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A generalization of Kruskal–Katona's theorem

Luca Amata and Marilena Crupi

Abstract

Let K be a field, E the exterior algebra of a finite dimensional K-vector space, and F a finitely generated graded free E-module with homogeneous basis $g_1, \dots, g_r$ such that $\deg g_1 \leq \deg g_2 \leq \cdots \leq \deg g_r$. We characterize the Hilbert functions of graded E–modules of the type F/M, with M graded submodule of F. The existence of a unique lexicographic submodule of F with the same Hilbert function as M plays a crucial role.

1 Introduction

The extremal properties of Hilbert functions have been studied in a lot of papers. Indeed, such a subject is related to combinatorics, commutative algebra and algebraic geometry, and encodes important information. There are many well–known results, that date back to Macaulay [20], on the classification of Hilbert functions in various contexts. The Macaulay's key idea about the existence of highly structured monomial ideals, the lexicographic ideals, which attain all Hilbert functions of quotients of polynomial rings, revealed crucial. The pivotal property is that a lexicographic ideal grows as slowly as possible. Macaulay's theorem has been the inspiration for many similar classifications. Stanley wrote Macaulay's theorem in its modern form in [21] (see also [6]). Kruskal proved a theorem on bounding the f-vectors of simplicial complexes in a way similar to Macaulay's theorem [18]. Katona independently proved

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an equivalent result phrased in terms of Sperner families [19]. The Kruskal–Katona theorem may be also interpreted as a theorem on Hilbert functions of quotients of exterior algebras in [4]. Finally, Macaulay's theorem has been extended to modules by many authors, in particular by Hulett [17] and Gasharov in [11].

Let K be a field, V a K-vector space with basis $e_1, \ldots, e_n$, and E the exterior algebra of V. Let $F = \oplus_{i=1}^r Eg_i$ be a finitely generated graded free E-module with homogeneous basis $g_1, \ldots, g_r$ such that $\deg g_1 \leq \deg g_2 \leq \cdots \leq \deg g_r$.

In this paper, we generalize the combinatorial Kruskal–Katona theorem [4, Theorem 4.1] for finitely generated modules over exterior algebras. More precisely, we describe the possible Hilbert functions of graded E–modules of the form F/M, with M graded submodule of F. Our result bounds the growth of Hilbert function of such a kind of modules. A key role in our context is played by the class of lexicographic submodules.

If I is a graded ideal of E, the construction of the lexicographic ideal I^{lex} with the same Hilbert function of I proceeds as follows: for each graded component I_j of I, let I_j^{lex} be the K–vector space spanned by the (unique) lexicographic segment L_j with $|L_j| = \dim_K I_j$. Then one defines $I^{\text{lex}} = \oplus_j I_j^{\text{lex}}$. Such a construction, with suitable modifications, can be applied if one wants to get the unique lexicographic submodule attaining the Hilbert function of the given graded submodule in F (Theorem 4.2).

The paper is organized as follows. Section 2 contains preliminary notions and results. In Section 3, we discuss in details the Hilbert functions of quotients of graded free E-modules. The study of the behavior of these functions is crucial for the development of the paper. In Section 4, we state a new expression for such Hilbert functions (Proposition 4.1) and we give their characterization (Theorem 4.2) via the lexicographic submodules. Finally, Section 5 contains some examples illustrating our results.

All the examples are constructed by a *Macaulay2* package created by the authors of this article.

2 Preliminaries and notations

Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n \rangle$ the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$. For any subset $\sigma = \{i_1, \ldots, i_d\}$ of $\{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_d$ we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$. The set of monomials in E forms a K-basis of E of cardinality 2^n.

In order to simplify the notation, we put $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$. An ideal I is called *graded* if I is generated by homogeneous elements. If I is graded, then $I = \oplus_{j \geq 0} I_j$, where I_j is the K-vector space of all homogeneous elements $f \in I$ of degree j. We denote by $\operatorname{indeg}(I)$ the *initial degree* of I, that is, the minimum s such that $I_s \neq 0$.

For any not empty subset S of E (respectively, of F), we denote by $\operatorname{Mon}(S)$ the set of all monomials in S (respectively, of F), and we denote its cardinality by $|S|$. Moreover, we denote by $\operatorname{Mon}_d(S)$ the set of all monomials of degree d in S.

Let $\mathcal{M}$ be the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$.

If $M \in \mathcal{M}$, the function $H_M : \mathbb{Z} \to \mathbb{Z}$ given by $H_M(d) = \dim_K M_d$ is called the Hilbert function of M ([6, Chapter 4], [7, Chapter 1, § 1.9]).

Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \ldots, g_r$, where $\deg(g_i) = f_i$ for each $i = 1, \ldots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. We write $F = \oplus_{i=1}^r E g_i$. The elements of the form $e_\sigma g_i$, where $e_\sigma \in \operatorname{Mon}(E)$, are called *monomials* of F, and $\deg(e_\sigma g_i) = \deg(e_\sigma) + \deg(g_i)$.

When we write $F = E^r$, we mean that F is the free E-module $F = \oplus_{i=1}^r E g_i$ with homogeneous basis $g_1, \ldots, g_r$, where g_i $(i = 1, \ldots, r)$ is the r-tuple where the unique non zero–entry is 1 in the i–th position, and such that $\deg(g_i) = 0$, for all i.

Definition 2.1. A graded submodule M of F is a monomial submodule if M is a submodule generated by monomials of F, *i.e.*,

$$M = I_1 g_1 \oplus \cdots \oplus I_r g_r,$$

with I_i a monomial ideal of E, for each i.

If we order the monomials of F with respect to the *degree reverse lexicographic order*, $>_{\mathrm{degrevlex}_F}$ ([5, Section 5], [7, Chapter 15, § 15.7]) and M is a graded submodule of F, denoting by $\operatorname{in}(M)$ the submodule of F generated by the initial terms of elements of M, one has that $H_{F/M} = H_{F/\operatorname{in}(M)}$. Hence, since $\operatorname{in}(M)$ is a monomial submodule of F with the same Hilbert function as M, one may assume M itself to be a monomial submodule without changing the Hilbert function.

In the study of the behavior of the Hilbert function of a graded E-module, the class of lexicographic modules plays a fundamental role.

Let $\operatorname{Mon}_d(E)$ be the set of all monomials of degree $d \geq 1$ in E. Denote by $>_{\mathrm{lex}}$ the *lexicographic order* (lex order, for short) on $\operatorname{Mon}_d(E)$, *i.e.*, if $e_\sigma = e_{i_1} e_{i_2} \cdots e_{i_d}$ and $e_\tau = e_{j_1} e_{j_2} \cdots e_{j_d}$ are monomials belonging to $\operatorname{Mon}_d(E)$ with $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_d \leq n$, then $e_\sigma >_{\mathrm{lex}} e_\tau \quad \text{if} \quad i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$ and $i_s < j_s$ for some $1 \leq s \leq d$.

Definition 2.2. A nonempty subset M of $\mathrm{Mon}_d(E)$ is called a *lexicographic segment* (lex segment, for short) of degree d if for all $v \in M$ and all $u \in \mathrm{Mon}_d(E)$ such that $u >_{\mathrm{lex}} v$, we have that $u \in M$.

Definition 2.3. A monomial ideal I of E is called a *lexicographic ideal* (lex ideal, for short) if for all monomials $v \in I$ and all monomials $u \in E$ with $\deg v = \deg u$ and $u >_{\mathrm{lex}} v$, then $u \in I$, *i.e.*, $\mathrm{Mon}_d(I)$ is a lex segment, for all d.

Remark 2.4. The trivial ideals of E are monomial lex ideals.

It is well–known that if I is a graded ideal of E, then there exists a unique lex segment ideal of E, usually denoted by I^{lex}, such that $H_{E/I} = H_{E/I^{\mathrm{lex}}}$ [4, Theorem 4.1] (see also [14, Theorem 6.3.1]).

Now, let F_d be the part of degree d of $F = \oplus_{i=1}^{r} Eg_i$, *i.e.*, the K-vector space of homogeneous elements of F of degree d, and let $\mathrm{Mon}_d(F)$ be the set of all monomials of degree d of F. We order such a set by the ordering $>_{\mathrm{lex}_F}$ defined as follows:
if ug_i and vg_j are monomials of F such that $\deg(ug_i) = \deg(vg_j)$, then $ug_i >_{\mathrm{lex}_F} vg_j$ if $i < j$ or $i = j$ and $u >_{\mathrm{lex}} v$.

For instance, if $E = K\langle e_1, e_2, e_3\rangle$ and $F = Eg_1 \oplus Eg_2$, with $\deg g_1 = 2$ and $\deg g_2 = 3$, the monomials of F, with respect to $>_{\mathrm{lex}_F}$, are ordered as follows:

$\mathrm{Mon}_2(F)$	g_1
$\mathrm{Mon}_3(F)$	$e_1g_1 >_{\mathrm{lex}_F} e_2g_1 >_{\mathrm{lex}_F} e_3g_1 >_{\mathrm{lex}_F} g_2$
$\mathrm{Mon}_4(F)$	$e_1e_2g_1 >_{\mathrm{lex}_F} e_1e_3g_1 >_{\mathrm{lex}_F} e_2e_3g_1 >_{\mathrm{lex}_F} e_1g_2 >_{\mathrm{lex}_F} e_2g_2 >_{\mathrm{lex}_F} e_3g_2$
$\mathrm{Mon}_5(F)$	$e_1e_2e_3g_1 >_{\mathrm{lex}_F} e_1e_2g_2 >_{\mathrm{lex}_F} e_1e_3g_2 >_{\mathrm{lex}_F} e_2e_3g_2$
$\mathrm{Mon}_6(F)$	$e_1e_2e_3g_2$

Definition 2.5. A nonempty subset N of $\mathrm{Mon}_d(F)$ is called a *lexicographic segment* of F (lex_F segment, for short) of degree d if for all $v \in N$ and all $u \in \mathrm{Mon}_d(F)$ such that $u >_{\mathrm{lex}_F} v$, then $u \in N$.

Definition 2.6. Let L be a monomial submodule of F. L is a *lexicographic submodule* (lex submodule, for short) if for all $u, v \in \mathrm{Mon}_d(F)$ with $v \in L$ and $u >_{\mathrm{lex}_F} v$, one has $u \in L$, for every d, *i.e.*, $\mathrm{Mon}_d(L)$ is a lex_F segment of degree d, for each degree d.

Definition 2.6 is equivalent to the following one [2, Proposition 3.12] (see also [9, Proposition 3.8]).

Definition 2.7. Let L be a graded submodule of F. L is a lex submodule of F if $L = \oplus_{i=1}^r I_i g_i$, with I_i lex ideals of E $(i = 1, \dots, r)$, and $(e_1, \dots, e_n)^{\rho_i + f_i - f_{i-1}} \subseteq I_{i-1}$, for $i = 2, \dots, r$, with $\rho_i = \operatorname{indeg} I_i$.

Example 2.8. *Let* $E = K\langle e_1, e_2, e_3, e_4\rangle$ *and* $F = Eg_1 \oplus Eg_2 \oplus Eg_3$*, with* $\deg g_1 = -2$*,* $\deg g_2 = -1$ *and* $\deg g_3 = 3$*. The submodule*

$$L = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3, e_1e_4, e_2e_3)g_2 \oplus (e_1e_2e_3, e_1e_2e_4)g_3$$

is a lex submodule of F*.*

3 The Hilbert function of graded E-modules

In this Section, we discuss the Hilbert functions of quotients of free E–modules.

We make the following conventions:

$$\binom{m}{k} = 0 \quad \text{if} \quad m < k \quad \text{or} \quad k < 0.$$

One can observe that if $E = K\langle e_1, \dots, e_n\rangle$, then $H_E(d) = \operatorname{Mon}(E_d) = \binom{n}{d}$, where $\binom{n}{d}$ is the number of monomials of degree d in E. Hence, if I is a graded ideal of E, it follows that

$$H_{E/I}(d) + H_I(d) = \binom{n}{d}.$$

Furthermore, if $F = \oplus_{i=1}^r Eg_i$, we have that

$$H_F(d) = \sum_{i=1}^r H_{Eg_i}(d) = \sum_{i=1}^r \binom{n}{d - f_i},$$

and consequently, if M is a graded submodule of F, one has

$$H_{F/M}(d) + H_M(d) = \sum_{i=1}^r \binom{n}{d - f_i}.$$

Let a and i be two positive integers. Then a has the unique i-th Macaulay expansion [14, Lemma 6.3.4]

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \dots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \dots > a_j \geq j \geq 1$. We define

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}.$$

We also set $0^{(i)} = 0$ for all $i \geq 1$.

We quote next result from [4].

Theorem 3.1. ([4, Theorem 4.1]) *Let $(h_1, \ldots, h_n)$ be a sequence of nonnegative integers. Then the following conditions are equivalent:*

(a) $1 + \sum_{i=1}^{n} h_i t^i$ *is the Hilbert series of a graded K-algebra E/I;*

(b) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n-1$.

Theorem 3.1 is known as the Kruskal-Katona theorem.

From now on, if $1 + \sum_{i=1}^{n} h_i t^i$ is the Hilbert series of a graded K-algebra E/I, $I \subsetneq E$, the sequence $(1, h_1, \ldots, h_n)$ is called the *Hilbert sequence* of E/I. We will denote it by $Hs_{E/I}$.

From the Kruskal-Katona theorem, one can deduce that a sequence of nonnegative integers $(h_0, h_1, \ldots, h_n)$ is the Hilbert sequence of a graded K–algebra E/I, with $I \subsetneq E$ graded ideal of initial degree ≥ 1, if $h_0 = 1$, $h_1 \leq n$ and condition (b) in Theorem 3.1 holds. Note that if $I = 0$, then $Hs_{E/I} = Hs_E = (1, n, \binom{n}{2}, \cdots, \binom{n}{n})$.

Finally, we set $Hs_{E/I} = (\underbrace{0, \ldots, 0}_{n+1})$, if $I = E$.

Let us consider the graded E-module $F = \oplus_{i=1}^{r} E g_i$. One can quickly verify that

$$H_F(d) = \dim_K F_d = 0, \quad \text{for} \quad d < f_1 \quad \text{and} \quad d > f_r + n. \tag{1}$$

Now, we discuss the Hilbert function of a graded E–algebra F/M, with M submodule of F.

Discussion 3.2. Assume M is a monomial submodule of F. From (1), it follows that

$$H_{F/M}(t) = \sum_{i=f_1}^{f_r+n} H_{F/M}(i) t^i,$$

and we can associate to F/M the following sequence

$$(H_{F/M}(f_1), H_{F/M}(f_1+1), \ldots, H_{F/M}(f_r+n)) \in \mathbb{N}_0^{f_r+n-f_1+1}. \tag{2}$$

Such a sequence is called *the Hilbert sequence* of F/M, and denoted by $Hs_{F/M}$. The integers $f_1, f_1+1, \ldots, f_r+n$ are called the $Hs_{F/M}$-degrees. It is clear that $Hs_{F/M} \leq Hs_F$ component–wise.

Moreover, we define

$$\mathrm{indeg} Hs_{F/M} = \min\{d : H_{F/M}(d) \neq 0\}, \quad \text{for } d = f_1, \dots, f_r + n.$$

We use the standard notation $[p]$ for the set $\{1, 2, \dots, p\}$.

Consider the sequence $Hs_{F/M}$ defined in (2). The entries $H_{F/M}(f_i)$ ($i = 1, \dots, r$) are called the *critical values* of $Hs_{F/M}$. Moreover, we define

$$\mu_{f_i} = |\{s \in [r] : f_s = f_i\}|, \quad \text{for } \; i = 1, 2, \dots, r,$$

and we call μ_{f_i} the *multiplicity* of $H_{F/M}(f_i)$.

Now, let us consider the case $H_{F/M}(f_1) = 0$. In such a situation, one has:

$$M = Eg_1 \oplus T_2,$$

where T_2 is a submodule of $Eg_2 \oplus \cdots \oplus Eg_r$. Indeed, if $H_{F/M}(f_1) = 0$, then $M_{f_1} = F_{f_1}$ and so $M_j = F_j$, for $j = f_1, \dots, f_2 - 1$ (it is clear because $1_K g_1 \in M$). Hence, $H_{F/M}(j) = 0$, for $j = f_1, \dots, f_2 - 1$.

Now, let us consider the critical value $H_{F/M}(f_2)$.

If $H_{F/M}(f_2) = 0$, we can repeat the same reasoning done for $H_{F/M}(f_1) = 0$, *i.e.*, $H_{F/M}(j) = 0$, for $j = f_2, \dots, f_3 - 1$, and $M = Eg_1 \oplus Eg_2 \oplus T_3$, where T_3 is a submodule of $Eg_3 \oplus \cdots \oplus Eg_r$. And so on.

Now, let k be the minimum integer such that $H_{F/M}(f_k) \neq 0$, *i.e.*, $\mathrm{indeg} Hs_{F/M} = f_k$. Note that $M = Eg_1 \oplus \cdots \oplus Eg_{k-1} \oplus T_k$, where T_k is a submodule of $Eg_k \oplus \cdots \oplus Eg_r$.We have:

$$H_{F/M}(f_k) \leq \mu_{f_k},$$

and

$$H_{F/M}(f_k + 1) \leq n\mu_{f_k} + \mu_{f_k+1}.$$

The integer $H_{F/M}(f_k)$ is called the *initial critical value* (of F/M) and f_k the *initial critical degree* (of F/M).

4 The main result

In this Section, we state a generalization of the Kruskal–Katona theorem. We characterize the Hilbert functions of quotients of a fixed free E-module $F = \oplus_{i=1}^r Eg_i$.

Our first result gives a new expression for the Hilbert functions of graded E-modules.

Proposition 4.1. *Let M be a graded submodule of $F = \oplus_{i=1}^{r} Eg_i$ and let $H_{F/M}$ the Hilbert function of F/M. There exists an integer $N \leq r$ such that we have the unique expression*

$$H_{F/M}(d) = \sum_{i=N+1}^{r} \binom{n}{d-f_i} + \binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s},$$

where

$$\binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s} < \binom{n}{d-f_N}$$

and $a_0 > a_1 > \cdots > a_s$ and $a_i \geq d - f_N - i$, for all $0 \leq i \leq s$.

Then,

$$H_{F/M}(d+1) \leq \sum_{i=N+1}^{r} \binom{n}{d-f_i+1} + \binom{a_0}{d-f_N+1} + \binom{a_1}{d-f_N} + \cdots + \binom{a_s}{d-f_N-s+1},$$

for $d \geq \operatorname{indeg} Hs_{F/M} + 1$.

Proof. Since $\dim_K F_d = \sum_{i=1}^{r} \binom{n}{d-f_i}$, one has that

$$H_{F/M}(d) \leq \sum_{i=1}^{r} \binom{n}{d-f_i}.$$

Let N be the greatest positive integer less than or equal to r such that

$$H_{F/M}(d) = \sum_{i=N+1}^{r} \binom{n}{d-f_i} + a = \sum_{i=N+1}^{r} H_E(d-f_i) + a, \qquad a < \binom{n}{d-f_N}.$$

We may assume there exists a graded ideal I of E generated in degree $d - f_N$ such that $H_{E/I}(d - f_N) = a$. If

$$a = \binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s}$$

is the $(d - f_N)$–th Macaulay representation of a, one has:

$$H_{F/M}(d) = \sum_{i=N+1}^{r} \binom{n}{d-f_i} + \binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s},$$

for $d \geq \mathrm{indeg} Hs_{F/M} + 1$. Therefore, from Theorem 3.1, it follows that:

$$
\begin{aligned}
&H_{F/M}(d+1) = \sum_{i=N+1}^{r} H_E(d+1-f_i) + H_{E/I}(d+1-f_N) \\
&\leq \sum_{i=N+1}^{r} \binom{n}{d+1-f_i} + H_{E/I}(d-f_N)^{(d-f_N)} = \sum_{i=N+1}^{r} \binom{n}{d+1-f_i} + a^{(d-f_N)} \\
&= \sum_{i=N+1}^{r} \binom{n}{d+1-f_i} + \binom{a_0}{d-f_N+1} + \binom{a_1}{d-f_N} + \cdots + \binom{a_s}{d-f_N-s+1}.
\end{aligned}
$$

□

If T is a set of monomials of degree $d < f_r + n$ of F, we denote by $\mathrm{Shad}(T)$ the following set of monomials of degree $d+1$ of F:

$$\mathrm{Shad}(T) = \{(-1)^{\alpha(\sigma,j)} e_j e_\sigma g_i : e_\sigma g_i \in T,\ j \notin \mathrm{supp}(e_\sigma),\ j = 1, \ldots, n,\ i = 1, \ldots r\},$$

$\alpha(\sigma, j) = |\{r \in \sigma : r < j\}|$. Such a set is called the *shadow* of T (see [8], for the $r = 1$ case). Moreover, let us define the i-th *shadow* recursively by $\mathrm{Shad}^i(T) = \mathrm{Shad}(\mathrm{Shad}^{i-1}(T))$, $\mathrm{Shad}^0(T) = T$.

Furthermore, if M is a monomial submodule of F, and M_d ($d \geq f_1$) is the K-vector space generated by all monomials of degree d belonging to M, we set $\mathrm{Shad}(M_d) = \mathrm{Shad}(\mathrm{Mon}(M_d))$ and by $E_1 M_d$ the K-vector space spanned by $\mathrm{Shad}(M_d)$.

For $p, q \in \mathbb{Z}$ with $p < q$, let us define the following set:

$$[p, q] = \{j \in \mathbb{Z} : p \leq j \leq q\}.$$

Theorem 4.2. *Let $(f_1, f_2, \ldots, f_r) \in \mathbb{Z}^r$ be an r–tuple such that $f_1 \leq f_2 \leq \cdots \leq f_r$ and let $(h_{f_1}, h_{f_1+1}, \ldots, h_{f_r+n})$ be a sequence of nonnegative integers. Set*

$$s = \min\{k \in [f_1, f_r + n] : h_k \neq 0\},$$

and

$$\tilde{r}_j = |\{p \in [r] : f_p = s + j\}|, \quad \text{for} \quad j = 0, 1.$$

Then the following conditions are equivalent:

(a) *$\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of a graded E-module F/M, with $F = \oplus_{i=1}^{r} E g_i$ finitely generated graded free E-module with the basis elements g_i of degrees f_i;*

(b) $h_s \leq \tilde{r}_0$, $h_{s+1} \leq n\tilde{r}_0 + \tilde{r}_1$, $h_i = \sum_{j=N+1}^{r} \binom{n}{i-f_j} + a$, *where a is a positive integer less than* $\binom{n}{i-f_N}$, $0 < N \leq r$, *and* $h_{i+1} \leq \sum_{j=N+1}^{r} \binom{n}{i-f_j+1} + a^{(i-f_N)}$, $i = s+1, \ldots, f_r + n$;

(c) *there exists a unique lexicographic submodule L of a finitely generated graded free E-module $F = \oplus_{i=1}^{r} Eg_i$ with the basis elements g_i of degrees f_i and such that $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of F/L.*

Proof. (a)⇒ (b). It follows from Proposition 4.1 and Discussion 3.2. Note that s is the initial critical degree, $\widetilde{r}_0 = \mu_s$ and $\widetilde{r}_1 = \mu_{s+1}$.
(b)⇒ (c). We construct a lexicographic submodule L of F such that $H_{F/L}(t) = \sum_{i=s}^{f_r+n} h_i t^i$.

Setting $L_p = \langle \mathrm{Mon}(F_p) \rangle$ $(p = f_1, \ldots, s-1)$, let L_{s+j} be the K-vector space generated by the lex_F segment of length $\dim_K F_{s+j} - h_{s+j}$, $j = 0, 1$, where $h_s \leq \tilde{r}_0$ and $h_{s+1} \leq n\tilde{r}_0 + \tilde{r}_1$.

Now, suppose L_k, $s \leq k \leq i$, has already been constructed.

By hypothesis, $\dim_K F_i/L_i = h_i = \sum_{j=N+1}^{r} \binom{n}{i-f_j} + a$, where a is a positive integer less than $a < \binom{n}{i-f_N}$. Hence,

$$\dim_K F_{i+1}/E_1 L_i = \sum_{j=N+1}^{r} \binom{n}{i - f_j + 1} + a^{(i-f_N)}$$

and

$$h_{i+1} \leq \dim_K F_{i+1}/E_1 L_i. \tag{3}$$

Let L_{i+1} be the K-vector space spanned by the lex_F segment of length $\dim_K F_{i+1} - h_{i+1}$. From (3), one has

$$\dim_F L_{i+1} = \dim_K F_{i+1} - h_{i+1} \geq \dim_K F_{i+1} - \dim_K F_{i+1}/E_1 L_i = \dim_K E_1 L_i.$$

Hence $E_1 L_i \subseteq L_{i+1}$. It follows that $L = \oplus_d L_d$ is a submodule of F. The uniqueness of L is clear from the definition of lex submodules.
(c)⇒ (a). It follows immediately. □

Remark 4.3. We have obtained a generalization of Kruskal-Katona's theorem (Theorem 4.2) via results on ideals in an exterior algebra (Proposition 4.1). We believe that such a characterization could also be obtained using the same techniques as in [4], *i.e.*, extending [4, Theorem 4.2] to graded E–modules.

5 Examples

In this Section, we collect some examples in order to illustrate our results.

Let $p, q \in \mathbb{Z}$ such that $p < q$. A finite sequence H of nonnegative integers is called $[p, q]$–sequence if it is indexed by the set $[p, q]$:

$$H = (h_i)_{i\in[p,q]} = (h_p, h_{p+1}, \dots, h_q).$$

We set

$$H(j) = h_j, \quad \text{for} \quad j \in [p, q];$$

the integers j are called H–degrees.

One can observe that the sequence $Hs_{F/M}$ is a $[f_1, f_r + n]$–sequence, and the integers $j \in [f_1, f_r + n]$ are the $Hs_{F/M}$–degrees.

Moreover, if $p = 0$, then H is the $(q + 1)$–tuple $(h_0, h_1, \dots, h_q)$.

Example 5.1. *Let* $p = -2$ *and* $q = 1$. *Then* $[-2, 1] = \{-2, -1, 0, 1\}$. *If* $H = (0, 2, 7, 3)$ *is a* $[-2, 1]$*–sequence, one has* $H(-2) = 0$, $H(-1) = 2$, $H(0) = 7$, *and* $H(1) = 3$.

Example 5.2. Let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = E^3$, and consider the $[0, 4]$–sequence

$$H = (3, 11, 13, 3, 0) = (h_0, h_1, \dots, h_4).$$

Using the procedure described in Theorem 4.2, we can guess if H is a Hilbert sequence of a quotient F/M (M graded submodule of F), and we can also construct the lex submodule L of F such that $H_{F/L} = H$.

With the same notations as in Theorem 4.2. We have $s = f_1 = 0$, $\tilde{r}_0 = 3$ and $\tilde{r}_1 = 0$. In fact, the initial critical value is the first element of the sequence and has multiplicity equal to 3, and there do not exist critical degrees different from it. Therefore, the first two conditions in Theorem 4.2 (b) are realized:

$$h_0 = 3 \leq 3 = \tilde{r}_0,$$

$$h_1 = 11 \leq 12 = n\tilde{r}_0 + \tilde{r}_1.$$

By Proposition 4.1, we have to verify the following inequalities

$$\begin{aligned}
h_1 = 11 = \binom{4}{1} + \binom{4}{1} + \underbrace{\binom{3}{1}}_{a} \quad &\Rightarrow \quad h_2 = 13 \quad \leq \quad 15 = \binom{4}{2} + \binom{4}{2} + \underbrace{\binom{3}{2}}_{a^{(1)}},\\
h_2 = 13 = \binom{4}{2} + \binom{4}{2} + \underbrace{\binom{2}{2}}_{a} \quad &\Rightarrow \quad h_3 = 3 \quad \leq \quad 8 = \binom{4}{3} + \binom{4}{3} + \underbrace{\binom{2}{3}}_{a^{(2)}},\\
h_3 = 3 = \underbrace{\binom{3}{3} + \binom{2}{2} + \binom{1}{1}}_{a} \quad &\Rightarrow \quad h_4 = 0 \quad \leq \quad 0 = \underbrace{\binom{3}{4} + \binom{2}{3} + \binom{1}{2}}_{a^{(3)}}.
\end{aligned}$$

a is the integer defined in Proposition 4.1 (see its proof). Hence H is the Hilbert sequence of a quotient of F.

In order to assure this, we construct the lex submodule $L = \oplus_{d=0}^4 L_d$ of F such that $H_{F/L} = H$; L_d is the K–vector space generated by a lex segment of length $\dim_K F_d - h_d$, for $d = 0, \dots, 4$.

Firstly, one can observe that $\dim_K L_0 = \dim_K F_0 - h_0 = 0$. Hence $L_0 = 0$. Furthermore, $\dim_K L_1 = \dim_K F_1 - h_1 = 12 - 11 = 1$, and so

$$L_1 = \langle e_1 g_1 \rangle.$$

In degree 2, we have $\dim_K L_2 = \dim_K F_2 - h_2 = 3\binom{4}{2} - 13 = 5$. Since, $\mathrm{Shad}(L_1) = \{e_1e_2g_1, e_1e_3g_1, e_1e_4g_1\}$,

$$L_2 = \langle u \in \mathrm{Shad}(L_1), e_2e_3g_1,\ e_2e_4g_1 \rangle.$$

In degree 3, we have $\dim_K L_3 = \dim_K F_3 - h_3 = 3\binom{4}{3} - 3 = 9$. Since $|\mathrm{Shad}(L_2)| = 4$ ($e_\sigma g_1 \in \mathrm{Shad}(L_2)$, for all $e_\sigma \in E_3$), one has

$$L_3 = \langle u \in \mathrm{Shad}(L_2), e_1e_2e_3g_2,\ e_1e_2e_4g_2,\ e_1e_3e_4g_2,\ e_2e_3e_4g_2,\ e_1e_2e_3g_3 \rangle.$$

Finally, we have $\dim_K L_4 = \dim_K F_4 - h_4 = 3\binom{4}{4} = 3$ and all the monomials we need are in $\mathrm{Shad}(L_3)$, *i.e.*, $L_4 = \langle u \in \mathrm{Shad}(L_3) \rangle$.

Hence, we have constructed the unique lex submodule $L = \oplus_{i=1}^r I_i g_i$ with $H_{F/L} = (3, 11, 13, 3, 0)$. More in details:

$$L = (e_1, e_2e_3, e_2e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3)g_3.$$

A more general example can be given if one considers a free-module F with a basis in different degrees.

Example 5.3. Let $E = K\langle e_1, e_2, e_3, e_4 \rangle$, $F = \oplus_{i=1}^3 Eg_i$ with $f_1 = -2, f_2 = 0, f_3 = 3$, and let us consider the $[-2, 7]$–sequence

$$H = (1, 4, 5, 4, 5, 2, 4, 3, 1, 0) = (h_{-2}, h_{-1}, \dots, h_7).$$

As in Example 5.2, we will verify that H is a Hilbert sequence, and then we will construct the lex submodule L of F such that $H_{F/L} = H$.

Since $s = f_1 = -2$, $\tilde{r}_{-2} = 1$ and $\tilde{r}_{-1} = 0$, we have:

$$h_{-2} = 1 \le 1 = \tilde{r}_{-2}, \qquad h_{-1} = 4 \le 4 = n\tilde{r}_{-2} + \tilde{r}_{-1}.$$

Moreover, next inequalities hold (Proposition 4.1):

$$h_{-1} = 4 = \binom{4}{-4} + \binom{4}{-1} + \binom{4}{1} \Rightarrow h_0 = 5 \leq 7 = \binom{4}{-3} + \binom{4}{0} + \binom{4}{2}$$

$$h_0 = 5 = \binom{4}{-3} + \binom{4}{0} + \underbrace{\binom{3}{2} + \binom{1}{1}}_{a} \Rightarrow h_1 = 4 \leq 5 = \binom{4}{-2} + \binom{4}{1} + \underbrace{\binom{3}{3} + \binom{1}{2}}_{a^{(2)}}$$

$$\begin{array}{llcll}
h_1 = 4 = \binom{4}{-2} + \binom{4}{1} & \Rightarrow & h_2 = 5 & \leq & 6 = \binom{4}{-1} + \binom{4}{2} \\
h_2 = 5 = \binom{4}{-1} + \underbrace{\binom{3}{2} + \binom{2}{1}}_{a} & \Rightarrow & h_3 = 2 & \leq & 3 = \binom{4}{0} + \underbrace{\binom{3}{3} + \binom{2}{2}}_{a^{(2)}} \\
h_3 = 2 = \binom{4}{0} + \underbrace{\binom{3}{3}}_{a} & \Rightarrow & h_4 = 4 & \leq & 4 = \binom{4}{1} + \underbrace{\binom{3}{4}}_{a^{(3)}} \\
h_4 = 4 = \binom{4}{1} & \Rightarrow & h_5 = 3 & \leq & 6 = \binom{4}{2} \\
h_5 = 3 = \underbrace{\binom{3}{2}}_{a} & \Rightarrow & h_6 = 1 & \leq & 1 = \underbrace{\binom{3}{3}}_{a^{(2)}} \\
h_6 = 1 = \underbrace{\binom{3}{3}}_{a} & \Rightarrow & h_7 = 0 & \leq & 0 = \underbrace{\binom{3}{4}}_{a^{(3)}}
\end{array}$$

It is worthy of being stressed that in order to get the right expression for the h_i's ($i = -1, \dots, 6$), we firstly compute the binomial coefficient $\binom{4}{i-f_3}$, then the other admissible ones.

For instance, $h_{-1} = 4 = \binom{4}{-1-3} + \binom{4}{-1-0} + \binom{4}{-1+2}$.

Now, we can construct the lex submodule $L = \oplus_{d=-2}^{7} L_d$ of F such that $H_{F/L} = H$, where L_d ($d = -2, \dots, 7$) is the K–vector space generated by a lex segment of length $\dim_K F_d - h_d$, for $d = -2, \dots, 7$.

At first, we observe that $\dim_K L_{-2} = \dim_K F_{-2} - h_{-2} = 0$. Moreover, $\dim_K L_{-1} = \dim_K F_{-1} - h_{-1} = 0$. Hence $L_{-2} = L_{-1} = 0$.
In degree 0, $\dim_K L_0 = \dim_K F_0 - h_0 = \binom{4}{-3} + \binom{4}{0} + \binom{4}{2} - 5 = 2$ and so

$$L_0 = \langle e_1 e_2 g_1, e_1 e_3 g_1 \rangle.$$

In degree 1, $\dim_K L_1 = \dim_K F_1 - h_1 = \binom{4}{-2} + \binom{4}{1} + \binom{4}{3} - 4 = 4$. Since $\mathrm{Shad}(L_0) = \{e_1e_2e_3g_1, e_1e_2e_4g_1, e_1e_3e_4g_1\}$, we choose

$$L_1 = \langle u \in \mathrm{Shad}(L_0), e_2e_3e_4g_1\rangle.$$

In degree 2, $\dim_K L_2 = \dim_K F_2 - h_2 = \binom{4}{-1} + \binom{4}{2} + \binom{4}{4} - 5 = 2$. Since $\mathrm{Shad}(L_1) = \{e_1e_2e_3e_4g_1\}$, we set

$$L_2 = \langle u \in \mathrm{Shad}(L_1), e_1e_2g_2\rangle.$$

In degree 3, $\dim_K L_3 = \dim_K F_3 - h_3 = \binom{4}{0} + \binom{4}{3} + \binom{4}{5} - 2 = 3$. Since $\mathrm{Shad}(L_2) = \mathrm{Shad}^2(L_1) \cup \{e_1e_2e_3g_2, e_1e_2e_4g_2\} = \{e_1e_2e_3g_2, e_1e_2e_4g_2\}$, we get

$$L_3 = \langle u \in \mathrm{Shad}(L_2), e_1e_3e_4g_2\rangle.$$

In degree 4, $\dim_K L_4 = \dim_K F_4 - h_4 = \binom{4}{1} + \binom{4}{4} + \binom{4}{6} - 4 = 1$. Since $\mathrm{Shad}(L_3) = \{e_1e_2e_3e_4g_2\}$, we have that $L_4 = \langle u \in \mathrm{Shad}(L_3)\rangle$.
In degree 5, $\dim_K L_5 = \dim_K F_5 - h_5 = \binom{4}{2} + \binom{4}{5} + \binom{4}{7} - 3 = 3$. Since $\mathrm{Shad}(L_4)$ is empty, we have

$$L_5 = \langle e_1e_2g_3, e_1e_3g_3, e_1e_4g_3\rangle.$$

In degree 6, $\dim_K L_6 = \dim_K F_6 - h_6 = \binom{4}{3} + \binom{4}{6} + \binom{4}{8} - 1 = 3$. Since $\mathrm{Shad}(L_5) = \{e_1e_2e_3g_3, e_1e_2e_4g_3, e_1e_3e_4g_3\}$, we set $L_6 = \langle u \in \mathrm{Shad}(L_5)\rangle$.
Finally, in degree 7, $\dim_K L_7 = \dim_K F_7 - h_7 = \binom{4}{4} + \binom{4}{7} + \binom{4}{9} - 0 = 1$. Since $\mathrm{Shad}(L_6) = \{e_1e_2e_3e_4g_3\}$, we have $L_7 = \langle u \in \mathrm{Shad}(L_6)\rangle$.

In so doing, we have determined the lex submodule $L = \oplus_i^r I_i g_i$ with $H_{F/L} = (1, 4, 5, 4, 5, 2, 4, 3, 1, 0)$. More in details:

$$L = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3e_4)g_2 \oplus (e_1e_2, e_1e_3, e_1e_4)g_3.$$

We close this Section with an example of a sequence of nonnegative integers H that is not a Hilbert sequence of a quotient of a free E–module.

Example 5.4. Let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = \oplus_{i=1}^3 Eg_i$ with $f_1 = -3, f_2 = -2, f_3 = 1$ and let us consider the [-2, 5]–sequence

$$H = (1, 3, 3, 4, 2, 4, 5, 1, 0) = (h_{-2}, h_{-1}, \ldots, h_5).$$

We proceed as in the previuos examples.

It is $s = f_1 = -3$, $\tilde{r}_{-3} = 1$ and $\tilde{r}_{-2} = 1$, and consequently

$$h_{-3} = 1 \leq 1 = \tilde{r}_{-3}, \qquad h_{-2} = 3 \leq 5 = n\tilde{r}_{-3} + \tilde{r}_{-2}.$$

By Proposition 4.1, we can test the required bounds:

$$h_{-2} = 3 = \binom{4}{-3} + \binom{4}{0} + \underbrace{\binom{2}{1}}_{a} \quad \Rightarrow \quad h_{-1} = 3 \quad \leq \quad 5 = \binom{4}{-2} + \binom{4}{1} + \underbrace{\binom{2}{2}}_{a^{(1)}},$$

$$h_{-1} = 3 = \binom{4}{-2} + \underbrace{\binom{3}{1}}_{a} \quad \Rightarrow \quad h_{0} = 4 \quad \not\leq \quad 3 = \binom{4}{-1} + \underbrace{\binom{3}{2}}_{a^{(1)}}$$

The integer h_0 does not satisfy the required inequality. We will see that there does not exist the lex submodule $L = \oplus_{d=-2}^{5} L_d$ of F such that $H_{F/L} = H$.

Indeed, $\dim_K L_{-3} = \dim_K F_{-3} - h_{-3} = 0$. Hence, $L_{-3} = 0$. Moreover, in degree -2, $\dim_K L_{-2} = \dim_K F_{-2} - h_{-2} = \binom{4}{-3} + \binom{4}{0} + \binom{4}{1} - 3 = 2$. Hence,

$$L_{-2} = \langle e_1 g_1, e_2 g_1 \rangle.$$

In degree -1, we have $\dim_K L_{-1} = \dim_K F_{-1} - h_{-1} = \binom{4}{-2} + \binom{4}{1} + \binom{4}{2} - 3 = 7$. On the other hand, $\mathrm{Shad}(L_{-2}) = \{e_1e_2g_1, e_1e_3g_1, e_1e_4g_1, e_2e_3g_1, e_2e_4g_1\}$, then

$$L_{-1} = \langle u \in \mathrm{Shad}(L_{-2}), e_3e_4g_1, e_1g_2 \rangle.$$

In degree 0, we have $\dim_K L_0 = \dim_K F_0 - h_0 = \binom{4}{-1} + \binom{4}{2} + \binom{4}{3} - 4 = 6$. Since, $\mathrm{Shad}(L_{-1}) = \{e_1e_2e_3g_1, e_1e_2e_4g_1, e_1e_3e_4g_1, e_2e_3e_4g_1, e_1e_2g_2, e_1e_3g_2, e_1e_4g_2\}$, then $|\mathrm{Shad}(L_{-1})| > 6$. This situation implies that it is not possible the construction of the lex submodule L with $H_{F/L} = (1, 3, 3, 4, 2, 4, 5, 1, 0)$.

On the contrary, one can verify that for $h_0 = 3$, there exists the lex submodule $L = \oplus_i^r I_i g_i$ of F with $H_{F/L} = (1, 3, 3, 3, 2, 4, 5, 1, 0)$.

Remark 5.5. The procedures described in this paper are part of two *Macaulay2* packages "ExteriorIdeals.m2" [1], "ExteriorModules.m2", and tested with Macaulay 1.10.

All the examples in this paper have been constructed by such packages.

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Luca AMATA,
Department of Mathematics and Computer Sciences,
Physics and Earth Sciences,
University of Messina,
Viale Ferdinando Stagno d'Alcontres 31, 98166 Messina, Italy
Email: lamata@unime.it

Marilena CRUPI,
Department of Mathematics and Computer Sciences,
Physics and Earth Sciences,
University of Messina,
Viale Ferdinando Stagno d'Alcontres 31, 98166 Messina, Italy
Email: mcrupi@unime.it