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Parallelism in Steiner systems*

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Abstract

The authors give a survey about the problem of parallelism in Steiner systems, pointing out some open problems.

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1 Introduction

A *Steiner system* $S(h, k, v)$ is a k-uniform hypergraph $\Sigma = (X, \mathcal{B})$ of order v, such that every subset $Y \subseteq X$ of cardinality h has degree $d(Y) = 1$ [4]. In the language of classical design theory, an $S(h, k, v)$ is a pair $\Sigma = (X, \mathcal{B})$ where X is a finite set of cardinality v, whose elements are called *points* (or *vertices*), and $\mathcal{B}$ is a family of k-subsets $B \subseteq X$, called *blocks*, such that for every subset $Y \subseteq X$ of cardinality h there exists exactly one block $B \in \mathcal{B}$ containing Y.

Using more modern terminology, if K_n^u denotes the complete u-uniform hypergraph of order n, then a Steiner system $S(h, k, v)$ is a K_k^h-*decomposition* of K_v^h, i.e. a pair $\Sigma = (X, \mathcal{B})$, where X is the vertex set of K_v^h and $\mathcal{B}$ is a collection of hypergraphs all isomorphic to K_k^h (blocks) such that every edge of K_v^h belongs to exactly one hypergraph of $\mathcal{B}$. An $S(2, 3, v)$ is usually called *Steiner Triple System* and denoted by STS(v); it is well-kown that an STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$, and contains $v(v-1)/6$ triples. An $S(3, 4, v)$ is usually called *Steiner Quadruple System* and denoted by SQS(v);

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it is well-kown that an $\mathrm{SQS}(v)$ exists if and only if $v \equiv 2, 4 \pmod 6$, and contains $v(v-1)(v-2)/24$ quadruples.

Given a Steiner system $\Sigma = (X, \mathcal{B})$, two distinct blocks $B', B'' \in \mathcal{B}$ are said to be *parallel* if $B' \cap B'' = \emptyset$. A *partial parallel class* of Σ is a family $\Pi \subseteq \mathcal{B}$ of parallel blocks. If Π is a partition of X, then it is said to be a *parallel class* of Σ. Of course not every Steiner system $S(h, k, v)$ has a parallel class (for example, when v is not a multiple of k) and so it is of considerable interest to determine in general just how large a partial parallel class a Steiner system $S(h, k, v)$ must have.

Open problem 1.1 (Parallelism Problem)**.** Let $2 \leq h < k < v$, determine the maximum integer $\pi(h, k, v)$ such that any $S(h, k, v)$ has at least $\pi(h, k, v)$ distinct parallel blocks.

In this paper we will survey known results on the *parallelism problem* and give some open problems, including Brouwer's conjecture.

2 A result of Lindner and Phelps

In [6] C. C. Lindner and K. T. Phelps proved the following result.

Theorem 2.1. *Any Steiner system $S(k, k+1, v)$, with $v \geq k^4 + 3k^3 + k^2 + 1$, has at least $\lceil \frac{v-k+1}{k+2} \rceil$ parallel blocks.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a Steiner system $S(k, k+1, v)$, with $v \geq k^4 + 3k^3 + k^2 + 1$. Let Π be a partial parallel class of maximum size, say t, and denote by P the set of vertices belonging to the blocks of Π. Since Π is a partial parallel class of maximum size, every $Y \subseteq X - P$, $|Y| = k$, is contained in one block $B \in \mathcal{B}$ which intersects P in exactly one vertex. Denote by Ω the set of all blocks having k elements in $X - P$ (and so the remaining vertex in P) and by A the set of all vertices belonging to P and to some block of Ω:

$$\Omega = \{B \in \mathcal{B} : |B \cap (X - P)| = k\},$$
$$A = \{x \in P : x \in B, B \in \Omega\}.$$

For every $x \in A$, set

$$T(x) = \{B - \{x\} : B \in \Omega\}.$$

We can see that $\Sigma' = (X - P, T(x))$ is a partial Steiner system of type $S(k-1, k, v - (k+1)t)$, with

$$|T(x)| \leq \frac{\binom{v-(k+1)t}{k-1}}{k},$$

and $\{T(x)\}_{x \in A}$ is a partition of $\mathcal{P}_k(X - P)$, i.e., the set of all k-subsets of $X - P$. Observe that, if B is a block of Π containing at least two vertices of A, then for each $x \in A \cap B$ we must have

$$|T(x)| \leq \frac{k\binom{v-(k+1)t-1}{k-2}}{k-1}.$$

Indeed, otherwise, let y be any other vertex belonging to $A \cap B$ and B_1 be a block of $T(y)$. Since at most $k\binom{v-(k+1)t-1}{k-2}/(k-1)$ of the blocks in $T(x)$ can intersect the block B_1, then $T(x)$ must contain a block B_2 such that $B_1 \cap B_2 = \emptyset$. Hence, the family $\Pi' =$

$(\Pi - \{B\}) \cup \{B_1, B_2\}$ is a partial parallel class of blocks having size $|\Pi'| > |\Pi|$, a contradiction. It follows that, for every block $B \in \Pi$ containing at least two vertices of A,

$$\sum_{x \in A \cap B} |T(x)| \leq \frac{(k+1)k\binom{v-(k+1)t-1}{k-2}}{k-1}.$$

Therefore, if we denote by r the number of blocks of Π containing at most one vertex of A and by s the number of blocks of Π containing at least two vertices of A, then

$$\binom{v-(k+1)t}{k} = \sum_{x \in A} |T(x)| \leq \left[\frac{(k+1)k\binom{v-(k+1)t-1}{k-2}}{k-1}\right] r + \left[\frac{\binom{v-(k+1)t}{k-1}}{k}\right] s.$$

Now, consider the following two cases:

Case 1. $\left[(k+1)k\binom{v-(k+1)t-1}{k-2}/(k-1)\right] \leq \binom{v-(k+1)t}{k-1}/k.$

It follows

$$\binom{v-(k+1)t}{k} = \sum_{x \in A} |T(x)| \leq \frac{(r+s)\binom{v-(k+1)t}{k-1}}{k} \leq t\left[\frac{\binom{v-(k+1)t}{k-1}}{k}\right],$$

from which $t \geq \frac{v-k+1}{k+2}$.

Case 2. $\left[(k+1)k\binom{v-(k+1)t-1}{k-2}/(k-1)\right] > \binom{v-(k+1)t}{k-1}/k.$

In this case, it follows $t \geq \left(v - k^3 - k^2\right)/(k+1)$ and so

$$t \geq \frac{v-k^3-k^2}{k+1} \geq \frac{v-k+1}{k+2},$$

for $v \geq k^4 + 3k^3 + k^2 + 1$.

Combining Cases 1 and 2 completes the proof of the theorem. □

For Steiner triple and quadruple systems Theorem 2.1 gives the following result.

Corollary 2.2.

(i) Any STS(v), with $v \geq 45$, has at least $\left\lceil \frac{v-1}{4} \right\rceil$ parallel blocks.

(ii) Any SQS(v), with $v \geq 172$, has at least $\left\lceil \frac{v-2}{5} \right\rceil$ parallel blocks.

Regarding STS(v)s, the cases of $v < 45$ has been studied by C. C. Lindner and K. T. Phelps in [6] and by G. Lo Faro in [7, 8], while for SQS(v)s, the cases of $v < 172$ has been examined by G. Lo Faro in [9]. Collecting together their results gives the following theorem.

Theorem 2.3.

(i) Any STS(v), with $v \geq 9$, has at least $\left\lceil \frac{v-1}{4} \right\rceil$ parallel blocks.

(ii) Any SQS(v) has at least $\left\lceil \frac{v-2}{5} \right\rceil$ parallel blocks, with the possible exceptions for $v = 20, 28, 34, 38$.

The following result due to D. E. Woolbright [12] improves the inequality of Lindner-Phelps for Steiner triple systems of order $v \geq 139$.

Theorem 2.4. *Any STS(v) has at least $\frac{3v-70}{10}$ parallel blocks.*

For large values of v (greater then $v' \approx 10000$), the above result in turn is improved by the following theorem which is due to A. E. Brouwer [1] and is valid for every admissible $v \geq 127$.

Theorem 2.5. *Any Steiner triple system of sufficiently large order v has at least $\left\lceil \frac{v-5v^{2/3}}{3} \right\rceil$ parallel blocks.*

In 1981 A. E. Brouwer stated the following open problem.

Open problem 2.6 (Brouwer's Conjecture)**.** Any STS(v) has at least $\lceil \frac{v-c}{3} \rceil$ parallel blocks, for a constant $c \in N$.

By similar arguments as in Theorem 2.1, C. C. Lindner and R. C. Mullin [11] proved a further result for an arbitrary Steiner system $S(h, k, v)$.

Theorem 2.7. *Any Steiner system $S(h, k, v)$, with*

$$v \geq \frac{2k[2k(k-1)^2(k-h) - (h-1)(k-h-1)] + h - 1}{k^2 - kh - h + 1},$$

has at least $\frac{2(v-h+1)}{(k+1)(k-h+1)}$ parallel blocks.

3 A result on parallelism in $S(k, k+1, v)$, for $k \geq 3$

For $k \geq 3$, in [3] (for $k = 3$) and in [2] (for $k > 3$) the author proved the following result.

Theorem 3.1. *Any Steiner system $S(k, k+1, v)$, with $k \geq 3$, has at least $\lfloor \frac{v+2}{2k} \rfloor$ parallel blocks.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a Steiner system $S(k, k+1, v)$, with $k \geq 3$, and Π be a family of parallel blocks of Σ such that

$$P = \bigcup_{B \in \Pi} B$$

and

$$|X - P| \geq (k-1)|\Pi| + 2(k-1),$$

which implies $v \geq 2k(|\Pi| + 1) - 2$. We will prove that Σ has a family Π' of parallel blocks such that $|\Pi'| > |\Pi|$. This is trivial if there exists a block $B \in \Sigma$ such that $B \subseteq X - P$. Therefore, we suppose that for every block $B \in \mathcal{B}$, $B \nsubseteq X - P$.

Note that, for any $Y \subseteq X - P, |Y| = k-1$, if $R = (X-P) - Y$, then there exists an injection $\varphi\colon R \to P$ defined as follows: for every $x \in R$, $\varphi(x)$ is the element of P such that $Y \cup \{x, \varphi(x)\} \in \mathcal{B}$. Now let

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k+1}\} \in \Pi, \text{ for } i = 1, 2, \ldots, r,$$

such that

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k}\} \subseteq \varphi(R);$$

let

$$\{b^j_{i,1}, b^j_{i,2}, \ldots, b^j_{i,k+1}\} \in \Pi, \text{ for } j = 1, 2, \ldots, k-1 \text{ and } i = 1, 2, \ldots, p_j,$$

such that

$$\{b^j_{i,1}, b^j_{i,2}, \ldots, b^j_{i,j}\} \subseteq \varphi(R) \quad \text{and} \quad \{b^j_{i,j+1}, \ldots, b^j_{i,k+1}\} \cap \varphi(R) = \emptyset;$$

and let

$$\{c_{i,1}, c_{i,2}, \ldots, c_{i,k+1}\} \in \Pi, \text{ for } i = 1, 2, \ldots, h,$$

such that

$$\{c_{i,1}, c_{i,2}, \ldots, c_{i,k+1}\} \cap \varphi(R) = \emptyset.$$

Necessarily,

$$(k+1)r + \sum_{i=1}^{k-1} ip_i \geq |\varphi(R)| = |X - P| - (k-1) \geq (k-1)t + k - 1.$$

Since $t = r + \sum_{i=1}^{k-1} p_i + h$, it follows that

$$(k+1)r + \sum_{i=1}^{k-1} ip_i \geq (k-1)r + (k-1)\sum_{i=1}^{k-1} p_i + (k-1)h + k - 1,$$

and so

$$r \geq \frac{1}{2}\left[\sum_{i=1}^{k-2} p_i(k-1-i) + h(k-1) + (k-1)\right].$$

Let $x_{i,j} \in R$ such that $\varphi(x_{i,j}) = a_{i,j}$ and let $y^j_{i,u} \in R$ such that $\varphi(y^u_{i,j}) = b^j_{i,u}$.

Case 1. Suppose $a_{i,k+1} \notin \varphi(R)$, for each $i = 1, 2, \ldots, r$. It follows that

$$|X - P| - (k-1) = \sum_{i=1}^{k-1} ip_i + kr.$$

Since $|X - P| - (k-1) \geq (k-1)t + (k-1)$ and $t = h + r + \sum_{i=1}^{k-1} p_i$, it follows

$$\sum_{i=1}^{k-1} ip_i + kr \geq (k-1)t + k - 1 = (k-1)h + (k-1)r + (k-1)\sum_{i=1}^{k-1} p_i + k - 1,$$

hence

$$r \geq \sum_{i=1}^{k-2} p_i(k-1-i) + h(k-1) + (k-1).$$

Now, consider the injection $\psi\colon R' \to P$, where $R' = \{x_{i,j} \in R : i \neq 1\}$, such that for all $x_{i,j} \in R', \psi(x_{i,j})$ is the element of $\varphi(R)$ satisfying the condition $\{x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, x_{i,j}, \psi(x_{i,j})\} \in \mathcal{B}$.

If Γ is the family of the blocks $\{x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, x_{i,j}, \psi(x_{i,j})\}$ and

$$L = \{c_{i,j} : i = 1, 2, \ldots, h,\ j = 1, 2, \ldots, k+1\} \cup \{b^1_{i,1} : i = 1, 2, \ldots, p_1\} \cup \{a_{1,k}\},$$

then $|\Gamma| = k(r-1)$ and $|L| = (k+1)h + p_1 + 1$, with

$$\begin{aligned}|\Gamma| &= k(r-1) = kr - k \\ &\geq \sum_{i=1}^{k-2} p_i(k-i-1) + hk(k-1) + k^2 - 2k > (k+1)h + p_1 + 1 = |L|,\end{aligned}$$

where we used the following inequalities, which hold for $k \geq 3$,

$$\begin{aligned} r &\geq \sum_{i=1}^{k-2} p_i(k-i-1) + h(k-1) + k - 1, \\ hk(k-1) &> h(k+1), \\ k^2 - k &> 1.\end{aligned}$$

Then, it is possible to find an element $x \in P - L$ such that

$$\{x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, \psi^{-1}(x), x\} \in \mathcal{B}.$$

Further, there exists at least an element $y \in \varphi(R)$, $y \neq x$, with x and y belonging to the same $B_{x,y} \in \Pi$. If

$$\Pi' = \Pi - \{B_{x,y}\} \cup \{\{x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, \psi^{-1}(x), x\}, Y \cup \{\varphi^{-1}(y), y\}\},$$

then Π' is a family of parallel blocks of $\mathcal{B}$ with $|\Pi'| > |\Pi|$.

Case 2. Suppose there is at least one element $a_{i,k+1}$ such that $\{a_{i,1}, a_{i,2}, \ldots, a_{i,k+1}\} \subseteq \varphi(R)$. Assume that

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k+1}\} \subseteq \varphi(R), \text{ for each } i = 1, 2, \ldots, r'$$

and

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k}\} \subseteq \varphi(R),\ a_{i,k+1} \notin \varphi(R), \text{ for each } i = r'+1, \ldots, r.$$

If $r \geq 2$, consider the injection $\mu\colon R'' \to P$, where

$$R'' = \left\{x_{i,j} \in R : (i,j) \neq (1,1), (1,2), \ldots, \left(1, \left\lceil \tfrac{k-1}{2} \right\rceil\right), (2,1), (2,2), \ldots, \left(2, \left\lceil \tfrac{k-1}{2} \right\rceil\right)\right\},$$

such that for every $x_{i,j} \in R''$, $\mu(x_{i,j})$ is the element of $\varphi(R)$ satisfying the condition

$$\left\{x_{1,1}, x_{1,2}, \ldots, x_{1,\left\lceil \frac{k-1}{2} \right\rceil}, x_{2,1}, x_{2,2}, \ldots, x_{2,\left\lfloor \frac{k-1}{2} \right\rfloor}, x_{i,j}, \mu(x_{i,j})\right\} \in \mathcal{B}.$$

If Γ' is the family of blocks

$$\left\{x_{1,1}, x_{1,2}, \ldots, x_{1,\left\lceil \frac{k-1}{2} \right\rceil}, x_{2,1}, x_{2,2}, \ldots, x_{2,\left\lfloor \frac{k-1}{2} \right\rfloor}, x_{i,j}, \mu(x_{i,j})\right\}$$

and

$$L' = \{c_{i,j} : i = 1, 2, \ldots, h,\ j = 1, 2, \ldots, k+1\} \cup \{b^1_{i,1} : i = 1, 2, \ldots, p_1\},$$

it follows that

$$\begin{aligned}
|\Gamma'| &\geq (k-1)t - \sum_{i=1}^{k-1} ip_i = (k-1)(r + \sum_{i=1}^{k-1} p_i + h) - \sum_{i=1}^{k-1} ip_i \\
&= (k-1)r + (k-1)h + \sum_{i=1}^{k-2}(k-1-i)p_i \\
&\geq \frac{k+1}{2}\sum_{i=1}^{k-2}(k-1-i)p_i + \frac{h(k^2-1)}{2} + \frac{(k-1)^2}{2} \\
&> (k+1)h + p_1 + 1 = |L'| + 1,
\end{aligned}$$

where we used

$$\begin{aligned}
&t = r + h + \sum_{i=1}^{k-1} p_i, \\
&r \geq \frac{1}{2}\left[\sum_{i=1}^{k-2} p_i(k-i-1) + h(k-1) + (k-1)\right], \\
&k \geq 3.
\end{aligned}$$

Therefore, it is possible to find at least two distinct elements x', x'' belonging to two distinct blocks B', B'' of Γ':

$$\begin{aligned}
B' &= \left\{x_{1,1}, x_{1,2}, \ldots, x_{1,\lceil \frac{k-1}{2} \rceil}, x_{2,1}, x_{2,2}, \ldots, x_{2,\lfloor \frac{k-1}{2} \rfloor}, \mu^{-1}(x'), x'\right\}, \\
B'' &= \left\{x_{1,1}, x_{1,2}, \ldots, x_{1,\lceil \frac{k-1}{2} \rceil}, x_{2,1}, x_{2,2}, \ldots, x_{2,\lfloor \frac{k-1}{2} \rfloor}, \mu^{-1}(x''), x''\right\},
\end{aligned}$$

such that $x', x'' \in P - L'$. Since $x' \neq x''$, we can suppose that

$$x' \neq a_{2,\lceil \frac{k-1}{2} \rceil}.$$

Therefore, it is possible to find an element $y \in \varphi(R)$, $y \neq x'$, with x' and y belonging to the same block $B_{x,y}$ of Π. It follows that there exists a family Π' of parallel blocks with $|\Pi'| = |\Pi| + 1$.

If $r = 1$, then $r' = r = 1$. Since

$$r \geq \frac{1}{2}\left[\sum_{i=1}^{k-2} p_i(k-i-1) + h(k-1) + (k-1)\right],$$

then

$$\sum_{i=1}^{k-2} p_i(k-i-1) + h(k-1) + (k-1) \leq 2.$$

It follows necessarily: $k = 3$, $h = 0$, $p_1 = 0$. Hence $t = p_2 + 1$, $|X - P| = 2p_2 + 6$, and $v = 6p_2 + 10$.

If $p_2 = 0$, then $v = 10$ and $t = 1$, and it is well-known that the unique STS(10) has two parallel blocks.

If $p_2 \geq 1$, consider the blocks

$$B' = \{x_{1,1}, \varphi^{-1}(b^2_{1,1}), X_{1,2}, x'\},$$
$$B'' = \{x_{1,1}, \varphi^{-1}(b^2_{1,1}), X_{1,3}, x''\},$$

where $x', x'' \in \varphi(R)$. Since $x' \neq x''$, we can assume $x' \neq b^2_{1,2}$ and by applying the same technique as the previous cases we can find a family Π' of parallel blocks with $|\Pi'| = |\Pi| + 1$.

Therefore, it is proved that if $\Sigma = (X, \mathcal{B})$ is any $S(k, k+1, v)$, with $k \geq 3$, and Π is a family of parallel blocks of Σ such that $|\Pi| = t$ and $|X - P| \geq (k-1)t + 2(k-1)$, where $P = \bigcup_{B \in \Pi} B$, then Σ has a partial parallel class Π' of cardinality $|\Pi'| > |\Pi|$. It follows that, if $t = \lfloor \frac{v-2(k-1)}{2k} \rfloor$, then Σ has a partial parallel class of cardinality $t' = t + 1 = \lfloor \frac{v+2}{2k} \rfloor$. □

By applying the same technique used in the previous proof, M. C. Marino and R. S. Rees [10] improved the lower bound stated by Theorem 3.1 to $\left\lfloor \frac{2(v+2)}{3(k+1)} \right\rfloor$.

4 Open problems

(a) *Remove the exceptions of Theorem 2.3.*

It is known that $\pi(3, 4, v) = \lfloor \frac{v}{4} \rfloor$ for $v = 4, 8, 10, 14$. In [5] by means of an exhaustive computer search the authors classified the Steiner quadruple systems of order 16 up to isomorphism; following a private conversation, it turned out that the computer search showed that every SQS(16) has a parallel class and so $\pi(3, 4, 16) = 4$.

(b) *Determine the smallest v such that $\pi(3, 4, v) \neq \lfloor \frac{v}{4} \rfloor$.*

Concerning the parallelism in Steiner systems, an interesting question arises when we consider resolvable systems. A Steiner system $\Sigma = (X, \mathcal{B})$ is said to be *resolvable* provided $\mathcal{B}$ admits a partition $\mathcal{R}$ (*resolution*) into parallel classes. A resolvable Steiner triple system is called *Kirkman Triple System* (KTS, in short). It is well-known that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$ (any resolution contains $(v-1)/2$ parallel classes of size $v/3$).

(c) **Problem of A. Rosa** (1978)**:** *Let $\Sigma = (X, \mathcal{B})$ be any KTS(v) and $\mathcal{R}$ be a resolution of Σ. Determine a lower bound for the size of partial parallel classes of Σ in which no two triples come from the same parallel class of $\mathcal{R}$.*

The problem of A. Rosa can be posed for any resolvable Steiner systems $S(h, k, v)$:

(c') **Problem of A. Rosa:** *Let $\Sigma = (X, \mathcal{B})$ be any Steiner system $S(h, k, v)$ and $\mathcal{R}$ be a resolution of Σ. Determine a lower bound for the size of partial parallel classes of Σ in which no two blocks come from the same parallel class of $\mathcal{R}$.*

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