DE GRUYTER Nonauton. Dyn. Syst. 2020; 7:53–64

Research Article Open Access

Martin Bohner*, Giuseppe Caristi, Fariba Gharehgazlouei, and Shapour Heidarkhani

Existence and Multiplicity of Weak Solutions for a Neumann Elliptic Problem with $\vec{p}(x)$-Laplacian

https://doi.org/10.1515/msds-2020-0108
Received May 6, 2020; accepted June 28, 2020

Abstract: We are interested in the existence of multiple weak solutions for the Neumann elliptic problem involving the anisotropic $\vec{p}(x)$-Laplacian operator, on a bounded domain with smooth boundary. We work on the anisotropic variable exponent Sobolev space, and by using a consequence of the local minimum theorem due to Bonanno, we establish existence of at least one weak solution under algebraic conditions on the nonlinear term. Also, we discuss existence of at least two weak solutions for the problem, under algebraic conditions including the classical Ambrosetti–Rabinowitz condition on the nonlinear term. Furthermore, by employing a three critical point theorem due to Bonanno and Marano, we guarantee the existence of at least three weak solutions for the problem in a special case.

Dedicated to the Memory of Professor Constantin Corduneanu
(July 26, 1928 - December 26, 2018)

Keywords: $\vec{p}(x)$-Laplacian operator, Neumann elliptic problem, weak solution, variational principle, anisotropic variable exponent Sobolev space.

MSC: 34C27, 34K14, 35B15, 35K57, 37A30

1 Introduction

Our study is conducted in the framework of the anisotropic variable exponent Lebesgue–Sobolev space. In this article, we are interested in the existence of multiple weak solutions for the Neumann $\vec{p}(x)$-elliptic problem of the type

$$\begin{cases} -\Delta_{\vec{p}(x)}u + a(x)|u|^{p_0(x)-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \gamma} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with boundary $\partial\Omega$ of class C^1, and where $\vec{v}$ is the outward unit normal to $\partial\Omega$. Let $a(\cdot) \in L^\infty(\Omega)$, $a_0 := \operatorname{ess\,inf}_{x\in\Omega} a(x) > 0$, suppose $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and $\lambda, \mu > 0$ are parameters.

In the last few decades, one of the topics from the field of partial differential equations that has continuously attracted interest is that concerning the Sobolev space with variable exponents, $W^{1,p(\cdot)}(\Omega)$, where p is a function depending on x, see, for example, the monograph [16] and the references therein. Naturally,

***Corresponding Author: Martin Bohner:** Missouri S&T, Department of Mathematics and Statistics, Rolla, MO 65409, USA, E-mail: bohner@mst.edu
Giuseppe Caristi: University of Messina, Department of Economics, Messina, Italy, E-mail: gcaristi@unime.it
Fariba Gharehgazlouei, Shapour Heidarkhani: Razi University, Department of Mathematics, Faculty of Sciences, 67149 Kermanshah, Iran, E-mail: f.gharehgazloo@yahoo.com and sh.heidarkhani@razi.ac.ir

problems involving the $p(\cdot)$-Laplace operator

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$$

were intensively studied. At the same time, due to the development of the theory regarding the anisotropic Sobolev space, the anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ has captured the attention of many researchers, and a new operator has taken its place in the mathematical literature, namely

$$\Delta_{\vec{p}(x)} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \right).$$

It is clear that this $\vec{p}(\cdot)$-Laplace operator is a generalization of the $p(\cdot)$-Laplace operator. The interest in transposing the problems into new problems with variable exponents is linked to a large scale of applications that are involving some nonhomogeneous materials. It was established that for an appropriate treatment of these materials we cannot rely on the classical Sobolev space, and that we have to allow the exponent to vary instead. Working with variable exponents, hence working in the framework of variable exponent spaces, opens the door for multiple applications. We can refer here to electrorheological fluids or to thermorheological fluids that have multiple applications to hydraulic valves and clutches, brakes, shock absorbers, robotics, space technology, tactile displays etc. (see [5, 28] and their references). In addition, the variable exponent spaces are involved in studies hat provide other types of applications, e.g., in image restoration [15] and contact mechanics [13]. Recently, this theory has been expanded by many researchers, see [3, 18]. For example, Fan and Ji have treated in [19] the problem

$$\begin{cases} -\Delta_{p(x)} u + a(x)|u|^{p(x)-2} u = f(x,u) + g(x,u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \gamma} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and they proved the existence of infinitely many weak solutions of the problem under weaker hypotheses by applying a variational principle due to B. Ricceri and the theory of the variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$. Ahmed, Hjiaj, and Touzani have studied in [3] the Neumann $\vec{p}(x)$-elliptic problem

$$\begin{cases} -\Delta_{\vec{p}(x)} u + a(x)|u|^{p_0(x)-2} u = f(x,u) + g(x,u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \gamma} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and they proved the existence of infinitely many weak solutions in the anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ under some hypotheses. For other related results, we refer to [2, 4, 11, 17, 20–22, 25–27, 29].

Here, we deal with the problem (1.1) when the nonlinearity f satisfies a subcritical growth condition. Via variational methods, we obtain the existence of at least one, two, and three weak solutions whenever the parameters λ and μ are in explicitly given positive intervals. The main tools are critical points theorems established in [9, 10, 12]. Variants of such theorems have been successfully applied for other problems, see, e.g., [1, 7, 8].

This paper is organized as follows: In Section 2, we present some preliminary knowledge on the anisotropic Sobolev spaces with variable exponent. Section 3 contains the main results and the proofs of the main results. For our Neumann elliptic problem, we prove the existence of one weak solution in Theorem 3.1, the existence of two weak solutions in Theorem 3.3, and the existence of three weak solutions in Theorem 3.4.

2 Preliminaries

In this section, we introduce some definitions and results which will be used in the next section.

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 1$. We define

$$\mathcal{C}_+ = \left\{ p(\cdot) : \overline{\Omega} \to \mathbb{R} \text{ is measurable} : \ 1 < p^- \leq p^+ < \infty \right\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\}.$$

We define the variable exponent Lebesgue space $\mathrm{L}^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} \mathrm{d}x$$

is finite. We define a norm, the so-called Luxemburg norm, on this space by the formula

$$\|u\|_{p(\cdot)} = \inf\left\{\gamma > 0 : \rho_{p(\cdot)}\left(\frac{u}{\gamma}\right) \le 1\right\}.$$

The space $\left(\mathrm{L}^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\right)$ is a separable and reflexive Banach space. Moreover, the space $\mathrm{L}^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $\mathrm{L}^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}$$

for all $u \in \mathrm{L}^{p(\cdot)}(\Omega)$ and $v \in \mathrm{L}^{p'(\cdot)}(\Omega)$. An important role in manipulating the generalized Lebesgue spaces is played by the $\rho_{p(\cdot)}$-modular of the space $\mathrm{L}^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1 (See [31]). *If $u \in \mathrm{L}^{p(\cdot)}(\Omega)$, $u_n \in \mathrm{L}^{p(\cdot)}(\Omega)$, and $p^+ < \infty$, then*
(i) $\|u\|_{p(\cdot)} > 1$ *implies* $\|u\|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^+}$,
(ii) $\|u\|_{p(\cdot)} < 1$ *implies* $\|u\|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-}$,
(iii) $\lim_{n\to\infty} \|u_n - u\|_{p(\cdot)} = 0$ *iff* $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

We define the variable exponent Sobolev space $\mathrm{W}^{1,p(\cdot)}(\Omega)$ by

$$\mathrm{W}^{1,p(\cdot)}(\Omega) = \left\{u \in \mathrm{L}^{p(\cdot)}(\Omega) : |\nabla u| \in \mathrm{L}^{p(\cdot)}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $\left(\mathrm{W}^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ is a separable and reflexive Banach space. Now, we present the anisotropic variable exponent Sobolev space used in the study of the main problem. Let $p_0(\cdot), p_1(\cdot), \dots, p_N(\cdot)$ be $N+1$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We denote

$$\vec{p}(\cdot) = \{p_0(\cdot), p_1(\cdot), \dots, p_N(\cdot)\}, \quad D^0 u = u, \quad D^i u = \frac{\partial u}{\partial x_i}.$$

We define

$$\underline{p} = \min\{p_i^- : i = 0, 1, \dots, N\}, \quad \text{so} \quad \underline{p} > 1$$

and

$$\overline{p} = \max\{p_i^+ : i = 0, 1, \dots, N\}.$$

The anisotropic variable exponent Sobolev space $\mathrm{W}^{1,\vec{p}(\cdot)}(\Omega)$ is defined by

$$\mathrm{W}^{1,\vec{p}(\cdot)}(\Omega) = \left\{u \in \mathrm{L}^{p_0(\cdot)}(\Omega) : D^i u \in \mathrm{L}^{p_i(\cdot)}(\Omega) \text{ for } i = 1, 2, \dots, N\right\},$$

endowed with the norm

$$\|u\| = \|u\|_{1,\vec{p}(\cdot)} = \sum_{i=0}^{N} \left\|D^i u\right\|_{p_i(\cdot)}.$$

The space $\left(\mathrm{W}^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|\right)$ is a separable and reflexive Banach space (cf. [6, 24]). Throughout the rest of this paper, we assume

$$\underline{p} > N.$$

Remark 2.2. Since $W^{1,\vec{p}(\cdot)}(\Omega)$ is continuously embedded in $W^{1,\underline{p}}(\Omega)$, and since $W^{1,\underline{p}}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ (the space of continuous functions), the continuous embedding of $W^{1,\vec{p}(x)}(\Omega)$ in $C^0(\overline{\Omega})$ is compact.

We set

$$c_0 = \sup_{u\in W^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|_{1,\vec{p}(\cdot)}}.$$

Assume that $f, g : \Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying

$$\sup_{|t|\leq s}|f(\cdot,t)|\in L^1(\Omega) \quad \text{and} \quad \sup_{|t|\leq s}|g(\cdot,t)|\in L^1(\Omega) \quad \text{for each} \quad s>0.$$

We set

$$F(x,t)=\int_0^t f(x,\xi)\mathrm{d}\xi \quad \text{and} \quad G(x,t)=\int_0^t g(x,\xi)\mathrm{d}\xi \quad \text{for all} \quad (x,t)\in\Omega\times\mathbb{R}.$$

We define, for any $u\in X:=W^{1,\vec{p}(x)}(\Omega)$, the functionals $\Phi, \Psi_{\lambda,\mu} : X\to\mathbb{R}$ by

$$\Phi(u) := \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x + \int_\Omega \frac{a(x)}{p_0(x)}|u|^{p_0(x)}\mathrm{d}x \tag{2.1}$$

and

$$\Psi_{\lambda,\mu}(u) := \int_\Omega F(x,u)\mathrm{d}x + \frac{\mu}{\lambda}\int_\Omega G(x,u)\mathrm{d}x. \tag{2.2}$$

We say that a function $u\in X$ is a weak solution of (1.1) if

$$\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\mathrm{d}x + \int_\Omega a(x)|u|^{p_0(x)-2}uv\mathrm{d}x - \lambda\int_\Omega f(x,u)v\mathrm{d}x - \mu\int_\Omega g(x,u)v\mathrm{d}x = 0 \quad \text{for all} \quad v\in X.$$

As in [3, Proposition 4.1], using our assumptions, we need the following proposition in the proof of Theorem 3.1.

Proposition 2.3. *The functional $\Phi(u)$ defined in* (2.1) *is coercive, that is,*

$$\Phi(u)\to\infty \quad \textit{as} \quad \|u\|\to\infty \quad \textit{and} \quad u\in X.$$

Proof. We have

$$\begin{aligned}
\Phi(u) &= \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x + \int_\Omega \frac{a(x)}{p_0(x)}|u|^{p_0(x)}\mathrm{d}x \\
&\geq \sum_{i=1}^N \frac{1}{p_i^+}\left(\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)}^{\underline{p}} - 1\right) + \frac{a_0}{p_0^+}\left(\|u\|_{p_0(x)}^{\underline{p}} - 1\right) \\
&\geq \frac{\min\{1,a_0\}}{\overline{p}(N+1)^{\underline{p}-1}}\|u\|^{\underline{p}} - \frac{N+a_0}{\underline{p}} \\
&\geq K_1\|u\|^{\underline{p}} - K_2,
\end{aligned}$$

where $K_1, K_2>0$ are constants. Thus, if $\|u\|\to\infty$, then $\Phi(u)\to\infty$. □

Definition 2.4. Let Φ and Ψ be two continuously Gâteaux-differentiable functionals defined on a real Banach space X. Fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais–Smale condition cut off upper at r, in short $(\mathrm{PS})^{[r]}$, if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

$$\{I(u_n)\} \text{ is bounded}, \quad \lim_{n\to\infty} \left\| I'(u_n) \right\|_{X^*} = 0, \quad \Phi(u_n) < r \text{ for each } n \in \mathbb{N}$$

has a convergent subsequence.

The following three theorems are the main tools in the next section to prove the results. The first one is used to prove the existence of at least one weak solution, the second one for the existence of at least two weak solutions, and the third one for the existence of at least three weak solutions. While the first two results are due to Bonanno, the third one is due to Bonanno and Marano.

Theorem 2.5 (See [10, Theorem 2.3]). *Let X be a real Banach space. Assume $\Phi, \Psi : X \to \mathbb{R}$ are two continuously Gâteaux-differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{v} \in X$ with $0 < \Phi(\overline{v}) < r$ such that*
(E_1) $\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,
(E_2) *for all $\lambda \in \Lambda := \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)<r} \Psi(u)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(\mathrm{PS})^{[r]}$ condition.*

Then, for each $\lambda \in \Lambda$, there is $u_{0,\lambda} \in \Phi^{-1}((0, r))$ such that $I'_\lambda(u_{0,\lambda}) = \vartheta_{X^*}$ and $I_\lambda(u_{0,\lambda}) < I_\lambda(u)$ for all $u \in \Phi^{-1}((0, r))$.

Theorem 2.6 (See [10, Theorem 3.2]). *Let X be a real Banach space. Assume $\Phi, \Psi : X \to \mathbb{R}$ are two continuously Gâteaux-differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}((-\infty,r))} \Psi(u)}\right),$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the (PS) condition and is unbounded from below. Then, for each

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}((-\infty,r))} \Psi(u)}\right),$$

the functional I_λ admits two distinct critical points.

Theorem 2.7 (See [12, Theorem 3.6]). *Let X be a reflexive real Banach space. Assume $\Phi : X \to \mathbb{R}$ is a coercive and continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional, whose Gâteaux derivative admits a continuous inverse on X^*. Assume $\Psi : X \to \mathbb{R}$ is a continuously Gâteaux-differentiable functional, whose derivative is compact. Suppose*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\overline{v} \in X$ with $r < \Phi(\overline{v})$ such that
(E_3) $\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,
(E_4) *for all $\lambda \in \Lambda := \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)<r} \Psi(u)}\right)$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

3 Main Results

We start by giving the existence of one weak solution for (1.1).

Theorem 3.1. *Assume that there exist two positive constants τ and δ such that*
(H_1) $\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\overline{p}},\delta^{\underline{p}}\} < \underline{p}\min\{1,a_0\}\tau^{\underline{p}}$,
(H_2) $\frac{\int_\Omega \sup_{|t|\le\tau} F(x,t)\mathrm{d}x}{\tau^{\underline{p}}} < \frac{\underline{p}\min\{1,a_0\}\int_\Omega F(x,\delta)\mathrm{d}x}{\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\overline{p}},\delta^{\underline{p}}\}}$,
(H_3) $F(x,t)\ge 0$ *for each* $(x,t)\in\Omega\times\mathbb{R}^+$.

Then, for each $\lambda\in\Lambda_w$, given by

$$\left(\frac{\|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\overline{p}},\delta^{\underline{p}}\}}{\underline{p}\int_\Omega F(x,\delta)\mathrm{d}x}, \frac{\min\{1,a_0\}\tau^{\underline{p}}}{\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\int_\Omega \sup_{|t|\le\tau}F(x,t)\mathrm{d}x}\right) \tag{3.1}$$

and for each $g:\Omega\times\mathbb{R}\to\mathbb{R}$, there exists $\delta_{\lambda,g}>0$, given by

$$\min\left\{\frac{\min\{1,a_0\}\tau^{\underline{p}}-\lambda\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\int_\Omega \sup_{|t|\le\tau}F(x,t)\mathrm{d}x}{\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\int_\Omega \sup_{|t|\le\tau}G(x,t)\mathrm{d}x}, \frac{\lambda\underline{p}\int_\Omega F(x,\delta)\mathrm{d}x-\|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\overline{p}},\delta^{\underline{p}}\}}{\underline{p}\int_\Omega G(x,\delta)\mathrm{d}x}\right\},$$

such that for each $\mu\in[0,\delta_{\lambda,g})$, the problem (1.1) *admits at least one nontrivial weak solution $u_\lambda\in X$ such that $\|u_\lambda\|_\infty\le\tau$.*

Proof. Our goal is to apply Theorem 2.5 to (1.1). To this end, take the real Banach space $X=\mathrm{W}^{1,\vec{p}(x)}(\Omega)$ with the norm as defined in Section 2, with fixed λ and μ as in the conclusion, Φ, $\Psi_{\lambda,\mu}$ defined in (2.1) and (2.2). We can see that $\Phi,\Psi_{\lambda,\mu}\in \mathrm{C}^1(X,\mathbb{R})$ (see [24, Lemma 3.4]) with derivatives given by

$$\left\langle \Phi'(u),v\right\rangle=\sum_{i=1}^N\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\mathrm{d}x+\int_\Omega a(x)|u|^{p_0(x)-2}uv\mathrm{d}x$$

and

$$\left\langle \Psi'_{\lambda,\mu}(u),v\right\rangle=\int_\Omega f(x,u)v\mathrm{d}x+\frac{\mu}{\lambda}\int_\Omega g(x,u)v\mathrm{d}x.$$

The functionals Φ, $\Psi_{\lambda,\mu}$ are sequentially weakly lower semicontinuous [14, Lemma 3]. Moreover, $\Psi'_{\lambda,\mu}:X\to X^*$ is a compact operator. Indeed, it is enough to show that $\Psi'_{\lambda,\mu}$ is strongly continuous on X. To show this, for fixed $u\in X$, if $u_n\to u$ weakly in X as $n\to\infty$, then $u_n(x)$ converges uniformly to $u(x)$ on Ω as $n\to\infty$, see [30]. Since f, g are continuous in $\mathbb{R}$ for every $x\in\Omega$, we get

$$f(x,u_n)+\frac{\mu}{\lambda}g(x,u_n)\to f(x,u)+\frac{\mu}{\lambda}g(x,u)\quad\text{as}\quad n\to\infty.$$

Thus, $\Psi'_{\lambda,\mu}(u_n)\to\Psi'_{\lambda,\mu}(u)$ as $n\to\infty$. Hence, by [30, Proposition 26.2], $\Psi'_{\lambda,\mu}$ is a compact operator. Moreover, Φ' admits a continuous inverse on X^*. Indeed, according to [30, Theorem 26.A(d)], it is enough to verify that Φ' is coercive, hemicontinuous, and uniformly monotone. By Proposition 2.3, it is clear that for any $u\in X$, we have

$$\frac{\left\langle\Phi'(u),u\right\rangle}{\|u\|}=\frac{\sum_{i=1}^N\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x+\int_\Omega a(x)|u|^{p_0(x)}\mathrm{d}x}{\|u\|}\ge\frac{k_3\|u\|^{\underline{p}}-k_4}{\|u\|},$$

where k_3, k_4 are positive constants. Thus,

$$\lim_{\|u\|\to\infty}\frac{\left\langle\Phi'(u),u\right\rangle}{\|u\|}=\infty,$$

i.e., Φ' is coercive. The fact that Φ' is hemicontinuous can be verified using standard arguments. Finally, we show that Φ' is uniformly monotone. In fact, for any $\xi_i,\psi_i\in\mathbb{R}$, we have the inequality (see [23])

$$\left(|\xi_i|^{r_i-2}\xi_i-|\psi_i|^{r_i-2}\psi_i\right)(\xi_i-\psi_i)\ge 2^{-r_i}|\xi_i-\psi_i|^{r_i}\quad\text{for all}\quad r_i>2. \tag{3.2}$$

Thus, for every $u, v \in X$, we deduce that

$$
\begin{aligned}
&\left\langle \Phi'(u) - \Phi'(v), u - v \right\rangle \\
&= \sum_{i=1}^{N} \int_{\Omega} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial v}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \mathrm{d}x \\
&\quad + \int_{\Omega} a\left(x\right)(|u|^{p_0(x)-2}u - |v|^{p_0(x)-2}v\right)(u - v)\mathrm{d}x \\
&\geq 2^{-\overline{p}} \left(\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|^{p_i(x)} \mathrm{d}x + \int_{\Omega} a(x)|u - v|^{p_0(x)}\mathrm{d}x \right) \\
&\geq \min\{2^{-\overline{p}}, a_0 2^{-\overline{p}}\} \sum_{i=0}^{N} \int_{\Omega} |D^i(u - v)|^{p_i(x)}\mathrm{d}x \\
&\geq \begin{cases} C_1 \sum_{i=0}^{N} \left\| D^i(u - v) \right\|_{p(\cdot)}^{\underline{p}} & \text{if} \quad \|u - v\|_{p(\cdot)}, \left\| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right\|_{p(\cdot)} > 1, \\ C_2 \sum_{i=0}^{N} \left\| D^i(u - v) \right\|_{p(\cdot)}^{\overline{p}} & \text{if} \quad \|u - v\|_{p(\cdot)}, \left\| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right\|_{p(\cdot)} < 1, \end{cases}
\end{aligned}
$$

where the last inequality is obtained from Proposition 2.1. Thus, by [9, Proposition 2.1], the functional $I_{\lambda,\mu} = \Phi - \lambda\Psi_{\lambda,\mu}$ satisfies the $(\mathrm{PS})^{[r]}$ condition for each $r > 0$, and so (E_2) from Theorem 2.5 is satisfied. Therefore, it remains to verify (E_1) from Theorem 2.5. To this end, we put $r := \frac{\min\{1,a_0\}}{\overline{p}(N+1)^{\underline{p}-1}} \left(\frac{\tau}{c_0}\right)^{\underline{p}}$ and pick $w \in X$, defined as

$$
w(x) = \begin{cases} \delta & \text{if} \quad x \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}
$$

Then we have

$$
\Phi(w) = \int_{\Omega} \frac{a(x)}{p_0(x)} |w(x)|^{p_0(x)}\mathrm{d}x \leq \frac{\mathrm{meas}(\Omega)\|a\|_\infty}{\underline{p}} \max\{\delta^{\overline{p}}, \delta^{\underline{p}}\}. \tag{3.4}
$$

Hence, it follows from (H_1) that $0 < \Phi(w) < r$. Now, let $u \in X$ be such that $u \in \Phi^{-1}([0, r])$. Then, by Proposition 2.1(i), for any $u \in X$ with $\|u\| > 1$, we obtain

$$
\frac{\min\{1, a_0\}}{\overline{p}(N + 1)^{\underline{p}-1}} \left(\frac{\tau}{c_0}\right)^{\underline{p}} \geq \Phi(u) \geq \frac{\min\{1, a_0\}}{\overline{p}(N + 1)^{\underline{p}-1}} \|u\|^{\underline{p}} - \frac{N + a_0}{\overline{p}}.
$$

Similarly, by Proposition 2.1(ii), for any $u \in X$ with $\|u\| < 1$, we obtain

$$
\frac{\min\{1, a_0\}}{\overline{p}(N + 1)^{\underline{p}-1}} \left(\frac{\tau}{c_0}\right)^{\underline{p}} \geq \Phi(u) \geq \frac{\min\{1, a_0\}}{\overline{p}(N + 1)^{\overline{p}-1}} \|u\|^{\overline{p}} - \frac{N + a_0}{\overline{p}}.
$$

Then,

$$
\|u\| \leq \max\left\{\frac{\tau}{c_0}, (N + 1)^{\frac{\overline{p}}{\underline{p}}-1}\left(\frac{\tau}{c_0}\right)^{\frac{\underline{p}}{\overline{p}}}\right\} = \frac{\tau}{c_0}.
$$

Hence, we obtain

$$
|u(x)| \leq \|u\|_{\mathrm{L}^\infty(\Omega)} \leq c_0\|u\| \leq \tau \quad \text{for all} \quad x \in \Omega.
$$

Therefore, one has

$$
\begin{aligned}
\frac{\sup_{\Phi(u)<r} \Psi_{\lambda,\mu}(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}((-\infty,r))} \left(\int_\Omega F(x, u)\mathrm{d}x + \frac{\mu}{\lambda}\int_\Omega G(x, u)\mathrm{d}x\right)}{\frac{\min\{1,a_0\}}{\overline{p}(N+1)^{\underline{p}-1}} \left(\frac{\tau}{c_0}\right)^{\underline{p}}} \\
&\leq \frac{c_0^{\underline{p}}\overline{p}(N + 1)^{\underline{p}-1} \int_\Omega \sup_{|t|\leq\tau} F(x, u)\mathrm{d}x}{\min\{1, a_0\}\tau^{\underline{p}}}
\end{aligned}
$$

$$+\frac{\mu}{\lambda}\frac{c_0^{\underline{p}}\overline{p}(N+1)^{\underline{p}-1}\int_\Omega \sup_{|t|\le\tau} G(x,u)\mathrm{d}x}{\min\{1,a_0\}\tau^{\underline{p}}}.$$

Moreover, thanks to (H_3) and (3.4), one has

$$\frac{\Psi(w)}{\Phi(w)} \ge \frac{\underline{p}\int_\Omega F(x,\delta)\mathrm{d}x}{\|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\underline{p}},\delta^{\overline{p}}\}} + \frac{\mu}{\lambda}\frac{\underline{p}\int_\Omega G(x,\delta)\mathrm{d}x}{\|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\underline{p}},\delta^{\overline{p}}\}}.$$

Since $\mu < \delta_{\lambda,\mu}$, we have

$$\mu < \frac{\min\{1,a_0\}\tau^{\underline{p}} - \lambda\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\int_\Omega \sup_{|t|\le\tau} F(x,t)\mathrm{d}x}{\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\int_\Omega \sup_{|t|\le\tau} G(x,t)\mathrm{d}x} \tag{3.5}$$

and

$$\mu < \frac{\lambda\underline{p}\int_\Omega F(x,\delta)\mathrm{d}x - \|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\overline{p}},\delta^{\underline{p}}\}}{-\underline{p}\int_\Omega G(x,\delta)\mathrm{d}x}. \tag{3.6}$$

From (3.5) and (3.6), we get

$$\frac{\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\int_\Omega \sup_{|t|\le\tau} F(x,t)\mathrm{d}x}{\min\{1,a_0\}\tau^{\underline{p}}} + \frac{\mu}{\lambda}\frac{\overline{p}c_0^{\underline{p}}(N+1)^{\underline{p}-1}\int_\Omega \sup_{|t|\le\tau} G(x,t)\mathrm{d}x}{\min\{1,a_0\}\tau^{\underline{p}}} < \frac{1}{\lambda}$$

and

$$\frac{\underline{p}\int_\Omega F(x,\delta)\mathrm{d}x}{\|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\overline{p}},\delta^{\underline{p}}\}} + \frac{\mu}{\lambda}\frac{\underline{p}\int_\Omega G(x,\delta)\mathrm{d}x}{\|a\|_\infty \operatorname{meas}(\Omega)\max\{\delta^{\overline{p}},\delta^{\underline{p}}\}} > \frac{1}{\lambda}.$$

Then,

$$\frac{\sup_{\Phi(x)\le r}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Phi(w)}{\Psi(w)}.$$

Therefore, (E_1) from Theorem 2.5 is satisfied. Now, since $\lambda \in \left(\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(u)}\right)$, Theorem 2.5 with $\bar{v} = w$ guarantees the existence of a local minimum point u_λ for the functional $I_{\lambda,\mu}$ such that $0 < \Phi(u_\lambda) < r$, and so u_λ is a nontrivial weak solution of (1.1) such that $\|u_\lambda\|_\infty < \tau$. □

The following example is an application of Theorem 3.1.

Example 3.2. Given the domain $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\}$. Let $p_i(x)$ for $i = 0, 1, 2$ be the functions defined by

$$\begin{aligned} p_0(x_1,x_2) &= 4 + x_1x_2 \quad \text{for} \quad (x_1,x_2) \in \mathbb{R}^2,\\ p_1(x_1,x_2) &= 2(2 + x_1 + x_2) \quad \text{for} \quad (x_1,x_2) \in \mathbb{R}^2,\\ p_2(x_1,x_2) &= 2(2 + x_1x_2) \quad \text{for} \quad (x_1,x_2) \in \mathbb{R}^2. \end{aligned}$$

Then, $\underline{p} = 4$, $\overline{p} = 6$, and $\operatorname{meas}(\Omega) = \pi$. for all $((x_1,x_2),t) \in \Omega\times\mathbb{R}$, put $f((x_1,x_2),t) = p_i(x_1,x_2)e^{\frac{t}{2}}$. By the expression of f, we have

$$F((x_1,x_2),t) = 2p_i(x_1,x_2)e^{\frac{t}{2}} \quad \text{for all} \quad ((x_1,x_2),t) \in \Omega\times\mathbb{R}.$$

By choosing $a(x_1,x_2) = 1$, $\delta = \frac{1}{12}$, $\tau = 1$, and $c_0 = 4\pi^{-\frac{1}{4}}$, by simple calculations, obviously all assumptions of Theorem 3.1 are satisfied. Hence, by applying Theorem 3.1, for every

$$\lambda \in \left(\frac{1}{4\times 12^5 e^{\frac{1}{24}}}, \frac{1}{2\times 12^5 e^{\frac{1}{2}}}\right)$$

and for each $g : \Omega\times\mathbb{R} \to \mathbb{R}$, there exists $\delta_{\lambda,g} > 0$ such that for each $\mu \in [0, \delta_{\lambda,g})$, the problem (1.1) admits at least one nontrivial weak solution $u_\lambda \in X$ such that $\|u_\lambda\|_\infty \le 1$.

Next, our goal is to obtain the existence of two distinct weak solutions for (1.1). The following result is obtained by applying Theorem 2.6. Note that (H_4) below is the well-known Ambrosetti–Rabinowitz condition.

Theorem 3.3. *Assume that there exist two positive constants τ and δ such that (H_1) from Theorem 3.1 holds. Moreover, assume*
(H_4) *there exist $\rho > \overline{p}$ and $R > 0$ such that*

$$0 < \rho F(x,t) < tf(x,t) \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad |t| \geq R.$$

Then, for each

$$\lambda \in \Lambda_r := \left(0, \frac{\min\{1, a_0\}\tau^{\underline{p}}}{\overline{p} c_0^{\underline{p}}(N+1)^{\underline{p}-1} \int_\Omega \sup_{|t|\leq\tau} F(x,t)\mathrm{d}x}\right)$$

and for each Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying
(H_5) *there exist $c_1, c_2 > 0$ and $r \in \mathcal{C}_+(\Omega)$ with $0 < r(x) \leq r^+ < \underline{p}$ such that*

$$|g(x,t)| \leq c_1 + c_2|t|^{r(x)-1} \quad \text{for every} \quad (x,t) \in \Omega \times \mathbb{R},$$

there exists $\delta_{\lambda,g} > 0$, given as in Theorem 3.1, such that for each $\mu \in [0, \delta_{\lambda,g})$, the problem (1.1) *admits at least two nontrivial weak solutions.*

Proof. Let Φ, Ψ be the functionals defined in Theorem 3.1, which satisfy all regularity assumptions requested in Theorem 2.6. Arguing as in the proof of Theorem 3.1, choose $r = \frac{\min\{1,a_0\}}{\overline{p}(N+1)^{\underline{p}-1}}(\frac{\tau}{c_0})^{\underline{p}}$ and pick $w \in X$. Now, from (H$_4$), by standard computations, there is a positive constant m such that

$$F(x,t) \geq m|t|^\rho \quad \text{for all} \quad x \in \Omega.$$

Hence, for every $\lambda \in \Lambda_r$, $u \in X \setminus \{0\}$, and $t > 1$, we obtain

$$\begin{aligned} I_{\lambda,\mu}(tu) &= \Phi(tu) - \lambda \int_\Omega F(x,tu)\mathrm{d}x - \mu \int_\Omega G(x,tu)\mathrm{d}x \\ &\leq \frac{t^{\overline{p}}}{\underline{p}}\left(\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x + \int_\Omega a(x)|u|^{p_0(x)}\mathrm{d}x\right) \\ &\quad -m\lambda t^\rho \int_\Omega |u(k)|^\rho + c_1 t \int_\Omega |u|\mathrm{d}x + c_2 t^{r^+} \int_\Omega |u|^{r(x)}\mathrm{d}x. \end{aligned}$$

Since $\rho > \overline{p} > r^+$, this condition guarantees that $I_{\lambda,\mu}$ is unbounded from below. We recall that $I_{\lambda,\mu}$ is a Gâteaux-differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $I'_{\lambda,\mu}(u) \in X^*$ given by

$$\begin{aligned} I'_{\lambda,\mu}(u)(v) &= \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\mathrm{d}x + \int_\Omega a(x)|u|^{p_0(x)-2}uv\mathrm{d}x \\ &\quad -\lambda \int_\Omega f(x,u)v\mathrm{d}x - \mu \int_\Omega g(x,u)v\mathrm{d}x \end{aligned}$$

for every $v \in X$. Finally, we verify that $I_{\lambda,\mu}$ satisfies the (PS) condition. Indeed, if $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{I_{\lambda,\mu}(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_{\lambda,\mu}(u_n) \to 0$ in X^* as $n \to +\infty$, Then there exists a positive constant s_0 such that

$$\left|I_{\lambda,\mu}(u_n)\right| \leq s_0 \quad \text{and} \quad \left\|I'_{\lambda,\mu}(u_n)\right\| \leq s_0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Using also (H$_4$), (H$_5$), and the definition of $I'_{\lambda,\mu}$, we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &s_0 + s_1 \|u_n\| \geq \rho I_{\lambda,\mu}(u_n) - I'_{\lambda,\mu}(u_n)u_n \\ &\geq \left(\frac{\rho}{\overline{p}} - 1\right)\left(\frac{\min\{1, a_0\}}{(N+1)^{\underline{p}-1}} \|u_n\|^{\underline{p}} - N_0 - a_0\right) \end{aligned}$$

$$
\begin{aligned}
&-\mu\rho\int_\Omega G(x,u_n)\mathrm{d}x+\mu\int_\Omega g(x,u_n)u_n\mathrm{d}x\\
\geq\ &\left(\frac{\rho}{\overline{p}}-1\right)\left(\frac{\min\{1,a_0\}}{(N+1)^{\underline{p}-1}}\|u_n\|^{\underline{p}}-N_0-a_0\right)\\
&-\mu c_3\|u_n\|_\infty-\mu c_4\|u_n\|_\infty^{r^+}\\
\geq\ &\left(\frac{\rho}{\overline{p}}-1\right)\left(\frac{\min\{1,a_0\}}{(N+1)^{\underline{p}-1}}\|u_n\|^{\underline{p}}-N_0-a_0\right)-s_2\|u_n\|-s_3\|u_n\|^{r^+}
\end{aligned}
$$

for some $s_1, s_2, s_3 > 0$. Since $\rho > \overline{p}$, it follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, since X is a reflexive Banach space, we have, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in} \quad X.$$

By $I'_{\lambda,\mu}(u_n)\to 0$ and $u_n \rightharpoonup u$ in X, we get

$$\left(I'_{\lambda,\mu}(u_n)-I'_{\lambda,\mu}(u)\right)(u_n-u)\to 0.$$

From the continuity of f and g, we have

$$\int_\Omega \left(f(x,u_n)-f(x,u)\right)(u_n-u)\mathrm{d}x\to 0 \quad \text{as} \quad n\to\infty$$

and

$$\int_\Omega \left(g(x,u_n)-g(x,u)\right)(u_n-u)\mathrm{d}x\to 0 \quad \text{as} \quad n\to\infty.$$

Moreover, an easy computation shows

$$
\begin{aligned}
&\left(I'_{\lambda,\mu}(u_n)-I'_{\lambda,\mu}(u)\right)(u_n-u)\\
=\ &\sum_{i=1}^N\int_\Omega\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u_n}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)\mathrm{d}x\\
&+\int_\Omega a(x)\left(|u_n|^{p_0(x)-2}u_n-|u|^{p_0(x)-2}u\right)(u_n-u)\mathrm{d}x\\
&-\lambda\int_\Omega\left(f(x,u_n)-f(x,u)\right)(u_n-u)\mathrm{d}x\\
&-\mu\int_\Omega\left(g(x,u_n)-g(x,u)\right)(u_n-u)\mathrm{d}x\\
\geq\ &\min\{2^{-\overline{p}},a_0 2^{-\overline{p}}\}\sum_{i=0}^N\int_\Omega\left|D^i(u_n-u)\right|^{p_i(x)}\mathrm{d}x,
\end{aligned}
$$

where the last inequality is obtained by using (3.2). Combining the last relation with Proposition 2.1(iii), we find that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in X. Therefore, $I_{\lambda,\mu}$ satisfies the (PS) condition, and so all hypotheses of Theorem 2.6 are satisfied. Hence, applying Theorem 2.6, for each $\lambda\in\Lambda_r$, the function $I_{\lambda,\mu}$ admits at least two distinct critical points that are the weak solutions of (1.1). □

Finally, we discuss the existence of at least three weak solutions for (1.1).

Theorem 3.4. *Assume*
(H_6) *there exist* $c > 0$ *and* $r\in\mathcal{C}_+$ *with* $0 < r(x) \leq r^+ < \underline{p}$ *such that*

$$F(x,t),\ \left|G(x,t)\right| \leq c\left(1+|t|^{r(x)}\right) \quad \textit{for all} \quad (x,t)\in\Omega\times\mathbb{R}.$$

Assume that there exist two positive constants τ and η such that
(H_7) $a_0 \overline{p} c_0^{\underline{p}} (N+1)^{\underline{p}-1} \operatorname{meas}(\Omega) \min\{\delta^{\overline{p}}, \delta^{\underline{p}}\} > \min\{1, a_0\} \tau^{\underline{p}}$

and let (H_2) *and* (H_3) *from Theorem 3.1 hold. Then, for every* $\lambda \in \Lambda_w$ *as in* (3.1) *and for each* $g : \Omega \times \mathbb{R} \to \mathbb{R}$*, there exists* $\delta_{\lambda,g} > 0$*, given as in Theorem 3.1, such that for each* $\mu \in [0, \delta_{\lambda,g})$*, the problem* (1.1) *admits at least three distinct weak solutions.*

Proof. Our aim is to apply Theorem 2.7. We consider the functionals Φ and $\Psi_{\lambda,\mu}$, which, as seen before, satisfy the regularity assumptions requested in Theorem 2.7. Now, arguing as in the proof of Theorem 3.1, put w as in (3.3) and $r = \frac{\min\{1,a_0\}}{\overline{p}(N+1)^{\underline{p}-1}} \left(\frac{\tau}{c_0}\right)^{\underline{p}}$. Bearing in mind ($H_7$), we obtain

$$\Phi(w) > r > 0.$$

Therefore, according to the proof of Theorem 3.1, (E_3) from Theorem 2.7 holds. Now, we prove that, for each $\lambda \in \Lambda_w$, the functional $I_{\lambda,\mu}$ is coercive. By using (H_6) and the Sobolev embedding theorem, we easily obtain for all $u \in X$

$$\begin{aligned} I_{\lambda,\mu}(u) \quad \geq \quad & \frac{\min\{1, a_0\}}{\overline{p}(N+1)^{\underline{p}-1}} \|u\|^{\underline{p}} - \frac{N + a_0}{\underline{p}} \\ & - \frac{c\lambda}{r^+} \left(\|u\|^{r^+} + \frac{\mu}{\lambda} \|u\|^{r^+} \right), \end{aligned}$$

which implies $I_{\lambda,\mu} \to \infty$ as $\|u\| \to \infty$. Hence, the functional $I_{\lambda,\mu}$ is coercive, and (E_4) holds. So, for each $\lambda \in \Lambda_w$, Theorem 2.7 implies that the functional $I_{\lambda,\mu}$ admits at least three critical points in X, and these are weak solutions of (1.1). □

References

[1] Ghasem A. Afrouzi, Martin Bohner, Giuseppe Caristi, Shapour Heidarkhani, and Shahin Moradi. An existence result for impulsive multi-point boundary value systems using a local minimization principle. *J. Optim. Theory Appl.*, 177(1):1–20, 2018.

[2] Ahmed Ahmed, Mohamed Saad Bouh Elemine Vall, Abdelmoujib Benkirane, and Abdelfattah Touzani. Existence of infinitely many solutions to the Neumann problem for quasilinear elliptic systems involving the $p(\cdot)$ and $q(\cdot)$-Laplacian. *Nonlinear Stud.*, 24(3):687–698, 2017.

[3] Ahmed Ahmed, Hassane Hjiaj, and Abdelfattah Touzani. Existence of infinitely many weak solutions for a Neumann elliptic equations involving the $\vec{p}(x)$-Laplacian operator. *Rend. Circ. Mat. Palermo (2)*, 64(3):459–473, 2015.

[4] Ahmed Ahmed, Abdelfettah Touzani, and Youssef Akdim. Infinitely many solutions for an elliptic Neumann problem in weighted variable exponent Sobolev spaces. *J. Nonlinear Evol. Equ. Appl.*, Paper No. 2, 15–34, 2019.

[5] Stanislav N. Antontsev and José F. Rodrigues. On stationary thermo-rheological viscous flows. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 52(1):19–36, 2006.

[6] Mostafa Bendahmane, Moussa Chrif, and Said El Manouni. An approximation result in generalized anisotropic Sobolev spaces and applications. *Z. Anal. Anwend.*, 30(3):341–353, 2011.

[7] Martin Bohner, Giuseppe Caristi, Shapour Heidarkhani, and Shahin Moradi. A critical point approach to boundary value problems on the real line. *Appl. Math. Lett.*, 76:215–220, 2018.

[8] Martin Bohner, Shapour Heidarkhani, Amjad Salari, and Giuseppe Caristi. Existence of three solutions for impulsive multi-point boundary value problems. *Opuscula Math.*, 37(3):353–379, 2017.

[9] Gabriele Bonanno. A critical point theorem via the Ekeland variational principle. *Nonlinear Anal.*, 75(5):2992–3007, 2012.

[10] Gabriele Bonanno. Relations between the mountain pass theorem and local minima. *Adv. Nonlinear Anal.*, 1(3):205–220, 2012.

[11] Gabriele Bonanno and Antonia Chinnì. Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent. *J. Math. Anal. Appl.*, 418(2):812–827, 2014.

[12] Gabriele Bonanno and Salvatore A. Marano. On the structure of the critical set of non-differentiable functions with a weak compactness condition. *Appl. Anal.*, 89(1):1–10, 2010.

[13] Maria-Magdalena Boureanu, Andaluzia Matei, and Mircea Sofonea. Nonlinear problems with $p(\cdot)$-growth conditions and applications to antiplane contact models. *Adv. Nonlinear Stud.*, 14(2):295–313, 2014.

[14] Maria-Magdalena Boureanu and Vicenţiu D. Rădulescu. Anisotropic Neumann problems in Sobolev spaces with variable exponent. *Nonlinear Anal.*, 75(12):4471–4482, 2012.

[15] Yunmei Chen, Stacey Levine, and Murali Rao. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66(4):1383–1406, 2006.

[16] Lars Diening, Petteri Harjulehto, Peter Hästö, and Michael Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[17] Mohamed Saad Bouh Elemine Vall and Ahmed Ahmed. Multiplicity of solutions for a class of Neumann elliptic systems in anisotropic Sobolev spaces with variable exponent. *Adv. Oper. Theory*, 4(2):497–513, 2019.

[18] Xianling Fan. Anisotropic variable exponent Sobolev spaces and $p(\vec{x})$-Laplacian equations. *Complex Var. Elliptic Equ.*, 56(7-9):623–642, 2011.

[19] Xianling Fan and Chao Ji. Existence of infinitely many solutions for a Neumann problem involving the $p(x)$-Laplacian. *J. Math. Anal. Appl.*, 334(1):248–260, 2007.

[20] Shapour Heidarkhani, Ghasem A. Afrouzi, and Armin Hadjian. Multiplicity results for elliptic problems with variable exponent and nonhomogeneous Neumann conditions. *Math. Methods Appl. Sci.*, 38(12):2589–2599, 2015.

[21] Shapour Heidarkhani, Anderson L. A. de Araujo, Ghasem A. Afrouzi, and Sina Pourali Roudbari. Variational approaches to equations of $p(x)$-Laplace type operators with variable exponents and nonlinear boundary conditions. 2020. Submitted.

[22] Somayeh Khademloo, Ghasem A. Afrouzi, and T. Norouzi Ghara. Infinitely many solutions for anisotropic variable exponent problems. *Complex Var. Elliptic Equ.*, 63(9):1353–1369, 2018.

[23] Satyanad Kichenassamy and Laurent Véron. Singular solutions of the p-Laplace equation. *Math. Ann.*, 275(4):599–615, 1986.

[24] Blaise Kone, Stanislas Ouaro, and Sado Traore. Weak solutions for anisotropic nonlinear elliptic equations with variable exponents. *Electron. J. Differential Equations*, No. 144, 11 pages, 2009.

[25] Jongrak Lee, Jae-Myoung Kim, and Yun-Ho Kim. Existence and multiplicity of solutions for Kirchhoff-Schrödinger type equations involving $p(x)$-Laplacian on the entire space $\mathbb{R}^N$. *Nonlinear Anal. Real World Appl.*, 45:620–649, 2019.

[26] Tongxing Li, Nicola Pintus, and Giuseppe Viglialoro. Properties of solutions to porous medium problems with different sources and boundary conditions. *Z. Angew. Math. Phys.*, 70(3):Paper No. 86, 18, 2019.

[27] Ying Li and Lin Li. Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$. *Bull. Malays. Math. Sci. Soc.*, 40(4):1455–1463, 2017.

[28] Michael Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[29] Weihong Xie and Haibo Chen. Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^n$. *Math. Nachr.*, 291(16):2476–2488, 2018.

[30] Eberhard Zeidler. *Nonlinear functional analysis and its applications. II/B*. Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.

[31] Dun Zhao and Xianling Fan. On the generalised Orlicz–Sobolev space $W^{k,p(x)}(\omega)$. *J. Gansu Educ. College*, 12(1):1–6, 1998.

Professor M. Bohner is an Editor of the Journal Nonautonomous Dynamical Systems, and therefore this submission was handled by another Editor of the journal.