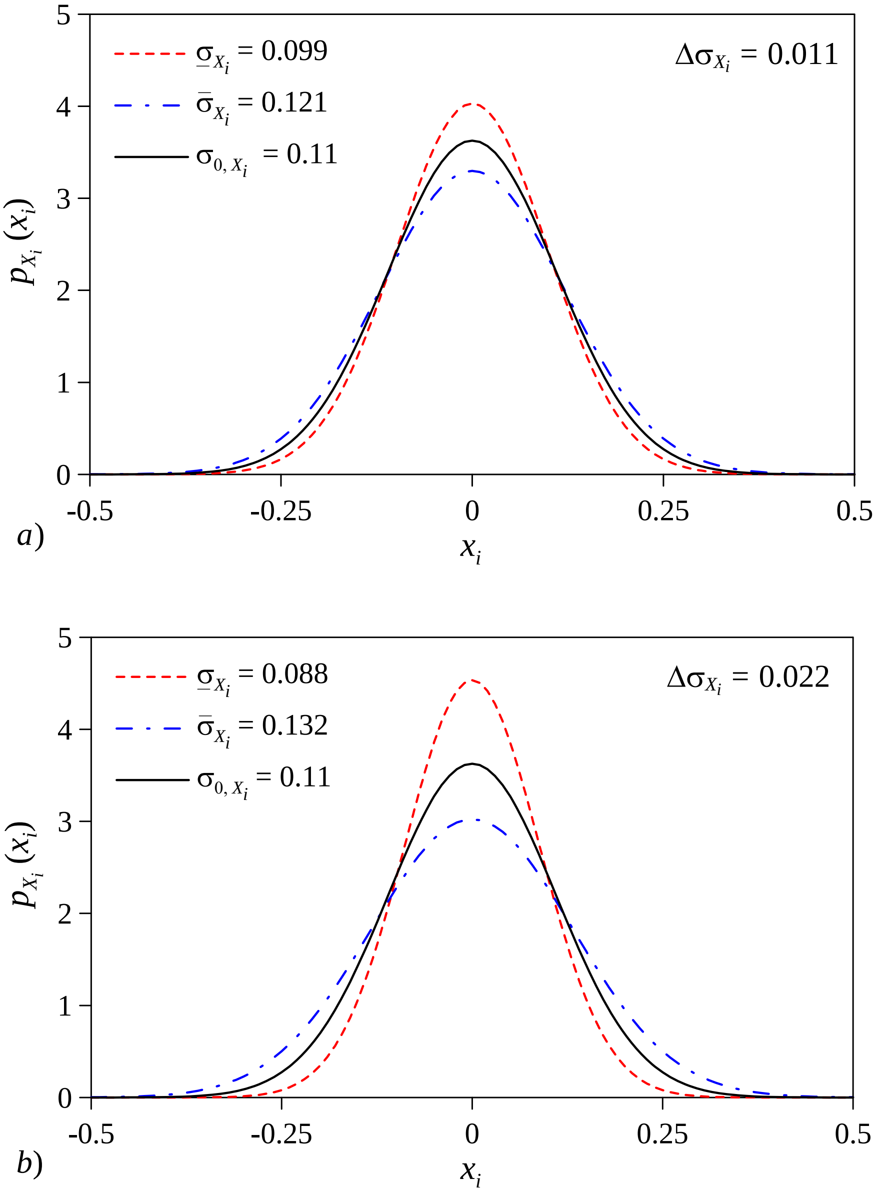


Figure 1. Realizations of the Gaussian *imprecise PDF* of the dimensionless fluctuation of the uncertain axial stiffness: a) $\Delta\sigma_{X_i} = 0.011$; b) $\Delta\sigma_{X_i} = 0.022$.

Figure 2

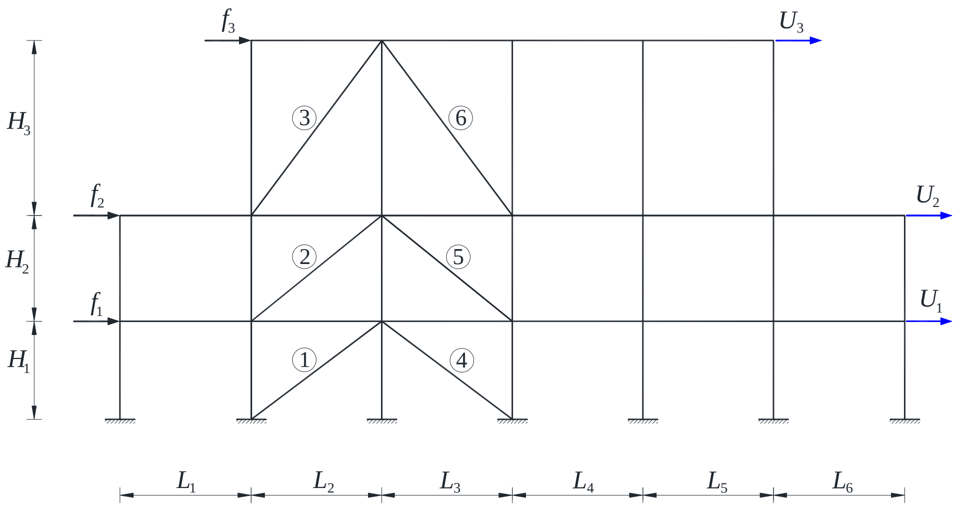


Figure 2. Braced shear-type frame with uncertain axial stiffness of the braces.

Figure 3

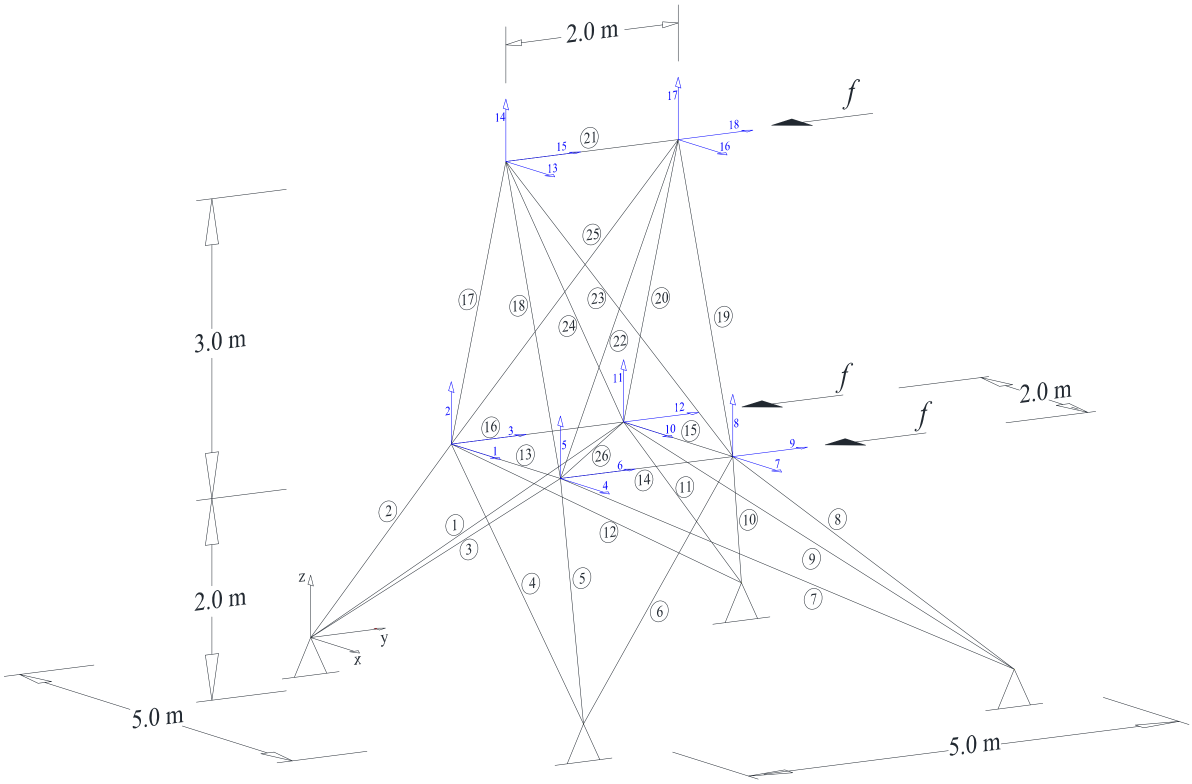


Figure 3. 3D truss with uncertain Young's moduli.

Figure 4

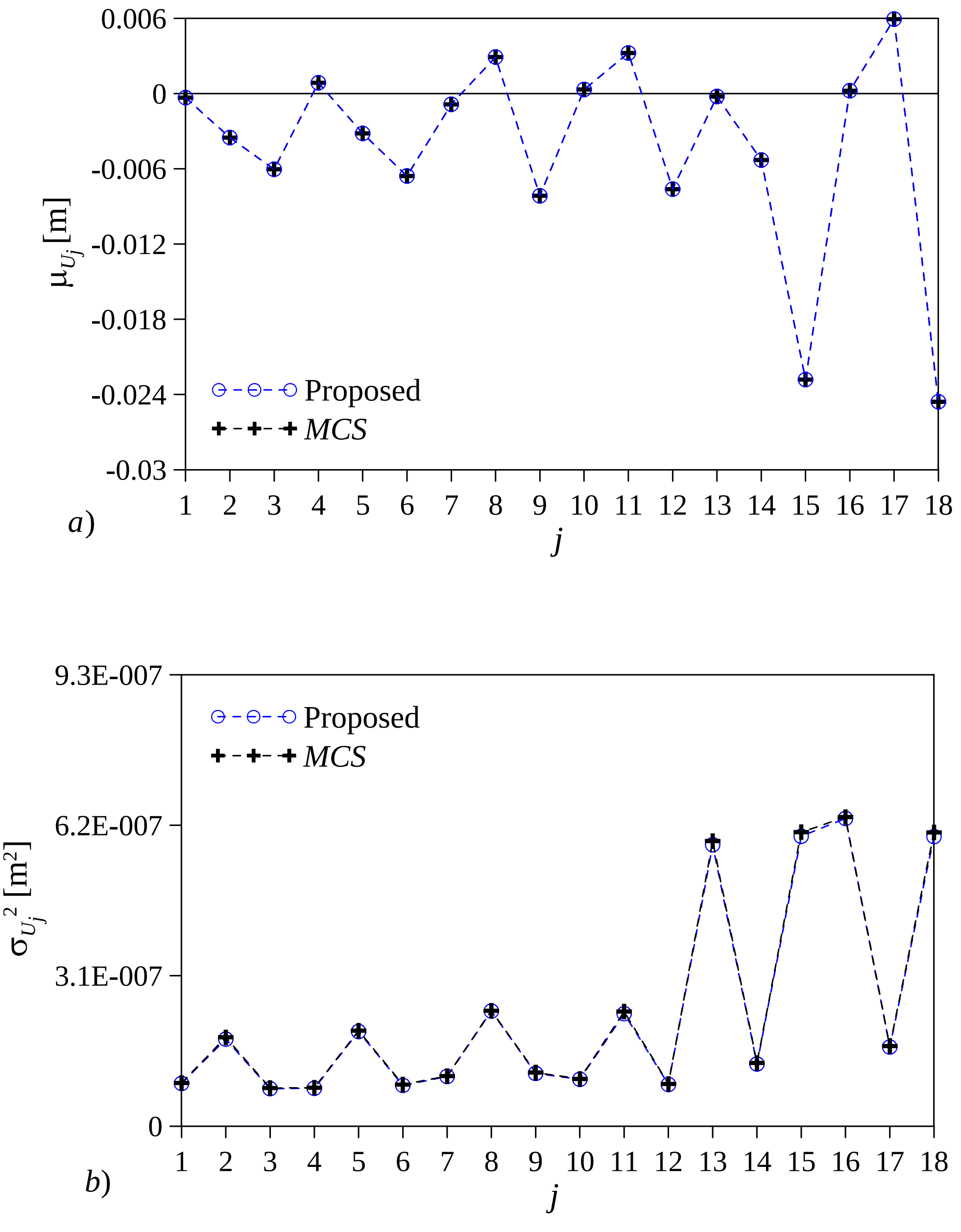


Figure 4. a) Mean-value and b) variance of the nodal displacements of the 3D truss with random Young's moduli described by Gaussian *PDF* ($\sigma_{0,X_i} = 0.11$): comparison between the proposed estimates and *MCS* data.

Figure 5

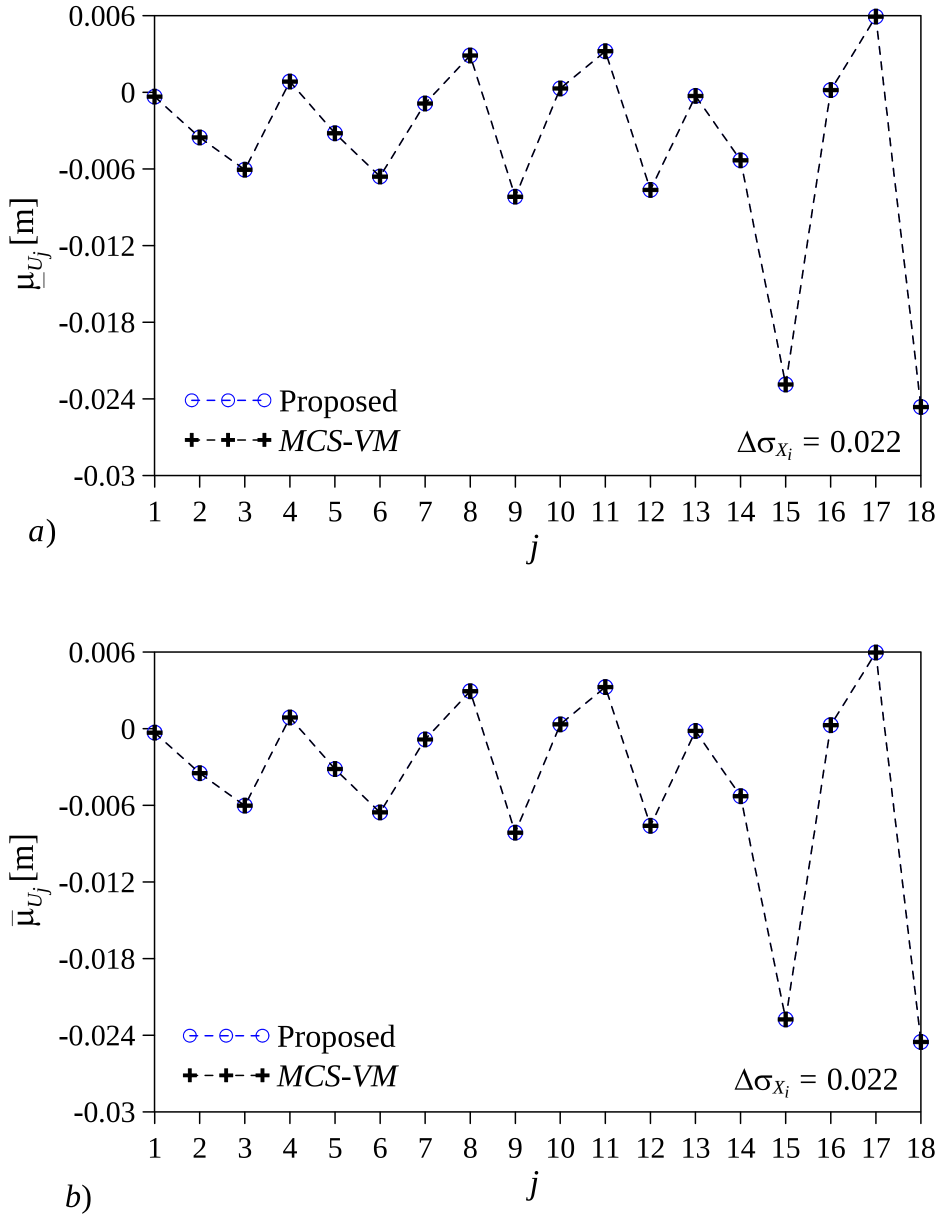
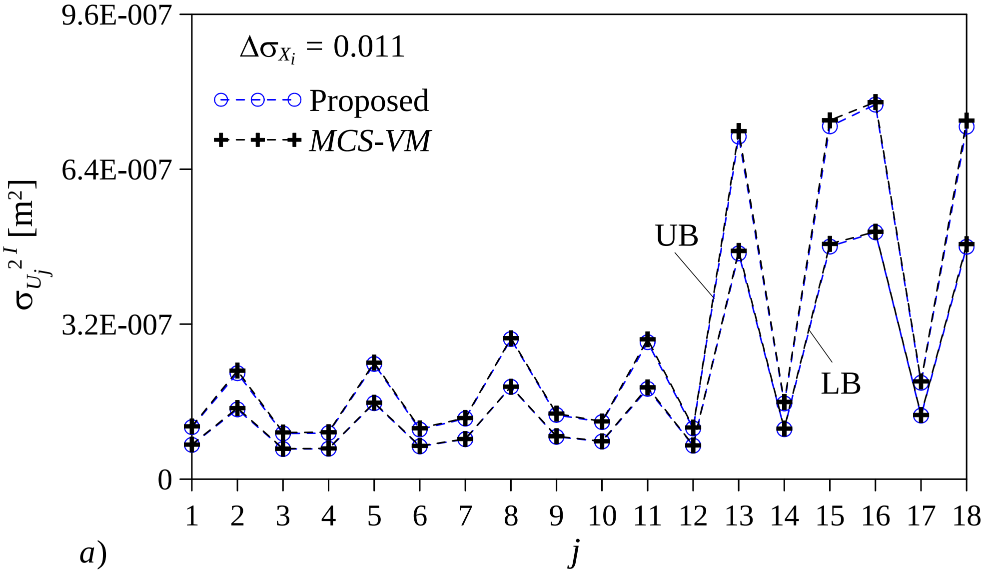


Figure 5. a) Lower bound and b) upper bound of the interval mean-value of the nodal displacements of the 3D truss with random Young's moduli described by Gaussian *imprecise PDF* ($\Delta\sigma_{X_i} = 0.022$): comparison between the proposed estimates and those obtained by the joint application of *MCS* and the *VM*.

Figure 6



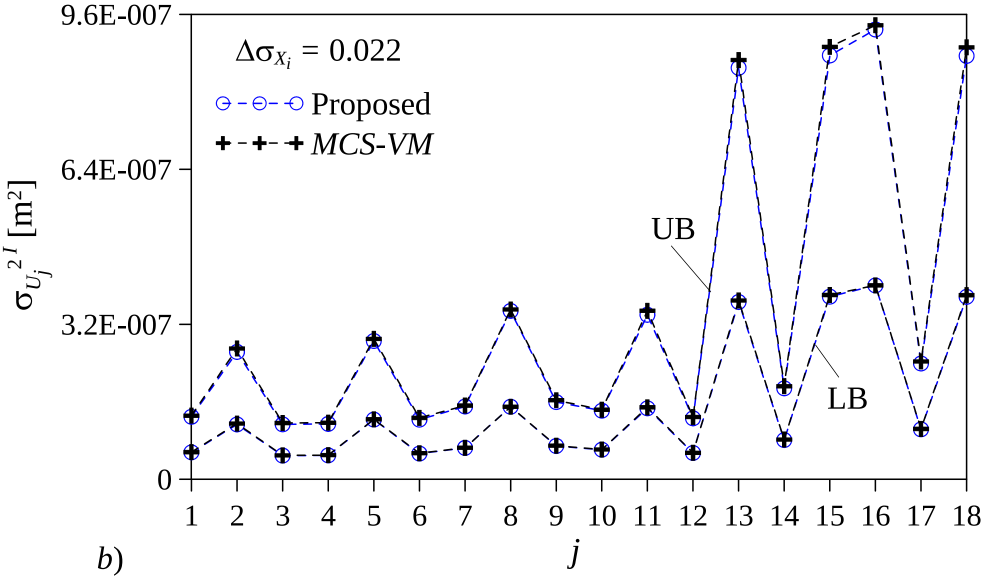


Figure 6. Lower bound and upper bound of the interval variance of the nodal displacements of the 3D truss with random Young's moduli described by Gaussian *imprecise PDF*: comparison between the proposed estimates and those obtained by the joint application of *MCS* and the *VM* for: a) $\Delta\sigma_{X_i} = 0.011$, b) $\Delta\sigma_{X_i} = 0.022$.

Figure 7

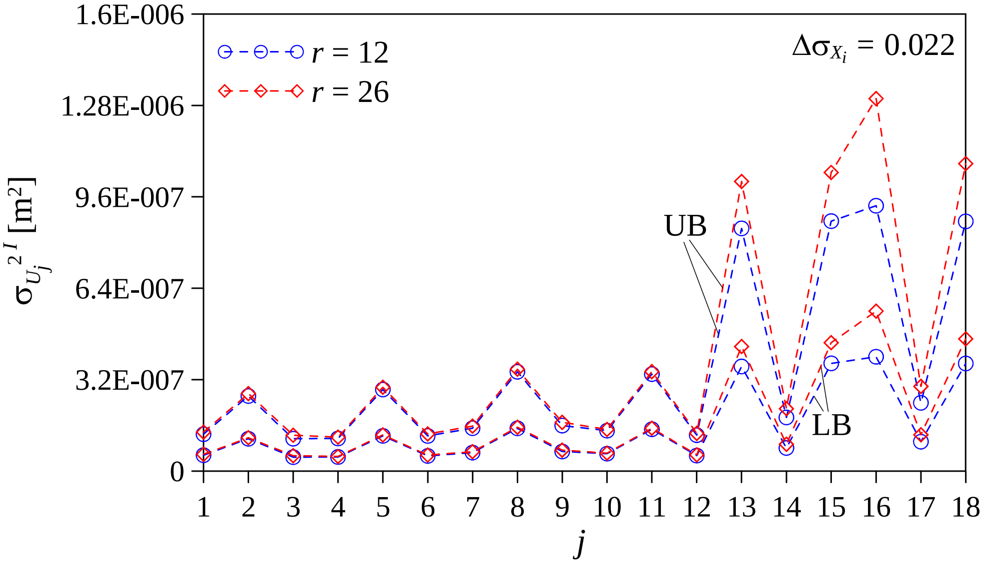


Figure 7. Influence of the number r of bars with random Young's moduli on the proposed LB and UB of the interval variance of the nodal displacements of the 3D truss.

Table 1. Mean-value and variance of floor displacements of the braced shear-type frame with random axial stiffness of braces described by a Gaussian *PDF* ($\sigma_{0,X_i} = 0.11$): comparison between the proposed estimates and *MCS* data.

	$\mu_{U_j} \times 10^{-2}$ [m]		$\sigma^2_{U_j} \times 10^{-8}$ [m^2]	
	Proposed	*MCS*	Proposed	*MCS*
U_1	0.17107	0.17103	0.08654	0.08593
U_2	0.34253	0.34250	0.19778	0.19726
U_3	0.72460	0.72448	2.86757	2.87197

Table 2. LB and UB of the interval mean-value of floor displacements of the braced shear-type frame with random axial stiffness of braces described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11, \Delta\sigma_{X_i} = 0.022$): comparison between the proposed estimates and the reference ones provided by the *MCS-VM*.

	$\underline{\mu}_{U_j} \times 10^{-2}$ [m]		$\overline{\mu}_{U_j} \times 10^{-2}$ [m]	
	Proposed	*MCS-VM*	Proposed	*MCS-VM*
U_1	0.17105	0.17102	0.17109	0.17105
U_2	0.34249	0.34247	0.34258	0.34255
U_3	0.72431	0.72421	0.72496	0.72482

Table 3. LB and UB of the interval variance of floor displacements of the braced shear-type frame with random axial stiffness of braces described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11$, $\Delta\sigma_{X_i} = 0.022$): comparison between the proposed estimates and the reference ones provided by the *MCS-VM*.

	$\underline{\sigma}^2_{U_j} \times 10^{-8}$ [m^2]		$\overline{\sigma}^2_{U_j} \times 10^{-8}$ [m^2]	
	Proposed	*MCS-VM*	Proposed	*MCS-VM*
U_1	0.05536	0.05496	0.12468	0.12383
U_2	0.12652	0.12615	0.28497	0.28434
U_3	1.83061	1.82883	4.14215	4.16140

Table 4. Proposed *coefficient of interval uncertainty* of the mean-value and variance of the floor displacements of the braced shear-type frame for two different deviation amplitudes, $\Delta\sigma_{X_i}$, of the standard deviation of the random variables X_i ($i = 1, 2, \ldots, r = 6$) described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11$).

	$\Delta\sigma_{X_i} = 0.011$		$\Delta\sigma_{X_i} = 0.022$	
	c.i.u$[\mu^I_{U_j}]\times 100$	c.i.u$[\sigma^{2\ I}_{U_j}]\times 100$	c.i.u$[\mu^I_{U_j}]\times 100$	c.i.u$[\sigma^{2\ I}_{U_j}]\times 100$
U_1	0.0062	19.8247	0.0123	38.5019
U_2	0.0070	19.8286	0.0140	38.5087
U_3	0.0226	19.9373	0.0452	38.7015

Table 5. Proposed *coefficient of interval uncertainty* of the mean-value and variance of the nodal displacements in the load direction of the 3D truss for two different deviation amplitudes $\Delta\sigma_{X_i}$ of the standard deviation of the random variables X_i $(i=1,2,\dots,r=12)$ described by Gaussian *imprecise PDF* ($\sigma_{0,X_i}=0.11$).

	$\Delta\sigma_{X_i}=0.011$		$\Delta\sigma_{X_i}=0.022$	
	$\text{c.i.u}[\mu_{U_j}^I]\times 100$	$\text{c.i.u}[\sigma_{U_j}^{2\ I}]\times 100$	$\text{c.i.u}[\mu_{U_j}^I]\times 100$	$\text{c.i.u}[\sigma_{U_j}^{2\ I}]\times 100$
U_3	0.1703	20.5474	0.3408	39.7798
U_6	0.1699	20.5388	0.3401	39.7647
U_9	0.1370	20.5608	0.2743	39.8034
U_{12}	0.1456	20.5259	0.2913	39.7420
U_{15}	0.1241	20.5445	0.2484	39.7747
U_{18}	0.1153	20.5445	0.2308	39.7747

Table 6. Proposed *coefficient of interval uncertainty* of the mean-value and variance of the nodal displacements in the load direction of the 3D truss with uncertain Young's moduli of all the bars described by Gaussian *imprecise PDF* ($\sigma_{0,X_i} = 0.11$; $\Delta\sigma_{X_i} = 0.022$, $i = 1, 2, \ldots, r = 26$).

	c.i.u$[\mu_{U_j}^{I}]\times 100$	c.i.u$[\sigma_{U_j}^{2\ I}]\times 100$
U_3	0.4519	39.8494
U_6	0.4211	39.8029
U_9	0.3526	39.8516
U_{12}	0.3569	39.7798
U_{15}	0.3909	39.8386
U_{18}	0.3685	39.8458