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A Deterministic and Stochastic Variational Approach to Multistage Equilibrium Problems Under Uncertainty and Application to an Electricity Market

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Mathematical methods of economy, finance, and actuarial sciences

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Introduction

The central aim of this thesis is the study of a Radner equilibrium model, also known as *equilibrium of plans, prices, and price expectations*.
The modern theory of general economic equilibrium has its founder in Leon Walras [66] in 1874. Walras recognized the importance of dealing with models closer to reality and provided a sequence of models, each taking into account more aspects of a real economy. This led him to introduce a suitable price-adjustment mechanism to model the *law of supply and demand*. Subsequently, Arrow [4] and after Debreu [18], enabled the theory of general economic equilibrium to be reinterpreted to cover the case of uncertainty about the availability of resources and about consumption and production possibilities. Indeed, Debreu [19] presented, in 1959, a unified treatment of time and uncertainty by introducing an economic equilibrium model that evolves in a sequence of markets under uncertainty on future conditions. The uncertainty is formalized by means of *states of the world*. In this way, the elements in the market are distinguishable not only by their physical characteristics and the location and dates of their availability and/or use, but also for the state of the world. However, all trades are assumed to take place simultaneously, and before the uncertainty is revealed. Radner [51], in 1972, presented a model of exchange, consumption, and production under *time* and *uncertainty* that generalizes the Debreu equilibrium model to make the market institutions more realistic. Indeed, Radner equilibrium is characterized by:

(i) the central role of information, progressively revealed in the market;

(ii) the possibility of agents to transfer wealth among all possible future times, before the uncertainty is revealed;

(iii) the possibility to trade, at each possible time and in each possible state that can occur, after the uncertainty is revealed and the market reopens.

We approach the study of this equilibrium model by means of variational inequalities theory: introduced by Stampacchia and Fichera in 1964, it provides powerful and handy tools to perform quantitative and qualitative studies in relation to optimization problems, equilibrium problems, system of equations, etc. In the last two decades, this theory has been developed in order to capture also the uncertainty and randomness involved in many applications, for instance in economics, management, and finance. Stochastic variational inequalities are been introduced as a natural extension of deterministic variational inequalities problems. In particular, *one-stage* stochastic variational inequality problems have been studied and various analytical formulations have been considered, in which, however, opportunities for recourse decisions are not allowed. Nevertheless, in many real-life applications, the decision-maker has to make sequential decisions, motivating the interest in stochastic variational inequalities problems of multistage nature. Indeed, to capture the dynamics that are essential to stochastic decision processes in response to an increasing level of information, in 2016 Rockafellar and Wets [59] introduced a multistage extension of a stochastic variational problem. This formulation provides innovative and flexible tools

(i) to study real-life problems complicated by *time*, *uncertainty* and *risk*;

(ii) to capture the role of *information* in the recursive decision processes;

(iii) to efficiently find the solution of *large scale problems* by means of parallel algorithms.

Supported by these tools of variational inequalities theory, both in a deterministic and stochastic framework, my contribution in this Ph.D. thesis has been to study and to weak some mathematical aspects, linked to the Radner equilibrium problem, to be as close as possible to the mechanisms governing real-world problems. In particular, the questions we pose and at which I want to answer are:

1. *What happens if the preferences of the agents cannot be represented by a utility function?*

   We remind that if the agents' preferences can be represented by means of a utility function, then the preference maximization is equivalent to an optimization problem of a real values function and, it is well known that it can be studied by means of a variational inequality problem. Indeed, maximizing a concave differentiable function f on a closed convex set is equivalent to solving a variational inequality problem where the operator is the gradient of f. If the function is not differentiable the gradient can be replaced by the supergradient. In the setting of quasiconcave functions, necessary and sufficient conditions include the normal cone to the superlevel set. However, in real-world problems, the considered assumptions are not sufficient to guarantee the existence of a utility function representing the preferences. So, we can not apply the results known in the literature. Motivated by this fact, we overcame this occurrence by introducing an opportune operator which involves the strictly upper counter set and the normal cone associated with it. This formulation allows us to study the preference maximization problem by means of a variational problem without representing the preferences by a utility function. In force of theoretical results obtained, in terms of existence and regularity of the solution map, we apply this formulation to study a Radner equilibrium problem by means of a quasi-variational problem where the agents' preferences can be not representable by a utility function.

2. *How to set the Radner equilibrium problem into a scenario framework to explicitly study how the increasing level of information influences the decision process of agents during the evolution of the market? and, how to compute efficiently the solution of these large scale problems complicated by time and uncertainty?*

   In force of Rockafellar and Wets approach, we rewrite the uncertainty quantity of the equilibrium model no more as vectors but as functions. This led to the introduction of an opportune functional setting relative to a finite set of final possible states and certain information fields. The key concept of this approach is the presence of nonanticipativity constraints on the variables of the problem. Variables are not based on the information not yet known, but they are related to the information field up to the considered time. In addition, nonanticipativity constraints provide a powerful tool in both theoretical and computational aspects as they can be dualized by multipliers, providing a tool for a point-wise decomposition of the original stochastic variational problem. The latter means that nonanticipativity formulation enables the decomposition of the original stochastic variational problem into a separate sub-problem for each scenario. In this way, we can provide a procedure to compute the equilibrium solution using the Progressive Hedging Algorithm, recently update [60] in this field of variational analysis. This procedure works in parallel and, so, it allows us to efficiently compute the solution of large scale problems.

3. *Can be given a specific economic application in which to take advantage of this stochastic variational approach, eventually complicated by future uncertain occurrences which vary with continuity?*

   To this end, we study a deregulated electricity market introduced in [15]. Indeed, from 1996, in many countries, the electric power industry has undergone a transformation from a government-regulated to a competitive regime, motivating the grown interest in the development of electricity market models. The main features are:

• the central role of information;

• the possibility of highly uncertain spot prices that causes volatility of profits and costs;

• the presence of tools to hedge against these risks, in terms of future markets, that is, forward contracts and options;

• a competitive regime in energy procurement.

We focus on the decision-making framework of large consumers, such as petrochemical industries, aluminum production complex or vehicle-assembling facilities, and set it in an *economy with multiple trading dates and a continuum of states* in order to be as much as close to the realistic case. Indeed, most real-world phenomena, such as interest rate, inflation, natural events, wind speed, etc, vary with continuity and affect the agents' decision process. We capture these dynamics by introducing in the formulation of the problem the *filtration*, a particular σ-algebra, relatively to which we constraint all the variables. Also in this case, a central role is played by the nonanticipativity constraints. The strength point of this approach is that, after a suitable discretization, we could use the parallel procedure introduced to efficiently perform qualitative studies.

Summarizing, the thesis is organized as follows. Chapter 1 and Chapter 2 are devoted to recalling the main literature of support. In particular, Chapter 1 focus on the introduction of the major historical developments of the general economic equilibrium theory while in Chapter 2 we deal with the presentation of the variational inequalities formulations, both a in deterministic and stochastic setting, with the related problems, links and the main existence results available. On this basis, Chapter 3, Chapter 4, and Chapter 5 are structured, respectively, to give an exhaustive and organic answer to each questions posed before. They represent the heart of this elaborate and my contribution in this Ph.d. Finally, in Chapter 6 possible future developments are proposed, in continuity with the research presented. At the end, in order to make the thesis self-contained, it is provided an Appendix on basic concepts of the set-valued maps, the generalized monotonicity, and the probability theory.

I would like to thanks Professor Monica Milasi for introducing me to this research field and for her constant and patient support, “at each stage ”of these years, which has allowed me to grow from an academic point of view.

Domenico Scopelliti

Chapter 1

Economic Equilibrium Problems

The aim of this Chapter is to introduce the main historical developments of the general economic equilibrium theory.
It was Leon Waras [66] who, in 1874, laid the fundamental ideas for the study of the general equilibrium theory, providing a succession of models, each taking into account more aspects of the real economy. It was in order to link his equilibrium model to the real world that Walras's writings developed the *tâtonnement* process, which is a way to model the *law of supply and demand*. This law is a central condition in economic theory and it states that the price of a commodity will increase when the demand for that commodity exceeds supply and that the price will decrease if supply exceeds demand. From a mathematical point of view, Walras "solved" it by writing a system of n equations in n unknowns, which he calls the *equations of exchange*, and a solution to this system is an equilibrium for this exchange economy. However, he does not prove the existence of solutions and so, for more than half a century, the equality of the number of equations and of the unknowns of his system remained the only remark made in favor of the existence of a competitive equilibrium.
The first rigorous results for the equilibrium existence, for a model of production and a model of exchange, are due to Wald [65]. Thereafter, mostly in the 1950s and 1960s, McKenzie [48], Arrow and Debreu [5], Gale [29], and other authors obtained several equilibrium existence results which are more general and with simpler proof than Wald's. In particular, Arrow and Debreu give a rigorous existence proof based on Nash's 1950 result on the existence of equilibria in N-person games that, in turn, based on Kakutani's 1941 Fixed Point theorem. Moreover, in the same years, firstly Arrow [4] and after Debreu [18], enabled the theory of general economic equilibrium to be reinterpreted to cover the case of uncertainty about the availability of resources and about consumption and production possibilities. Indeed, Debreu [19] presented, in 1959, a unified treatment of time and uncertainty by introducing an economic equilibrium model that evolves in a sequence

of markets under uncertainty on future conditions. The uncertainty is formalized by means of *states of the world* and the key idea is the *contingency commodity*: a commodity whose delivery is conditioned on the realized state of the world. In this way, the Debreu model can be seen as a Walrasian equilibrium in which contingent commodities are traded.
If from one side Debreu setting provides a remarkable illustration of the power of general equilibrium theory, from the other side it is hardly realistic since it assumes that all trades take place simultaneously and before the uncertainty is revealed. In reality, trade is not a "one-shot affair", but it takes place to a large extent sequentially over time and frequently as a consequence of information disclosures. These dynamics were captured by Radner [51] who, in 1972, generalized the Debreu equilibrium model: he introduced the possibility of agents to transfer wealth among all possible future times and to trade at each possible contingency after the uncertainty is revealed and the market reopens. Two different market structures, forward and spot markets, are so considered. This results in a significant reduction in the number of ex-ante markets that must operate. By staring from this model, subsequent studies are been performed by other scholars, such as Cass, McKenzie, Duffie, Hart, Kreps, Sharf, Geanakopolos, Grossman, etc, mostly in the 1970s and 1980s. They give rigorous existence results in terms of Fixed Point arguments.
The Chapter is organized as follows. Section 1.1 is devoted to the introduction of a Walrasian equilibrium problem for a pure exchange market economy. On this basis, the connection with a Debreu model, which involves time and uncertainty, is considered in Section 1.2. Finally, in Section 1.3 a Radner equilibrium model of plans, prices, and price expectations is described: this is the equilibrium problem on which all the thesis is focus on.

1.1 Walrasian Equilibrium Problem

We introduce a marketplace consisting of H different commodities and I agents and we denote by $\mathcal{H} := \{1 \ldots, h \ldots, H\}$ and $\mathcal{I} := \{1, \ldots, i, \ldots, I\}$, respectively, the sets of commodities and agents. The notation just introduced will hold also for all the other models that will be introduced in the Chapter.
In the economy, only consumption and pure exchanges are assumed: the only activities of each agent are to trade his own commodities with each other agent and to consume. Each agent i is equipped with a nonnegative commodity vector e_i such that

$$e_i := (e_i^1, \ldots, e_i^h, \ldots, e_i^H) \in \mathbb{R}_+^H$$

where e_i^h represents her endowment of the commodity h, that is, the amount of commodity h that she can consume or trade with other agents. The consumption plan of the agent i is denoted by the nonnegative vector x_i such that

$$x_i := (x_i^1, \ldots, x_i^h, \ldots, x_i^H) \in \mathbb{R}_+^H$$

where x_i^h represents the quantity of commodity h consumed by i. At each commodity $h \in \mathcal{H}$ is associated a nonnegative price p^h and the price vector is denoted by

$$p := (p^1, \ldots, p^h, \ldots, p^H) \in \mathbb{R}_+^H \setminus \{0_H\}.$$

The price p is equal for all agents.
Agent's preferences for consuming different commodities are represented by a utility function

$$u_i : \mathbb{R}_+^H \to \mathbb{R}.$$

The aim of each agent i is to maximize her utility by performing pure exchange of the given commodities under the natural budget constraints at the current price p: the value of the consumption plan of agent i at current price p, $\langle p, x_i \rangle_H$, cannot exceed her wealth $\langle p, e_i \rangle_H$, that is, the value of her endowment. So, denote by $M_i(p)$ the set of the consumption vector available to agent i at current price p:

$$M_i(p) := \{x_i \in \mathbb{R}_+^H : \langle p, x_i \rangle_H \leq \langle p, e_i \rangle_H\}.$$

We denote by $\mathcal{E} := (u_i, e_i)_{i \in \mathcal{I}}$ the economy and we can formalize the equilibrium conditions.

Definition 1 (See [66]). *A Walasian equilibrium for the economy $\mathcal{E}$ is a vector $(\bar{x}, \bar{p}) \in \prod_{i \in \mathcal{I}} M_i(\bar{p}) \times \mathbb{R}_+^H \setminus \{0_H\}$ such that*

(i) for any $i \in \mathcal{I}$:

$$\max \ u_i(x_i) = u_i(\overline{x}_i)$$

$$s.t. \quad x_i \in M_i(\overline{p})$$

(ii) for any $h \in \mathcal{H}$:

$$\sum_{i \in \mathcal{I}} \bar{x}_i^h \leq \sum_{i \in \mathcal{I}} e_i^h \qquad \text{and} \qquad \langle \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i), \bar{p} \rangle_H = 0$$

In Definition 1, we have introduced a *free-disposal* equilibrium. It relies on the fact that the prices are assumed to be nonnegative. At the equilibrium, if the total supply of some commodity $h \in \mathcal{H}$ in the market exceeds its total demand, then the corresponding price $\bar{p}^h$ is zero. This means that it is allowed the excess supply of some commodities provided that they are free.

Prices Normalization

It is possible to manipulate in opportune manner prices and still have the equilibrium. Indeed, without loss of generality, one can perform such operations to limit the variability of prices, providing several mathematical advantages. This possibility of normalizing prices derives, as we are going to see, from the structure of the economic market in question.

1. *Nonnegative prices*: $p \in \mathbb{R}^H_+$. We observe that if we multiply prices by a positive constant still have an equilibrium. Then, without loss of generality, we can consider the equilibrium problem where the prices are in the simplex set:
$$\Delta := \{p \in \mathbb{R}^H_+ : \sum_{h\in\mathcal{H}} p^h = 1\} .$$
Indeed:
   - If $(\overline{x}, \overline{p})$, with $\overline{p} \in \Delta$, is an equilibrium, then $(\overline{x}, \widetilde{p})$, where $\widetilde{p} \in \mathbb{R}^H_+$, is an equilibrium, with $\widetilde{p} = \alpha\overline{p}$ for all $\alpha > 0$.
   Since
   $$\langle \widetilde{p}, x - e\rangle_H = \langle \alpha\overline{p}, x - e\rangle_H = \alpha\langle \overline{p}, x - e\rangle_H$$
   for all $\alpha > 0$, one has
   $$\langle \widetilde{p}, x - e\rangle_H \leq 0 \qquad \Leftrightarrow \qquad \langle \overline{p}, x - e\rangle_H \leq 0 .$$
   Hence, $(\overline{x}, \overline{p})$ is an equilibrium if and only if $(\overline{x}, \widetilde{p})$ is an equilibrium.
   - If $(\overline{x}, \widetilde{p})$, with $\widetilde{p} \in \mathbb{R}^H_+$, is an equilibrium, then there exist $\alpha > 0$ and $\overline{p} \in \Delta$ such that $\widetilde{p} := \alpha\overline{p}$ and $(\overline{x}, \overline{p})$ is an equilibrium.
   One can pose
   $$\widetilde{p} = \widetilde{p}\,\frac{\sum_{h\in\mathcal{H}} \widetilde{p}^h}{\sum_{h\in\mathcal{H}} \widetilde{p}^h} = \sum_{h\in\mathcal{H}} \widetilde{p}^h\Big(\frac{\widetilde{p}}{\sum_{h\in\mathcal{H}} \widetilde{p}^h}\Big) = \alpha\overline{p}$$
   where
   $$\alpha = \sum_{h\in\mathcal{H}} \widetilde{p}^h > 0 \quad \textit{and} \quad \overline{p} = \frac{\widetilde{p}}{\sum_{h\in\mathcal{H}} \widetilde{p}^h} \in \Delta .$$
   so that, since $M_i(\overline{p}) = M_i(\widetilde{p})$, it follows that $(\overline{x}, \overline{p})$ is an equilibrium.

   Thanks to this fact, one can consider the prices in the simplex set.

2. *positive prices*: $p \in \mathbb{R}^H_{++}$. One can normalize the prices by considering the price of one commodity, for instance commodity 1, to be 1. In this way, one can select $\alpha = \frac{1}{p^1}$ and multiply all prices by this constant. The normalized prices of the first commodity, often called *numerarie*, become $p^1 \frac{1}{p^1} = 1$, while all the others are set in terms of the first one.
   The main problem with the numerarie price normalization is that it treats commodities asymmetrically, which may not be very satisfactory from a mathematical perspective.

As already quoted, Arrow and Debreu give a rigorous existence proof, by means of fixed point theory, of the Walrasian equilibrium problem by requiring, for all $i \in \mathcal{I}$, the following assumptions.

Assumptions A

(A.1) u_i is continuous and semistrictly quasiconcave;

(A.2) u_i is non-satiated: $\forall x_i \in X_i \; \exists \tilde{x}_i \in X_i$ s.t. $u_i(\tilde{x}_i) > u_i(x_i)$;

(A.3) *survivability* : $e_i \in \mathbb{R}^H_{++}$.

From an economic point of view, assumption (A.3) ensures that each agent is endowed with each commodity $h \in \mathcal{H}$.

Theorem 1 (See [5], Th.1.1.5). *For each $i \in \mathcal{I}$, under assumptions A, there exists a equilibrium vector for economy $\mathcal{E}$.*

1.2 Debreu Equilibrium Problem

In this Section, we consider a *market economy under uncertainty* introduced in Ch.7 of [20]. In such an economy with uncertainty, commodities are to be distinguished not only by their physical characteristics and the location and dates of their availability and/or use, but also by the state of the world in which they are made available and/or used. This requires the introduction of a suitable setting in which to operate.

Time and Uncertain structure

Let us suppose that the market starts at time $t = 0$ and evolves in a finite sequence of T future dates. The sets $\mathcal{T} := \{1, \ldots, T\}$ and $\mathcal{T}_0 := \{0\} \bigcup \mathcal{T}$ denote the sets of time periods, respectively, without and with the initial date. At each time $t \in \mathcal{T}$ one or more than one situations are possible; at the final time T, S states of the world are possible. We can give a graphical representation of the evolution of the market through an *oriented graph* $\mathcal{G}$, consisting by a set of vertices $\Xi := \Xi_1 \cup \ldots \cup \Xi_T$ and $\Xi_0 := \{\xi_0\} \cup \Xi$, with $|\Xi_t| = k_t$ and $|\Xi_0| = N$, such that

- ξ_0 is the root vertex: it represents the initial situation and it is the unique vertex without immediate predecessor.
- For all $t \in \mathcal{T}$, the set $\Xi_t := \left\{\xi_t^1, \ldots, \xi_t^{k_t}\right\}$ is a finite set of vertices and represents all possible situations at time t. Each ξ_t^j has a unique immediate predecessor in Ξ_{t-1}.
- Ξ_T are the terminal nodes of the graph.

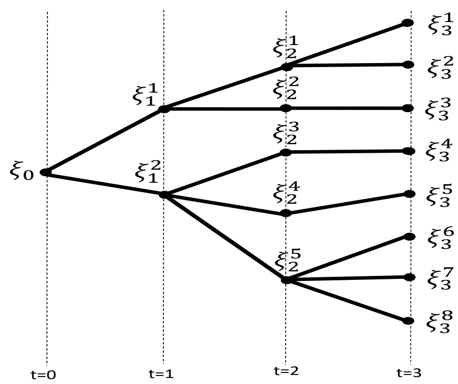


Figure 1.1: Example: oriented graph $\mathcal{G}$

Each node ξ_t^j of the graph represents a *contingency* of the market structure, that is, it identifies time and information. This leads to introducing the notion of a vector that describes all the characteristics.

Definition 2. *For every physical commodity $h \in \mathcal{H}$ and contingency $\xi_t^j \in \Xi_0$, a unit of state-contingent commodity is a title to receive a unit of the physical commodity h if and only if ξ_t^j occurs. Accordingly, a state-contingent commodity vector is specified by*

$$y := (y_0, \ldots, y_t, \ldots, y_T) \in \mathbb{R}^{G_0},$$

with $y_t := (y(\xi_t^0), \ldots, y(\xi_t^j), \ldots, y(\xi_t^{k_t})) = (y(\xi))_{\xi \in \Xi_t} \in \mathbb{R}^{G_t}$; $y(\xi_t^j) := (y^h(\xi_t^j))_{h \in \mathcal{H}} \in \mathbb{R}^H$. *If the component at contingency ξ_0 is not included one has that $y \in \mathbb{R}^G$.*

Economy under time and uncertainty

Now, in this structure of time and uncertainty, we can opportunely set a market economy. Every agent $i \in \mathcal{I}$ at every point in time has access to the same

information about the state of nature as has any other agent at that time. The market opens only once, at date 0, *before the beginning* of the physical history of the economic system. In this way, all market activities happen ex-ante, that is, all accounts are settled before the history of the economy begins, and there is no incentive to revise consumption plans, "reopen the market", or trade.
We set all elements introduced in Section 1.1 as state-contingency vectors. According to Definition 2:

$$x_i := (x_{i0}, x_{i1}, \dots, x_{it}, \dots, x_{iT}) \in \mathbb{R}^{G_0}_+ \ , \quad e_i := (e_{i0}, e_{i1}, \dots, e_{it}, \dots, e_{iT}) \in \mathbb{R}^{G_0}_+$$

$$p := (p_0, p_1, \dots, p_t, \dots, p_T) \in \mathbb{R}^{G_0}_+ \setminus \{0_{G_0}\} \ . \tag{1.2.1}$$

Agent's preferences for consuming different commodities are represented by a utility function

$$u_i : \mathbb{R}^{HN}_+ \to \mathbb{R}.$$

Moreover, the budget constraints set, at the current price p, must be considered at each contingency $\xi^j_t \in \Xi_0$; then it becomes:

$$M_i\,(p) := \{x_i \in \mathbb{R}^{HN}_+ : \langle p(\xi^j_t), x_i(\xi^j_t)\rangle_H \leq \langle p(\xi^j_t), e_i(\xi^j_t)\rangle_H \quad \forall \xi^j_t \in \Xi_t, t \in \mathcal{T}_0\}.$$

That is, for each contingency $\xi^j_t \in \Xi_0$, the value of the consumption plan of consumer i at current price $p(\xi^j_t)$, $\langle p(\xi^j_t), x_i(\xi^j_t)\rangle_H$, cannot exceed his wealth $\langle p(\xi^j_t), e_i(\xi^j_t)\rangle_H$. We denote by $\mathcal{E} := (\mathcal{G}, (u_i, e_i)_{i\in\mathcal{I}})$ the economy and we can formalize the equilibrium conditions.

Definition 3 (See [20], Ch.7)**.** *An Arrow-Debreu equilibrium for the economy $\mathcal{E}$ is a vector $(\bar{x}, \bar{p}) \in \prod_{i\in\mathcal{I}} M_i(\bar{p}) \times \mathbb{R}^{HN}_+ \setminus \{0_{HN}\}$ such that*

(i) for any $i \in \mathcal{I}$:

$$\begin{aligned} \max\ & u_i(x_i) = u_i(\overline{x}_i) \\ s.t.\ & \ x_i \in M_i(\overline{p}) \end{aligned} \tag{1.2.2}$$

(ii) for all $t \in \mathcal{T}_0$:

$$\sum_{i\in\mathcal{I}} \bar{x}_i(\xi^j_t) \leq \sum_{i\in\mathcal{I}} e_i(\xi^j_t) \quad \textit{and} \quad \langle \sum_{i\in\mathcal{I}} (\bar{x}_i(\xi^j_t) - e_i(\xi^j_t)), \bar{p}(\xi^j_t)\rangle_H = 0 \quad \forall \xi^j_t \in \Xi_t. \tag{1.2.3}$$

Prices Normalization

As in the previous section, since for each $\xi^j_t \in \Xi_0$ the budget constraint is homogenous of degree zero with respect to the prices, without loss of generality, we can limit the variability of the prices. So, with similar arguments of the previous Section, we can consider the following simplex set:

$$\Delta := \prod_{\xi_t^j \in \Xi_0} \Delta_{\xi_t^j} = \prod_{\xi_t^j \in \Xi_0} \{p(\xi_t^j) \in \mathbb{R}_+^H : \sum_{h \in \mathcal{H}} p^h(\xi_t^j) = 1\}.$$

Furthermore, since the Debreu model substantially is a Walrasian equilibrium model in which contingent commodities are traded, at t=0, then it results only in a growth of the dimensionality of the problem. Hence, the existence of a Debreu equilibrium vector for the economy $\mathcal{E}$ is still guaranteed by Theorem 1.

1.3 Radner Equilibrium Problem

In this Section, we present a model of exchange and consumption under uncertainty, introduced in [42] by Radner. As already stated, Radner generalized the Debreu equilibrium model to make the market institutions more realistic. Indeed, one of the major criticism posed by Radner to the Debreu model is that the latter requires that the economic agents possess capabilities of imagination and calculation that exceed reality by many orders of magnitude. One can imagine situations in which consumers may not be able to decide what commodities to consume many periods from now.
By using the same notation of the previous Section, as far as times, contingencies, agents, endowments and commodities and so on are concerned, now we introduce an economy that is characterized by two market structures: spot and forward markets.

Spot market: it *opens at each contingency* $\xi_t^j \in \Xi_0$ and agents consume or trade a certain amount of commodities $x_i(\xi_t^j) \in \mathbb{R}_+^H$ at prices $p(\xi_t^j)$; grouping in vectors, one has $x_i \in \mathbb{R}_+^{HN}$ and $p \in \mathbb{R}_+^{HN}$. Even if the vectors x_i and p are defined as (1.2.1), the main difference is from an economic point of view: each component vector $x_{it} := (x_i(\xi_t^0), \dots, x_i(\xi_t^{k_t})) = (x_i(\xi))_{\xi \in \Xi_t} \in \mathbb{R}_+^{Hk_t}$ represents the decisions that must be *made at time t* and no ex-ante as in Debreu model.

Forward market: at $t = 0$ a *further market opens* and offers participants the opportunity to reduce their exposure to future risks and randomness without, however, removing the incentive to trade and consume in the spot market that opens at each time period after the uncertainty is revealed. Indeed, at time $t = 0$, agents sign these forward contracts which allow them to transfer wealth in terms of commodity-1 among all future contingencies for immediate cash that will be used for spot consumption or for future contracts in other contingencies: at each subsequent time $t \in \mathcal{T}$ when the uncertainty is resolved, the contingency is reveled and contracts are executed. For each $i \in \mathcal{I}$, we set the forward contracts and the relative prices vectors as follows

$$z_i := (z_{i1}, \ldots, z_{it}, \ldots, z_{iT}) \in \mathbb{R}^{N-1}, \qquad q := (q_1, \ldots, q_t, \ldots, q_T) \in \mathbb{R}_+^{N-1} \quad (1.3.1)$$

so that $z_{it} := (z_i(\xi_t^1), \ldots, z_i(\xi_t^{k_t})) = (z_i(\xi))_{\xi \in \Xi_t} \in \mathbb{R}^{k_t}$ and $q_t := (q(\xi_t^1), \ldots, q(\xi_t^{k_t})) = (q(\xi))_{\xi \in \Xi_t} \in \mathbb{R}_+^{k_t}$, where $z_i(\xi_t^j)$ commodity-1 amount at ξ_t^j paid $q(\xi_t^j)$ at time 0. So, forward price $q(\xi_t^j)$ is not the price of commodity-1 at contingency ξ_t^j but it represents the price of the contract $z_i(\xi_t^j)$ signed at $t = 0$ relatively to the delivery or receipt of a certain amount of commodity-1 in the contingency ξ_t^j. Indeed, we observe that the components of z_i can be negative: if $z_i(\xi_t^j) < 0$, it is an amount to be delivered by agent i at ξ_t^j and $q(\xi_t^j)z_i(\xi_t^j)$ represents an *income* at ξ_0; while, if $z_i(\xi_t^j) > 0$, it is an amount to be received by agent i at ξ_t^j and $q(\xi_t^j)z_i(\xi_t^j)$ represents an *outcome* at ξ_0. Furthermore, it is allowed the possibility of *short selling*: $z_i(\xi_t^j) < -e_i^1(\xi_t^j)$, namely at $t = 0$ consumer i can sell commodity-1, available in the market, even if he does not own it.
Each consumer i has a preference on the *spot consumption* which is expressed by means of a utility function

$$\mathcal{U}_i : \mathbb{R}_+^{HN} \to \mathbb{R}.$$

In this model, the budget constraints set at the current price system (p, q) becomes:

$$\begin{aligned} M_i(p,q) :=\{ &(x_i, z_i) \in \mathbb{R}_+^{HN} \times \mathbb{R}^{N-1} : \\ &\langle p(\xi_0), x_i(\xi_0)\rangle_H + \langle q, z_i\rangle_{N-1} \le \langle p(\xi_0), e_i(\xi_0)\rangle_H \\ &\langle p(\xi_t^j), x_i(\xi_t^j)\rangle_H \le \langle p(\xi_t^j), e_i(\xi_t^j)\rangle_H + p^1(\xi_t^j) z_i(\xi_t^j) \quad \forall \xi_t^j \in \Xi_t, t \in \mathcal{T}\}. \end{aligned}$$

The first inequality represents the budget constraint at time 0 while, the second inequality represents the expected budget constraints at each contingency ξ_t^j, with $t \in \mathcal{T}$. In particular, one can observe that forward contracts are signed -sold or bought- at time $t = 0$ at current price q; $\langle q, z_i\rangle_{N-1}$ represents the value of these contracts. Furthermore, when each contingency is reveled and contracts are executed, the relative value will be $p^1(\xi_t^j)z_i(\xi_t^j)$, at current spot price $p^1(\xi_t^j)$. Furthermore, market-clearing conditions have to be satisfied: at each contingency ξ_t^j, the total spot consumption must not exceed the total endowment while the total forward contracts have to be zero. Indeed, since forward contracts are signed between the agents participating in the market and no one of them has an initial endowment of these contracts, then for each buyer there must be a seller. This means that, for each possible contingency, the total quantity bought must be equal to the total quantity sold.
We denote by $\mathcal{E}$ the economy $\mathcal{E} := \Big(\mathcal{G}, (\mathcal{U}_i, e_i)_{i\in\mathcal{I}}\Big)$ and we can, now, formalize the equilibrium conditions.

Definition 4. *[See [42], Def.16.3] An* equilibrium *of plans, prices, and price expectations for the economy $\mathcal{E}$ is a vector* $\left((\bar{x}_i, \bar{z}_i)_{i\in\mathcal{I}}, \bar{p}, \bar{q}\right) \in \prod_{i\in\mathcal{I}} M_i(\bar{p}, \bar{q}) \times \mathbb{R}_+^{HN} \times \mathbb{R}_+^{N-1}$*, such that*

(i) for any $i \in \mathcal{I}$:

$$\begin{aligned} &\max\ \mathcal{U}_i(x_i) = \mathcal{U}_i(\overline{x}_i) \\ &s.t.\ \ (x_i, z_i) \in M_i(\bar{p}, \bar{q}) \end{aligned} \tag{1.3.2}$$

(ii) for all $t \in \mathcal{T}_0$:

$$\sum_{i\in\mathcal{I}} \bar{x}_i(\xi_t^j) \le \sum_{i\in\mathcal{I}} e_i(\xi_t^j) \qquad \forall \xi_t^j \in \Xi_t; \tag{1.3.3}$$

(iii) for all $t \in \mathcal{T}$:

$$\sum_{i\in\mathcal{I}} \bar{z}_i(\xi_t^j) = 0 \qquad \forall \xi_t^j \in \Xi_t. \tag{1.3.4}$$

For all $i \in \mathcal{I}$, the following assumptions are introduced.

Assumptions B

(B.1) $\mathcal{U}_i$ is continuous and concave;

(B.2) $\mathcal{U}_i$ is non-satiated;

(B.3) *survivability* : $e_i \in \mathbb{R}_{++}^{HN}$;

(B.4) $\mathcal{U}_i$ is strictly increasing in the commodity-1: $\forall \tilde{x}_i, \tilde{\tilde{x}}_i \in \mathbb{R}_+^{HN}$ with $\tilde{x}_i \ge \tilde{\tilde{x}}_i$, then

$$\tilde{x}_i^1(\xi_t^j) > \tilde{\tilde{x}}_i^1(\xi_t^j) \quad \forall \xi_t^j \in \Xi_0 \quad \Rightarrow \quad \mathcal{U}_i(\tilde{x}_i) > \mathcal{U}_i(\tilde{\tilde{x}}_i)$$

From an economic point of view, assumption (B.3) ensures that each agent is endowed with each commodity $h \in \mathcal{H}$ in each contingency $\xi_t^j \in \Xi_0$. In Chapter 3 and Chapter 4, survival assumption will be always implicitly assumed, for brevity, in the set-up of the models.

Proposition 1. *For each $i \in \mathcal{I}$, let $\mathcal{U}_i$ be strictly increasing in commodity-1 and $(\bar{x}_i, \bar{z}_i)$ be maximal for $\mathcal{U}_i$ in $M_i(\bar{p}, \bar{q})$. Then, $\bar{p}^1(\xi_t^j) > 0$ for all $\xi_t^j \in \Xi_0$ and $\bar{q}(\xi_t^j) > 0$ for all $\xi_t^j \in \Xi$.*

Proof. We suppose that there exists $\xi_{t^*}^{j^*} \in \Xi_0$ such that $\bar{p}^1(\xi_{t^*}^{j^*}) = 0$. We pose $\tilde{x}_i$ such that $\tilde{x}_i(\xi_t^j) = \overline{x}_i(\xi_t^j)$ for all $\xi_t^j \neq \xi_{t^*}^{j^*}$ and

$$\tilde{x}_i(\xi_{t^*}^{j^*}) = \begin{cases} \bar{x}_i^1(\xi_{t^*}^{j^*}) + K \\ \bar{x}_i^h(\xi_{t^*}^{j^*}) \quad \forall h \neq 1 \end{cases}$$

with $K > 0$. It results $(\tilde{x}_i, \bar{z}_i) \in M_i(\bar{p}, \bar{q})$ and, since $\mathcal{U}_i$ is strictly increasing in commodity-1, one has $\mathcal{U}_i(\tilde{x}_i) > \mathcal{U}_i(\bar{x}_i)$ which contradicts the assumption.
The proof of $\bar{q}(\xi_t^j) > 0$, $\xi_t^j \in \Xi$, is closed to the latter. $\square$

Prices Normalization

Thanks to Proposition 1, without loss of generality, we can consider the prices in the simplex-set:

$$\begin{aligned} \Delta_{\xi_0} &:= \left\{ (p(\xi_0), q) \in \mathbb{R}_+^{HN} \times \mathbb{R}_+^{N-1} : \sum_{h \in \mathcal{H}} p^h(\xi_0) + \sum_{\xi_t^j \in \Xi} q(\xi_t^j) = 1 \right\}; \\ \Delta_{\xi_t^j} &:= \left\{ p(\xi_t^j) \in \mathbb{R}_+^H : \sum_{h \in \mathcal{H}} p^h(\xi_t^j) = 1 \right\} \qquad \forall \xi_t^j \in \Xi. \end{aligned} \tag{1.3.5}$$

so that we pose $\Delta := \Delta_{\xi_0} \times \prod_{\xi_t^j \in \Xi} \Delta_{\xi_t^j}$. Indeed:

- If $(\overline{x}, \overline{z}, \overline{p}, \overline{q})$, with $(\overline{p}, \overline{q}) \in \Delta$, is an equilibrium, then $(\overline{x}, \overline{z}, \widetilde{p}, \widetilde{q})$, where $(\widetilde{p}, \widetilde{q}) \in \mathbb{R}_+^{HN} \times \mathbb{R}_+^{N-1}$, is an equilibrium, with $(\widetilde{p}(\xi_0), \widetilde{q}) = (\alpha_{\xi_0}\overline{p}(\xi_0), \alpha_{\xi_0}\overline{q})$, $\widetilde{p}(\xi_t^j) = \alpha_{\xi_t^j}\overline{p}(\xi_t^j)$ for all $\xi_t^j \in \Xi_0$ and $\alpha_{\xi_t^j} > 0$. One can see that

$$\begin{aligned} &\langle \widetilde{p}(\xi_0), x_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \widetilde{q}, z_i \rangle_{N-1} = \langle \alpha_{\xi_0}\overline{p}(\xi_0), x_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \alpha_{\xi_0}\overline{q}, z_i \rangle_{N-1} = \\ &= \alpha_{\xi_0}(\langle \overline{p}(\xi_0), x_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \overline{q}, z_i \rangle_{N-1}) \end{aligned}$$

$$\begin{aligned} &\langle \widetilde{p}(\xi_t^j), x_i(\xi_t^j) - e_i(\xi_t^j) \rangle_H - \widetilde{p}^1(\xi_t^j) z_i(\xi_t^j) = \langle \alpha_{\xi_t^j}\overline{p}(\xi_t^j), x_i(\xi_t^j) - e_i(\xi_t^j) \rangle_H - \alpha_{\xi_t^j}\overline{p}^1(\xi_t^j) z_i(\xi_t^j) = \\ &= \alpha_{\xi_t^j}(\langle \overline{p}(\xi_t^j), x_i(\xi_t^j) - e_i(\xi_t^j) \rangle_H - \overline{p}^1(\xi_t^j) z_i(\xi_t^j)) \qquad \forall \xi_t^j \in \Xi \end{aligned}$$

for all $\alpha_{\xi_t^j} > 0$, with $\xi_t^j \in \Xi_0$, so that one has

$$\begin{aligned} \langle \widetilde{p}(\xi_0), x_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \widetilde{q}, z_i \rangle_{N-1} \leq 0 \quad &\Leftrightarrow \quad \langle \overline{p}(\xi_0), x_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \overline{q}, z_i \rangle_{N-1} \leq 0 \\ \langle \widetilde{p}(\xi_t^j), x_i(\xi_t^j) - e_i(\xi_t^j) \rangle_H - \widetilde{p}^1(\xi_t^j) z_i(\xi_t^j) \leq 0 \quad &\Leftrightarrow \quad \langle \overline{p}(\xi_t^j), x_i(\xi_t^j) - e_i(\xi_t^j) \rangle_H - \overline{p}^1(\xi_t^j) z_i(\xi_t^j) \leq 0 \end{aligned}$$

Hence, $(\overline{x}, \overline{z}, \overline{p}, \overline{q})$ is an equilibrium if and only if $(\overline{x}, \overline{z}, \widetilde{p}, \widetilde{q})$ is an equilibrium.

• If $(\overline{x},\overline{z},\widetilde{p},\widetilde{q})$, with $(\widetilde{p},\widetilde{q}) \in \mathbb{R}^{HN}_{+} \times \mathbb{R}^{N-1}_{+}$, is an equilibrium, then there exists $\alpha_{\xi^j_t} > 0$, for all $\xi^j_t \in \Xi_0$, and $(\overline{p},\overline{q}) \in \Delta$ such that $(\widetilde{p}(\xi_0),\widetilde{q}) := \alpha_{\xi_0}(\overline{p}_0,\overline{q})$, $\widetilde{p}_s(\xi^j_t) := \alpha_{\xi^j_t}\overline{p}(\xi^j_t)$ for each $\xi^j_t \in \Xi_0$ and $(\overline{x},\overline{z},\overline{p},\overline{q})$ is an equilibrium. One can pose

$$\begin{aligned}\widetilde{p}(\xi_0) =& \widetilde{p}(\xi_0)\,\frac{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t)}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t)} = \\ =&(\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t))\Big(\frac{\widetilde{p}(\xi_0)}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t)}\Big) = \alpha_{\xi_0}\overline{p}(\xi_0)\end{aligned}$$

$$\begin{aligned}\widetilde{q} =& \widetilde{q}\,\frac{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t)}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t)} = \\ =&(\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t))\Big(\frac{\widetilde{q}}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t)}\Big) = \alpha_{\xi_0}\overline{q}\end{aligned}$$

where $\alpha_{\xi_0} = \sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t) > 0$ and

$$(\overline{p}(\xi_0),\overline{q}) = \left(\frac{\widetilde{p}(\xi_0)}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t)}, \frac{\widetilde{q}}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi_0)+\sum_{\xi^j_t\in\Xi}\widetilde{q}(\xi^j_t)}\right) \in \Delta_{\xi_0}\,.$$

In similar way, $\widetilde{p}(\xi^j_t) = \widetilde{p}(\xi^j_t)\,\frac{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi^j_t)}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi^j_t)} = \sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi^j_t)\Big(\frac{\widetilde{p}(\xi^j_t)}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi^j_t)}\Big) = \alpha_{\xi^j_t}\overline{p}$ where

$$\alpha_{\xi^j_t} = \sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi^j_t) > 0 \quad \text{and} \quad \overline{p}(\xi^j_t) = \frac{\widetilde{p}(\xi^j_t)}{\sum_{h\in\mathcal{H}}\widetilde{p}^h(\xi^j_t)} \in \Delta_{\xi^j_t}\,.$$

for each $\xi^j_t \in \Xi$. Moreover, since $M_i(\overline{p},\overline{q}) = M_i(\widetilde{p},\widetilde{q})$, it follows that $(\overline{x},\overline{z},\overline{p},\overline{q})$ is an equilibrium.
For all up to now see, we can consider the prices in the simplex set (1.3.5).

As already quoted, Radner gives an equilibrium existence proof by using fixed point arguments and by requiring, for all $i \in \mathcal{I}$, the following assumptions.

Theorem 2 (See [51]). *For each $i \in \mathcal{I}$, under assumptions B, there exists a equilibrium vector of plans, prices and price expectations for economy $\mathcal{E}$.*

Chapter 2

Variational Inequalities Problems

Variational inequality theory has its origins in the calculus of variations associated with the minimization of infinite-dimensional functionals. The starting point was in the early 1960s by Fichera [27] and Stampacchia [62] in connection with partial differential equations to study equilibrium problems arising from elasticity and plasticity theory and from mechanics. Thereafter, its power was soon recognized in relation to the study of several equilibrium problems in optimization theory. Indeed, variational inequalities are tools for formulating and qualitatively analyzing several equilibrium problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and tools for computational purposes throughout the usage of suitable algorithms. Furthermore, several well-known problems from mathematical programming, such as systems of nonlinear equations, optimization problems, complementary problems, and fixed point problems, can be written in terms of a variational problem. Nowadays, the variational theory unifies a large range of applications arising in economics, finance, physics, game theory, control theory, operations research, and several branches of engineering sciences. However, although some practical problems involve only deterministic data, in mostly of real-world applications there are many important examples where problem data contain some uncertainty and randomness. Consequently, to reflect and capture these aspects, stochastic variational inequality problems are been recently introduced and studied as a natural extension of deterministic variational inequality models. The Chapter is organized as follows. Section 2.1 is devoted to the introduction of basic concepts and formulation of variational inequality problems. On this basis, it will be quoted the main results available in literature which make an explicit connection of it with the principal problems that we will be analyzed and studied in this thesis. In Section 2.2, we recall the main existence results, under different assumptions, of the solution to a variational and quasi-variational problem. Finally, in Section 2.3 stochastic variational problems are introduced.

2.1 Variational Inequalities

We start by considering variational inequalities in a finite dimensional Euclidean space $\mathbb{R}^G$.

Definition 5 (Stampacchia [62])**.** *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, and convex set and $F : K \to \mathbb{R}^G$ be a function. A variational inequality, associated with F and K, denoted by $VI(F, K)$, consists in the the following problem:*

$$\textit{Find } \bar{x} \in K \ \textit{ such that } \ \langle F(\bar{x}), x - \bar{x} \rangle_G \geq 0 \qquad \forall x \in K \tag{2.1.1}$$

Geometrically, a vector $\bar{x} \in K$ is a solution to (2.1.1) if and only if $F(\bar{x})$ forms a nonobtuse angle with all the feasible vectors emanating from $\bar{x}$. Equivalently, $\bar{x} \in K$ is a solution to $VI(F, K)$ if and only if $-F(\bar{x}) \in N_K(\bar{x})$, where

$$N_K(\bar{x}) = \left\{ d \in \mathbb{R}^G : \langle d, x - \bar{x} \rangle_G \leq 0 \qquad \forall x \in K \right\}$$

is the normal cone to K at the point $\bar{x}$.

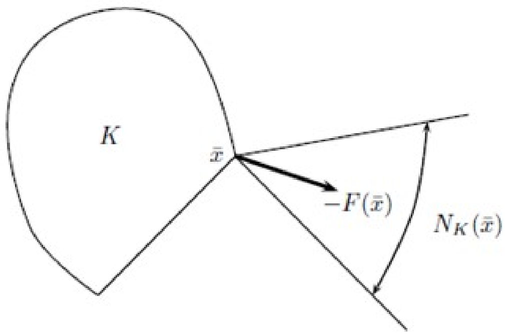


Figure 2.1: Geometric interpretation

In order to model complex phenomena, Definition 5 has been generalized[1] by Chan and Pang.

Definition 6 (Chan and Pang [12])**.** *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, and convex set. Let $\Phi : K \rightrightarrows \mathbb{R}^G$ and $S : K \rightrightarrows K$ be two set-valued maps. A generalized quasi-variational inequality, associated with F, S and K, denoted by $GQVI(\Phi, S, K)$, consists in the following problem:*

$$\textit{Find } \bar{x} \in S(\bar{x}) \ \textit{ such that } \ \exists \varphi \in \Phi(\bar{x}) \ \textit{ with } \ \langle \varphi, x - \bar{x} \rangle_G \geq 0 \qquad \forall x \in S(\bar{x}) \tag{2.1.2}$$

[1]See Appendix for concepts related to set-valued maps.

In particular, one has that

- when $S(x) = K$ for each $x \in K$, the problem (2.1.2) reduces to the *generalized Variational inequality* $GVI(\Phi, K)$:

$$\text{Find } \bar{x} \in K \text{ such that } \exists \varphi \in \Phi(\bar{x}) \text{ with } \langle \varphi, x - \bar{x} \rangle_G \geq 0 \quad \forall x \in K \ ;$$

- when Φ is single-valued, the problem (2.1.2) reduces to the *Quasi-Variational inequality* $QVI(F, S, K)$:

$$\text{Find } \bar{x} \in S(\bar{x}) \text{ such that } \langle F(\bar{x}), x - \bar{x} \rangle_G \geq 0 \quad \forall x \in S(\bar{x}) \ ;$$

- when both $\Phi(x)$ is a singleton and $S(x) = K$, for each $x \in K$, the problem (2.1.2) reduces to the *Variational inequality* $VI(F, K)$.

Related Problems

One of the strengths of variational inequalities is the flexibility and the adaptability with which they can be used to the study of several different mathematical problems.
Firstly, if we consider the following problem

$$\min_{x \in K} f(x) \tag{2.1.3}$$

then, we can link it with a specific variational problem under suitable assumptions on f.

Proposition 2 (See [40], Prop.5.1)**.** *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, convex set and $f : K \to \mathbb{R}$ be a continuously differentiable function. If $\bar{x}$ is a solution of the minimum problem (2.1.3), then $\bar{x}$ is a solution to $VI(\nabla f, K)$.*

We point out that Proposition 2 gives only a *necessary condition*, that is, it allows us to associate the constrained minimization problem (2.1.3) with a variational problem (2.1.1). In general, the converse does not hold. Indeed:

Example 1. *Let $f : [-1, 1] \to \mathbb{R}$ be a function such that for each $x \in [-1, 1]$ one has $f(x) = x^3$. If one consider ∇f evaluates at $x = 0$, then it follows that $\nabla f(0) = 0$ and thus $x = 0$ is a solution to (2.1.1). However, $x = 0$ is not a local minimum of f on $[-1, 1]$.*

The condition becomes *sufficient* when the function f is convex.
Now, let f be a convex function and $\partial f : K \rightrightarrows \mathbb{R}^G$ be the *subdifferential* map such that

$$\forall x \in \ K \qquad \partial f(x) = \left\{ h \in \mathbb{R}^G : f(y) - f(x) \geq \langle h, y - x \rangle_G \quad \forall y \in K \right\}$$

where $h \in \mathbb{R}^G$ is called *subgradient* of f at x.

Proposition 3 (See [3], Prop.8.3). *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, convex set and $f : K \to \mathbb{R}$ be a convex function. Then, $\bar{x}$ is a solution of the minimum problem (2.1.3) if and only if $\bar{x}$ is a solution to $GVI(\partial f, K)$.*

In this thesis, we will consider another broader class of functions for which *necessary and sufficient* optimality conditions can be obtained by replacing gradient or subgradient of the function by an opportune normal operator. We refer to the class of quasiconvex functions[2]. This class plays a central role in mathematical economics since it describes important features of the utility of agents.
For any $\alpha \in \mathbb{R}$, we denote by S_α and $S_\alpha^<$ the sublevel set and the strictly sublevel set, respectively, associated with f and α:

$$S_\alpha = \{y \in K : f(y) \leq \alpha\} \ , \quad S_\alpha^< = \{y \in K : f(y) < \alpha\} \ . \tag{2.1.4}$$

Definition 7 (See [2], Def.5.3). *Let $f : K \subseteq \mathbb{R}^G \to \mathbb{R}$ be any function. To any element $x \in$ dom f is associated the adjusted sublevel set $S_f^a(x)$ defined by*

$$S_f^a(x) := \begin{cases} S_{f(x)} \cap \overline{B}(S_{f(x)}^<, \rho_x) & if \ \ x \notin \operatorname{argmin}_K f \\ \\ S_{f(x)} & if \ \ x \in \operatorname{argmin}_K f. \end{cases}$$

where $\rho_x = \ dist(x, S_{f(x)}^<) = \inf_{y \in S_{f(x)}^<} \|x - y\|$ for any $x \in$ dom $f \setminus \operatorname{argmin}_K f$ and $\overline{B}(S_{f(x)}^<, \rho_x) = \left\{z \in K : \ dist(z, S_{f(x)}^<) \leq \rho_x\right\}$.

Proposition 4 (See [2], Prop.5.11). *A function f is quasiconvex if and only if the adjusted sublevel set $S_f^a(x)$ is nonempty convex, for each $x \in$ dom f.*

Let $N^a : K \rightrightarrows \mathbb{R}^G$ be the normal cone to S_f^a, then one introduces the set-valued map $G : \mathbb{R}^G \rightrightarrows \mathbb{R}^G$ such that, for all $x \in \mathbb{R}^G$,

$$G(x) := \begin{cases} \operatorname{conv}\left(N^a(x) \cap S(0,1)\right) & if \ \ x \notin \operatorname{argmin}_{\mathbb{R}^G} f \\ \\ \overline{B}(0,1) & if \ \ x \in \operatorname{argmin}_{\mathbb{R}^G} f. \end{cases}$$

where $\overline{B}(0,1) = \left\{x \in \mathbb{R}^G : \|x\| \leq 1\right\}$ and $S(0,1) = \left\{x \in \mathbb{R}^G : \|x\| = 1\right\}$.
If f is quasiconvex, since $S_f^a(x)$ is convex from Proposition 4, then N^a coincides with the normal cone to $S_f^a(x)$ at x, that is, for all $x \in$ *dom* f one has that $N^a(x) = N_{S_f^a}(x)$ is a convex cone.

Remark 1. *Let us denote by $N(x)$ and $N^<(x)$, respectively, the normal operator to $S_{f(x)}$ and $S_{f(x)}^<$. Since one has:*

$$S_{f(x)}^< \subseteq S_f^a(x) \subseteq S_{f(x)} \qquad \forall x \in dom \ f$$

[2]See Appendix for concepts related to quasiconvexity and quasimonotonicity.

it follows that

$$N(x) \subseteq N^a(x) \subseteq N^<(x) \qquad \forall x \in \mathit{dom}\ f$$

Moreover, if f is semistrictly quasiconvex, for all $x \in\ \mathit{dom}\ f \setminus \operatorname{argmin}_X f$, one has:

$$\overline{S}^<_{f(x)} = S^a_f(x) = S_{f(x)} \quad \Rightarrow \quad N(x) = N^a(x) = N^<(x)\ .$$

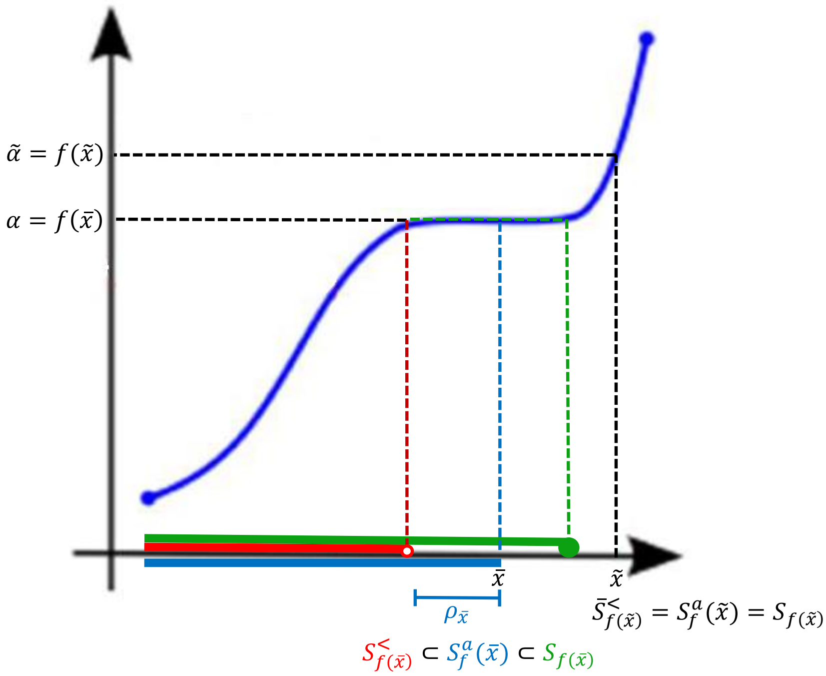


Figure 2.2: Example

Proposition 5 (See [7], Prop.4.1)**.** *Let $f : K \to \mathbb{R}$ be any function.*

(i) If f is quasiconvex, then G is non trivial on dom $f \setminus \operatorname{argmin}_{\mathbb{R}^G} f$, that is, $G(x)$ doesn't reduce to $\{0\}$.

(ii) If f is continuous, semistrictly quasiconvex and K is nonempty convex then $\bar{x} \in K$ is a solution to the previous $GVI(G, K)$ if and only if $\bar{x}$ is a solution to the optimization problem (2.1.3).

Remark 2. *Up to now, we focused on the connection between the minimization problem (2.1.3) and a suitable variational inequality problem, under different assumptions. We could do the same by considering the following constraint maximization problem:*

$$\max_{x \in K} f(x) = \min_{x \in K}(-f(x)) \tag{2.1.5}$$

In this case, in Proposition 2, Proposition 3 and Proposition 5 we should pose, respectively, $F = \nabla(-f)$, $F = \partial(-f)$ *and* $S^a_{(-f)}$.

Now, let us go back to problem (2.1.1). If $\bar{x}$ is a solution to $VI(F, K)$ and $\bar{x} \in intK$, then $F(\bar{x}) = 0$. Indeed, since $\bar{x}$ belong to the interior of K, there exists a $\tau > 0$ sufficiently small such that one can consider $x = \bar{x} - \tau F(\bar{x}) \in K$ and, from (2.1.1), one has:

$$\langle F(\bar{x}), -F(\bar{x})\rangle_G \geq 0 \quad \Rightarrow \quad -\|F(\bar{x})\|^2 \geq 0 \quad \Rightarrow \quad F(\bar{x}) = 0_G \ .$$

In force of this fact, one can link a variational inequality problem with the solution of *systems of nonlinear equations* when the function is defined on all the space $\mathbb{R}^G$. Furthermore, another important application arises in connection with fixed point theory. Let $K \subseteq \mathbb{R}^G$ be a nonempty set and $T : K \to K$ be a function. A *fixed point problem* is:

$$\text{Find } \ \bar{x} \in K \ \text{ such that } \ T(\bar{x}) = \bar{x}. \tag{2.1.6}$$

If one pose $F(x) = x - T(x)$ for each $x \in K$, then $VI(F, K)$ coincides with the fixed point problem (2.1.6).
The following result, based on fixed point arguments, gives the basis of many computational methods based on the projection operator.

Proposition 6 (See [40], Th.2.3)**.** *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, and convex set. An element $\bar{x} \in K$ is a solution to $VI(F, K)$ if and only if for any $\gamma > 0$, $\bar{x}$ is a fixed point of the mapping*

$$P_K(I - \gamma F) : K \to K$$

that is, $\bar{x} = P_K(\bar{x} - \gamma F(\bar{x}))$, where $P_K(\bar{x} - \gamma F(\bar{x}))$ denotes the projection of $(\bar{x} - \gamma F(\bar{x}))$ in K.

2.2 Existence Results

Now, we study under which assumptions a variational inequality problem admits solutions. We devote this Section to recall the main results in the literature on the existence for a variational and quasi-variational problem under different assumptions.

Variational Inequalities and Generalized Variational Inequalities

Theorem 3 (See [40], Th.2.3)**.** *Let $K \subseteq \mathbb{R}^G$ be a nonempty, compact, and convex set and $F : K \to \mathbb{R}^G$ be a continuous function. Then, the $VI(F, K)$ admits at least one solution.*

The next theorem gives a necessary and sufficient condition for the existence of a solution of a variational inequality. The study of variational inequalities over unbounded domains is usually based on *coercivity* conditions.

Theorem 4 (See [40], Th.4.2). *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, and convex set and $F : K \to \mathbb{R}^G$ be a continuous function. The $VI(F, K)$ admits at least one solution if and only if there exist $r > 0$ and a solution of the following problem:*

$$\textit{Find} \quad \bar{x}_r \in K_r := K \cap B(0,r) \quad \textit{such that} \quad \langle F(\bar{x}_r), x - \bar{x}_r \rangle_G \geq 0 \qquad \forall x \in K_r \, .$$

Corollary 1 (See [40], Cor.4.3). *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, and convex set and $F : K \to \mathbb{R}^G$ be a continuous function such that*

$$\frac{\langle F(x) - F(x_0), x - x_0 \rangle}{\|x - x_0\|} \to \infty \quad \textit{as} \quad \|x\| \to \infty, \quad x \in K$$

for some $x_0 \in K$. Then, the $VI(F, K)$ admits at least one solution.

Monotone functions and their generalizations play an important role in the theory of variational inequalities.

Theorem 5 (See [3], Prop.4.6). *Let $K \subseteq \mathbb{R}^G$ be a nonempty set and $F : K \to \mathbb{R}^G$ be strictly monotone. Then, the solution to $VI(F, K)$ is unique, if it exists.*

However, we observe that the strictly monotonicity does not ensure the existence of a solution to $VI(F, K)$. Indeed:

Example 2. *Let $K = \{x \in \mathbb{R} : x \geq 0\}$ and $F(x) = -e^{-x} - 1$. Since for each $x \in K$ one has $-e^{-x} - 1 < 0$, then it follows that*

$$(-e^{-\bar{x}} - 1)(x - \bar{x}) < 0 \qquad \forall x \in K \quad \textit{s.t.} \quad x > \bar{x}.$$

Hence, there is no $\bar{x} \in K$ such that $VI(F, K)$ holds.

Theorem 6 (See [3], Th.5.6). *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, and convex set and $F : K \to \mathbb{R}^G$ be a continuous function. If F is strongly monotone, then there exists a unique solution to $VI(F, K)$.*

When F is a set-valued map, the next result, due to Chang and Pang [12], gives the existence of a solution to $GVI(\Phi, K)$.

Theorem 7 (See [12], Cor.3.1). *Let $K \subseteq \mathbb{R}^G$ be a compact set and let $\Phi : K \rightrightarrows \mathbb{R}^G$ an upper semicontinuous set-valued map on K with compact and convex values. Then, the $GVI(\Phi, K)$ admits at least one solution.*

Quasi-Variational Inequalities and Generalized Quasi-Variational Inequalities

In contrast to the case of variational inequality problems, for which there exists rich literature dealing with the existence of a solution, there are only a few contributes for quasi-variational inequality problems. This is due to the fact that in a quasi-variational inequality problem the constraint set depends on the solution, making all more difficult, both from a theoretical and computational point of view.
Now, we are going to recall the main results available in the literature, distinguishing them on the basis of the different assumptions required. Firstly, we quote the result due to Tan.

Theorem 8 (See [64], Th.2). *Let $K \subseteq \mathbb{R}^G$ be a convex, compact, and nonempty set and $\Phi : K \rightrightarrows \mathbb{R}^G$ and $S : K \rightrightarrows K$ be two set-valued maps. Let us suppose that the following properties hold:*

(i) Φ is upper semicontinuous with nonempty, convex, and compact values;

(ii) S is closed, lower semicontinuous, and with nonempty, convex, and compact values.

Then, the $GQVI(\Phi, S, K)$ admits at least a solution.

One of the main assumptions of Theorem 8 is the upper semicontinuity of Φ. However, this requirement can be replaced with weaker continuity conditions and quasimonotonicity on Φ, as proved by Aussel and Cotrina in [7].
Another important result is due to Kien, Wong and Yao. They proved the existence of a solution to $GQVI(\Phi, S, K)$ without requiring any monotonicity and continuity assumption on Φ but that the set of the fixed points of S is closed.

Theorem 9 (See [39], Th.3.1). *Let $K \subseteq \mathbb{R}^G$ be a nonempty, compact, convex set and $S : K \rightrightarrows K$ and $\Phi : K \rightrightarrows \mathbb{R}^G$ be two set-valued maps. Let us suppose that the following properties hold:*

(i) S is lower semicontinuous with convex values and the set $M := \{x \in K : x \in S(x)\}$ is closed.

(ii) The set $\Phi(x)$ is nonempty, compact for each $x \in K$ and convex for each $x \in M$.

(iii) For each $y \in K$, the set $\{x \in K : \inf_{z \in \Phi(x)} \langle z, x - y \rangle_G \leq 0\}$ is closed.

Then, the $GQVI(\Phi, S, K)$ admits at least a solution.

Another important result, due to Harker, derives, under suitable assumptions, the existence of the solution to a quasi-variational inequality by that of an opportune variational inequality.

Theorem 10 (See [36], Th.3). *Let F and S be respectively a function and a set-valued map from $\mathbb{R}^G$ into itself. Suppose that there exists a nonempty, convex, and compact set $K \subset \mathbb{R}^G$ such that*

(i) $S(x) \subseteq K$ for all $x \in K$;

(ii) $x \in S(x)$ for all $x \in K$.

Then any solution to the variational inequality $VI(F, K)$ is a solution to the $QVI(F, S, K)$.

The key assumption of Theorem 10 is the condition (ii), which states that the graph of the mapping S contains the diagonal. This assumption rules out, for instance, mappings which are projections onto a proper subset of K.
Recently, Aussel, Sultana and Vetrivel studied the solution to $GQVI(\Phi, S, K)$ with non-self constraint map, that is, when $S(K) \not\subseteq K$, with the extreme situation when $S(K) \cap K \neq \emptyset$. They introduced a new concept of solution.

Definition 8 (Projected solution, see [9], Def.2.1). *Let $K \subseteq \mathbb{R}^G$ be a nonempty set, and $\Phi : \mathbb{R}^G \rightrightarrows \mathbb{R}^G$ and $S : K \rightrightarrows \mathbb{R}^G$ be two set-valued maps. A point $\bar{x} \in K$ is said to be a* projected solution *of $GQVI(\Phi, S, K)$ iff there exists $\bar{y} \in \mathbb{R}^G$ such that*

(i) $\bar{x}$ is a projection of $\bar{y}$ on K;

(ii) $\bar{y}$ is a solution of the following variational problem $QVI(\Phi, S(\bar{x}))$

$$\text{Find } \bar{y} \in S(\bar{x}) \text{ for which } \exists \bar{y}^* \in \Phi(\bar{x}) \text{ s.t. } \langle \bar{y}^*, z - \bar{y} \rangle_G \geq 0 \text{ for all } z \in S(\bar{x}).$$

The following result establishes the existence of projected solution to the quasi-variational problem, where the constraint map is not necessarily self-map.

Theorem 11 (See [9], Th.3.3). *Let $K \subseteq \mathbb{R}^G$ be a nonempty set. Let $\Phi : \mathbb{R}^G \rightrightarrows \mathbb{R}^G$ and $S : K \rightrightarrows \mathbb{R}^G$ be two set-valued maps, where $S(K)$ is relatively compact. Then, the $GQVI(\Phi, S, K)$ admits at least a projected solution if the following properties hold:*

(i) S is a closed, lower semicontinuous and convex-valued map with $intS(x) \neq \emptyset$, for all $x \in K$;

(ii) Φ is quasi-monotone, locally upper sign-continuous and dually lower semi-continuous on $convS(K)$.

Furthermore, we want to remark that for each of the Theorems 8, and 9 is available a corresponding result in infinite-dimensional spaces. In particular, now we quote only the result due to Tan since it will be useful in the sequel of the elaborate. We denote by X^* the topological dual of X.

Theorem 12 (See [64], Cor.). *Let X be a topological linear locally convex Hausdorff space, let $K \subseteq X$ be a convex, compact, and nonempty set and $\Phi : K \rightrightarrows X^*$ and $S : K \rightrightarrows K$ be two set-valued maps. Let us suppose that the following properties hold:*

(i) Φ is (norm-to-norm)upper semicontinuous with nonempty, convex, and compact values;

(ii) S is closed, lower semicontinuous, and with nonempty, convex, and compact values.

Then, the $GQVI(\Phi, S, K)$ admits at least a solution.

2.3 Stochastic Variational Inequalities

Although variational inequality problems allow for capturing a wide class of problems, however in most practical settings the data of such problems are affected by uncertainty. This allows us to examine a stochastic generalization of a variational problem. In the last two decades, *one-stage* stochastic variational inequality problems have been studied and various analytical formulations have been introduced, in which many of the challenges arise from computational aspects. Nevertheless, in many real-life applications, the decision-maker has to make sequential decisions, motivating the interest in stochastic variational inequality problems of multistage nature. So, the aim of this Section is to briefly describe the main analytical formulations of a stochastic variational problem, both of one-stage and multistage nature, and compare them.

Single-Stage SVI: Expected-Value and Almost-Sure Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A first formulation (see, e.g. [13], [31], [41], [68], [61], [63]) of a variational inequality in a stochastic environment is the following.

Definition 9 (Expected-Valued Formulation [41]). *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, and convex set, $\zeta : \Omega \to \mathbb{R}^d$ be a random vector defined on $(\Omega, \mathcal{A}, \mathbb{P})$*

and $F : \mathbb{R}^G \times \mathbb{R}^d \to \mathbb{R}^G$ *be a continuous function. A* stochastic Variational inequality in Expected-Valued Formulation*, associated with* F *and* K*, denoted by* $SVI(\mathbb{E}[F], K)$*, consists in the following problem:*

$$\textit{Find } \bar{x} \in K \textit{ such that } \langle \mathbb{E}\left[F\left(\bar{x};\zeta(\omega)\right)\right], x - \bar{x}\rangle_G \geq 0 \qquad \forall x \in K. \tag{2.3.1}$$

Example 3. *Let*

$$F\left(\bar{x};\zeta(\omega)\right) = \nabla g(\bar{x},\zeta(\omega)) \qquad \textit{so} \qquad \textit{that} \qquad \mathbb{E}\left[F\left(\bar{x};\zeta(\omega)\right)\right] = \nabla \mathbb{E}\left[g\left(\bar{x};\zeta(\omega)\right)\right]$$

then, $SVI(\mathbb{E}[F], K)$ *corresponds to minimize* $\mathbb{E}\left[g\left(x;\zeta(\omega)\right)\right]$ *over* K*.*

This formulation has been used to study several stochastic mathematical programs with equilibrium constraints and various equilibrium problems involving, for example, Stackelberg games [21] and Nash equilibrium games (see, e.g. [53],[52]).
When Ω is a discrete sample space, $\mathbb{E}[F]$ reduces to a finite summation of deterministic functions. In this way, the stochastic variational problem (2.3.1) becomes of deterministic type and numerical methods to deterministic variational inequality problems can be applied directly. It similarly occurs when Ω is a continuous sample space and $\mathbb{E}[F]$ is available in closed-form. However, in most stochastic regimes when Ω is continuous, it is difficult to evaluate exactly $\mathbb{E}[F]$ since this evaluation relies on a multidimensional integration. In addition, the distribution of ζ could be not available. To overcome this difficulty, some works in the literature rely on computer simulations and/or past data to get a sample of ζ. So, also thanks to the recent development of Monte Carlo sampling methods, we can find in the literature suitable methodologies, based on sampling, to find the solution to $SVI(\mathbb{E}[F], K)$.
Precursors of this approach have been King and Rockafellar [41]. Motivated by the asymptotic analysis of statistical estimators in stochastic programming, they studied the asymptotic behavior of the approximate solutions of stochastic generalized equations. Subsequently, Shapiro in [63] discussed the so called *Sample Average Approximation* approach, SAA, for stochastic variational inequality problems: let $\left\{\zeta^1, \ldots, \zeta^n, \ldots, \zeta^N\right\}$ be a sampling of ζ, then SAA approach approximates $\mathbb{E}\left[F\left(\bar{x};\zeta(\omega)\right)\right]$ by means of $\hat{F}^N(x) := \frac{1}{N}\sum_{n=1}^{N} F(x;\zeta^n)$. Indeed, thanks to the law of large number for random functions, it follows that $\hat{F}^N(x)$ converge with probability 1 to $\mathbb{E}\left[F\left(\bar{x};\zeta(\omega)\right)\right]$ (see [61]) when the sampling is independent and identically distributed. In this way, the problem $SVI(\mathbb{E}[F], K)$ is now replaced by the following

$$\text{Find } \bar{x} \in K \text{ such that } \langle \hat{F}^N(x), x - \bar{x}\rangle_G \geq 0 \qquad \forall x \in K. \tag{2.3.2}$$

Another methodology based on sampling and used to approximate solutions to $SVI(\mathbb{E}[F], K)$ is the so called *Sample Path Solution*, SPA, firstly introduced in this context in [31]. Through this approach, with simulations the expectation is estimated by a vector-valued stochastic process $\{f_n(x, \zeta(\omega)) : n = 1, 2, \ldots\}$. The resulting approximating problem is a deterministic problem and, if the simulation run length is sufficiently long, it has a solution with probability 1.

Alternatively to the Expected Value formulation of Definition 9, the following formulation has received widespread attention since it doesn't rely on expectation.

Definition 10 (Almost-Sure Formulation [13]). *Let $K \subseteq \mathbb{R}^G$ be a nonempty, closed, and convex set and $F : \mathbb{R}^G \times \mathbb{R}^d \to \mathbb{R}^G$ be a function. A* stochastic Variational inequality in Almost-Sure Formulation*, associated with F and K, consists in the following problem:*

$$\textit{Find} \;\; \bar{x} \in K \;\; \textit{s.t.} \quad \forall \omega \in \Omega \quad \langle F(\bar{x}, \zeta(\omega)), x - \bar{x} \rangle_G \geq 0 \qquad \forall x \in K(\xi(\omega)). \tag{2.3.3}$$

In general, a single $\bar{x}$, which satisfy (2.3.3) simultaneously for all $\omega \in \Omega$, it can not exist. For instance, if $F(\bar{x}, \zeta(\omega)) = \nabla g(\bar{x}, \zeta(\omega))$, we are looking for a $\bar{x}$ which minimizes $g(\cdot, \zeta(\omega))$ over $K(\zeta(\omega))$ simultaneously for each $\zeta(\omega)$. However, in general, this is impracticable.
Also in this case, methods to look for an approximate solution, and no an exact one, are been introduced in literature. One widely used approach is the so called *Expected Residual Method* (see [1]), ERM. Assumed $K(\zeta(\omega))$ to be the positive orthant, the variational problem (2.3.3) reduces to the following stochastic complementary problem:

$$\text{Find} \;\; \bar{x} \in \mathbb{R}^G \;\; \text{s.t.} \quad F(\bar{x}, \zeta(\omega)) \geq 0, \quad x \geq 0, \qquad \langle F(\bar{x}, \zeta(\omega)), \bar{x} \rangle_G = 0. \tag{2.3.4}$$

Associated to the complementary problem (2.3.4), one consider the following average problem:

$$\min_{x \in \mathbb{R}^G_+} \mathbb{E}\left[\|\Phi(x, \omega)\|\right], \quad \textit{where} \;\; \Phi(x, \omega) = (\varphi(F_1(\bar{x}, \zeta(\omega)), x_1), \ldots, \varphi(F_G(\bar{x}, \zeta(\omega)), x_G))^T.$$

It has been built by using an opportune *gap function* Φ that allows us to find a surrogate solution of the problem by transforming the stochastic complementary problem into a random minimization problem.

Single-Stage SVI: $\mathcal{L}^p$ Formulation

In this Section, we are going to present another approach to single-stage stochastic variational inequality problems, sometimes called *random variational inequalities*, in the framework of Lebesgue Spaces $\mathcal{L}^p$, where $1 \leq p \leq \infty$. In comparison with the Expected-Valued and Almost-Sure formulations introduced in the previous

Section, this approach does not rely on sample approximation techniques to compute the solution.
The first studies in this area date back to [24], in which the existence of a stochastic economic model is studied through a variational problem in $\langle L_1, L_\infty\rangle^*$, and to [30]. However, the main work that opened the path to a lot of theoretical and applicative developments is due to Gwinner[32], in which an existence and full discretization theory to study a class of variational inequalities in Separable Hilbert spaces are proposed with a linear random operator and an application to a unilateral boundary value problem arising from continuum mechanics is presented. In the same functional setting, a first extension was given in [34], in which authors have included randomness on the constraints.
Fixed $p = 2$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ be the Lebesgue space of 2-summable random vectors x from Ω to $\mathbb{R}^G$ such that

$$\mathbb{E}\,\|x\|^2 = \int_\Omega \|x(\omega)\|^2\, d\mathbb{P}(\omega) < \infty.$$

Throughout the expectation, the following bilinear form on $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})^* \times \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is defined

$$\langle\langle \tilde{x}, x\rangle\rangle_G = \int_\Omega \langle \tilde{x}(\omega), x(\omega)\rangle_G \, d\mathbb{P}(\omega)$$

where $\tilde{x} \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})^* = \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, $x \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\langle\cdot,\cdot\rangle_G$ is the inner product in $\mathbb{R}^G$. Let:

(i) $F : \Omega \times \mathbb{R}^G \to \mathbb{R}^G$ be a Caratheodory function, that is $F(\omega, x)$ is measurable in ω for each fixed x and continuous in x for each fixed ω;

(ii) $\mathcal{K} := \{x \in \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}) : x(\omega) \in K(\omega) \quad a.e.\ \omega \in \Omega\}$, with $K(\omega) \subseteq \mathbb{R}^G$ a nonempty, convex, and closed set $\ a.e.\ \omega \in \Omega$.

Then, the following problems are considered.

Problem 1

$$\text{Find } \ \bar{x} \in \mathcal{K} \ \text{ such that } \ \langle\langle \mathcal{F}(\bar{x}), x - \bar{x}\rangle\rangle_G \geq 0 \qquad \forall x \in \mathcal{K} \tag{2.3.5}$$

where $\mathcal{F}(\bar{x}) = F(\cdot, \bar{x})$.

Problem 2

$$\forall \omega \in \Omega \ \text{ find } \ \bar{x}(\omega) \in K(\omega) \ \text{ s.t. } \langle F(\omega, \bar{x}(\omega)), x(\omega) - \bar{x}(\omega)\rangle_G \geq 0 \quad \forall x(\omega) \in K(\omega)\ . \tag{2.3.6}$$

In a separable setting, it follows that the integral formulation (2.3.5) and the ω-formulation (2.3.6) are equivalent.

In the last decade, subsequent works have been done by different authors to develop further theoretical aspects in various directions (see, e.g. [10], [33], [26], [37]) to capture and better fit with the problems that could arise in applications. Indeed, this approach has opened to several different applications to nonlinear and generalized random traffic equilibrium problems, stochastic Nash equilibrium problems, such as Cournot Oligopoly, and market equilibrium problems (see, e.g. [16], [17], [35], [38],). Both quantitative and qualitative aspects are investigated in terms of measurability, existence, uniqueness and regularity results, with discretization-based approximation procedures through which the stochastic variational inequality can be reduced to a deterministic variational inequality and thus the stochastic variational inequality can numerically be solved. So, in contrast to the sampling approach introduced in the previous Section, $\mathcal{L}^p$ approach focus on *functional analytic methods* that allow us to obtain approximations of the random vector solution together with approximations of statistical quantities such as the mean and the variance of the random solution. However, as could be seen in [37] relatively to two network models, sampling models yield solutions that are quite far from the mean-value but are computational less expansive in comparison with the rigorous $\mathcal{L}^p$ approach, that so does not fit with large-scale models. Indeed, authors remark that only developments and extensions of numerical methods, such for example in the direction of parallelizations, could permit the treatment of problems with a large number of random variables.

Multistage SVI: Nonanticipativity Formulation

In this Section, we describe an approach introduced by Rockafellar and Wets [59] in 2016. This approach is able to study situations where the decisions have to interact dynamically with the availability of information. Indeed, in contrast to Expected-Value, Almost-Sure and $\mathcal{L}^p$ formulations in which opportunities for recourse decisions are not allowed, with this formulation it is possible to capture the dynamics that are essential to stochastic decision processes in response to increasing level of information.

The key concept of this new formulation turns out to be that relating to particular constraints, called *nonanticipativity*, that have to be included in the formulation of the problem in question. Nonanticipativity constraints go back to the foundations of stochastic programming (see, e.g. [54], [55]) and they are seen as explicit constraints on stochastic processes. Moreover, nonanticipativity constraints can be dualized by opportune multipliers.

Let $\mathcal{T} = \{1, \ldots, t, \ldots, T\}$ and $\mathcal{T}_0 = \{0\} \cup \mathcal{T}$ be the finite sets of stages, respectively, without and with the initial stage. At each stage $t \in \mathcal{T}$, Ξ_t denote the finite set of all uncertain situations that could occur, while ξ_0 represents the unique initial situation. In this way, we introduce the following sample space:

$$\Omega := \{\xi_0\} \times \Xi_1 \times \cdots \times \Xi_t \times \cdots \times \Xi_T \quad \text{s.t.}$$

$$\omega_s := (\xi_0, \xi_1, \ldots, \xi_t, \ldots, \xi_T) \in \{\xi_0\} \times \Xi_1 \times \cdots \times \Xi_t \times \cdots \times \Xi_T.$$

So, $\Omega = \{\omega_1, \ldots, \omega_s, \ldots, \omega_S\}$ is the finite set of all possible final occurrences on the entire history, called *scenarios*, and $\mathbb{P} = (\pi(\omega))_{\omega\in\Omega}$ a probability measure on them.
At this point, we pose

$$x := (x_0, x_1, \ldots, x_t, \ldots, x_T) \in \mathbb{R}^{G_0} \times \mathbb{R}^{G_1} \times \cdots \times \mathbb{R}^{G_t} \times \cdots \mathbb{R}^{G_T} = \mathbb{R}^G$$

where $G = G_0 + G_1 + \ldots + G_n + \ldots + G_N$ and we consider the following $T+1$-stage pattern:

$$\xi_0,\ x_0\ ,\ \xi_1,\ x_1\ ,\ \ldots\ ,\ \xi_t,\ x_t\ ,\ \ldots\ ,\ \xi_T,\ x_T \tag{2.3.7}$$

in which $\xi_t \in \Xi_t$ stands for the information revealed at the t-th stage when the decision x_t has to be made. To opportunely formalize this recursive nature of the decision processes, we introduce suitable information fields.

Definition 11. *A family of information-partitions of Ω is $\mathcal{P} := \{F_t\}_{t\in\mathcal{T}_0}$ where, for all $t \in \mathcal{T}_0$, $F_t := \left\{F_t^1, \ldots, F_t^{k_t}\right\}$ is a partition of Ω such that*

(i) $F_0 = \{\Omega\}$;

(ii) for all $t \in \mathcal{T}$, $F_{t+1} \subset F_t$, that is: if $F_{t+1}^j \in F_{t+1} \Rightarrow F_{t+1}^j \subset F_t^k$ for some $F_t^k \in F_t$;

(iii) $F_T = \Omega$.

For all $t \in \mathcal{T}_0$, the set F_t^j is called elementary event *and the partition F_t is called* event.

Condition (i) means that at stage $t = 0$ no uncertainty has resolved; condition (ii) means that information, about the environment are progressively revealed, i.e one has only partial information. Finally, (iii) tell us that all information are revealed at stage T.

To link time-uncertainty structure and the information partition, we can consider the oriented graph $\mathcal{G}$ as an *event-tree*: each pair (ω, t) identified in $\mathcal{P}$ corresponds a node ξ_t^j and at each ξ_t^j of the oriented graph $\mathcal{G}$, we tie the elementary event F_t^j, that is $F_t^j \cong \xi_t^j$.

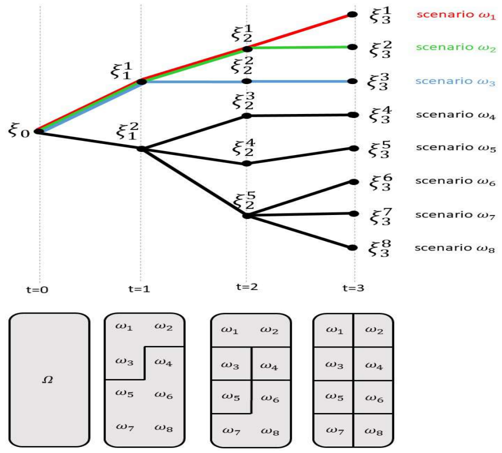


Figure 2.3: Example: Event-Tree $\mathcal{G}$

If two scenarios $\omega_s, \omega_c \in \Omega$ are in the same set $F_t^j \in F_t$, then they are indistinguishable at stage t on the basis of available information: because they share the same path up to stage t, the known information are the same, that is

$$\omega_s \cong (\xi_0, \xi_1^j, \ldots, \xi_t^j, \xi_{t+1}^s, \ldots, \xi_T^s) \qquad \omega_c \cong (\xi_0, \xi_1^j, \ldots, \xi_t^j, \xi_{t+1}^c, \ldots, \xi_T^c).$$

To study this time-uncertainty-information structure, authors introduced the following linear functional space

$$\mathcal{L}_G(\Omega, \mathbb{P}) := \mathcal{L}_G = \left\{\text{the collection of all functions } x : \Omega \to \mathbb{R}^G\right\}.$$

equipped with the following *expectational inner product* and the associated norm:

$$\langle\langle x, y\rangle\rangle_G := \mathbb{E}[\langle x, y\rangle] = \sum_{\omega\in\Omega} \pi(\omega)\langle x(\omega), y(\omega)\rangle_G, \qquad \|x\| := (\mathbb{E}[\langle x, x\rangle])^{\frac{1}{2}}. \quad (2.3.8)$$

The structure (5.1.1) makes $\mathcal{L}_G$ a finite-dimensions Hilbert space. Moreover, one has that $\mathcal{L}_G = \mathcal{L}_{G_0} \times \mathcal{L}_{G_1} \times \ldots \times \mathcal{L}_{G_t} \times \ldots \times \mathcal{L}_{G_T}$ where

$$\mathcal{L}_{G_t} = \left\{\text{the collection of all functions } x_t : \Omega \to \mathbb{R}^{G_t}\right\}.$$

Hence, for all $\omega \in \Omega$ we can consider $x(\omega) = (x_t(\omega))_{t\in\mathcal{T}_0}$.

Definition 12. *Given the information-partitions $\mathcal{P} := \{F_t\}_{t\in\mathcal{T}_0}$ of Ω, let $F_{\bar{t}} \in \mathcal{P}$; we say that $x \in \mathcal{L}_G$ is* $F_{\bar{t}}$-measurable *with respect to $\mathcal{P}$ if for all $j = 1, \ldots, k_{\bar{t}}$ one has:*

$$\forall \omega_s, \omega_c \in F_{\bar{t}}^j \qquad x_t(\omega_s) = x_t(\omega_c) \qquad \forall t = 0, \ldots, \bar{t}.$$

We say that $x \in \mathcal{L}_G$ is measurable *if it is F_t-measurable for all $F_t \in \mathcal{P}$ and $t \in \mathcal{T}_0$.*

On the basis of Definition 11 and Definition 12, we introduce the following subspace

$$\mathcal{N} := \{y \in \mathcal{L}_G : x_t \; is \quad F_t - measurable \qquad \forall t \in \mathcal{T}_0\}.$$

It is called *nonanticipativity* constrains subspace and it plays a central role in the following variational formulation.

Definition 13 (Basic Form, see [59]). *Let $\mathcal{K} := \{x \in \mathcal{L}_G : x(\omega) \in K(\omega)\}$, with $K(\omega)$ a nonempty, closed, and convex set for each $\omega \in \Omega$, and let $\mathcal{F} : \mathcal{L}_G \to \mathcal{L}_G$ be an operator. A stochastic variational inequality in* basic form*, associated with F and $\mathcal{K} \cap \mathcal{N}$, denoted by $SVI(\mathcal{F}, \mathcal{K} \cap \mathcal{N})$, consists in the following problem:*

$$\textit{Find } \; \bar{x} \in \mathcal{K} \cap \mathcal{N} \; \textit{ such that } \; \langle\langle \mathcal{F}, x - \bar{x} \rangle\rangle_G \geq 0 \qquad \forall x \in \mathcal{K} \cap \mathcal{N}. \tag{2.3.9}$$

We denote by $\mathcal{M}$ the following linear subspace

$$\mathcal{M} := \{\rho \in \mathcal{L}_G : \langle\langle y, \rho \rangle\rangle_G = 0 \;\; \forall y \in \mathcal{N}\}. \tag{2.3.10}$$

$\mathcal{M} = (\mathcal{N})^{\perp}$ is called *nonanticipativity multipliers subspace.* In this way, the following formulation can be introduced.

Definition 14 (Extensive Form, see [59]). *Let $\mathcal{K} := \{x \in \mathcal{L}_G : x(\omega) \in K(\omega)\}$, with $K(\omega)$ a nonempty, closed, and convex set for each $\omega \in \Omega$, and let $\mathcal{F} : \mathcal{L}_G \to \mathcal{L}_G$ be an operator. A stochastic variational inequality in* extensive form *consists in the following problem:*

$$\textit{Find } \bar{x} \in \mathcal{N} \textit{ for which } \exists \bar{\rho} \in \mathcal{M} \textit{ such that}$$

$$\forall \omega \in \Omega \qquad \langle F(\omega, \bar{x}(\omega)) + \bar{\rho}(\omega), x(\omega) - \bar{x}(\omega) \rangle_G \geq 0 \qquad \forall x(\omega) \in K(\omega). \tag{2.3.11}$$

We underline that the operator in (2.3.11) differs from that in the Almost-Sure formulation (2.3.3) for the presence of the linear term of the nonanticipativity multipliers ρ. This fact makes a huge difference since it can be seen as a *correction* needed, under opportune assumptions, to ensure the existence of an exact solution which otherwise would not be guaranteed following formulation (2.3.3).

Theorem 13 (Basic-extensive equivalence, see [59], Th.3.2). *If $\bar{x} \in \mathcal{L}_G$ solves (2.3.11), then $\bar{x}$ solves (2.3.9). Conversely, if $\bar{x} \in \mathcal{L}_G$ solves (2.3.9), then $\bar{x}$ is sure also to solve (2.3.11) if*

$$\textit{there exists some } \hat{x} \in \mathcal{N} \textit{ such that } \hat{x}(\omega) \in ri\, K(\omega) \textit{ for all } \omega \in \Omega.$$

This constraint qualification is superfluous if the sets $K(\omega)$ are all polyhedral.

The strength point of this approach is that the extensive formulation (2.3.11) enables the decomposition of the original stochastic variational problem into a separate problem for each scenario. This gives us the basis to apply one of the most effective solution methods, the *progressive hedging algorithm*, recently update to the variational analysis framework in [60]. This algorithm allows us to solve *efficiently and in parallel* stochastic multistage problems with recourse also of large-scale type.

Theorem 14 (See [59], Th.3.6)**.** *Let $\mathcal{F} : \mathcal{L}_G \to \mathcal{L}_G$ be a continuous operator and $\mathcal{K} = \{x \in \mathcal{L}_G : x(\omega) \in K(\omega) \quad \forall \omega \in \Omega\}$ be a nonempty, closed, and convex subspace of $\mathcal{L}_G$. The set of solutions to the multistage stochastic variational inequality*

$$\langle\langle \mathcal{F}(\bar{x}), x - \bar{x} \rangle\rangle_G \leq 0 \qquad \forall x \in \mathcal{K} \cap \mathcal{N}$$

is always closed. It is sure to be bounded and nonempty if $\mathcal{K} \cap \mathcal{N} \neq \emptyset$ and the sets $K(\omega)$ are bounded. Furthermore, under monotonicity of $\mathcal{F}$ relatively to $\mathcal{K}$, the set of solutions to $SVI(\mathcal{F}, \mathcal{K} \cap \mathcal{N})$ is convex.

All up to now seen can be opportunely rewritten and generalized (see, e.g. [58]) in the more general framework of Lebesgue spaces $\mathcal{L}^p$, for $p \in [1, \infty]$, under the usual duality pairing between $\mathcal{L}^p$ and $\mathcal{L}^q$, with $\frac{1}{p} + \frac{1}{q} = 1$. In Chapter 5, indeed, a stochastic variational formulation in a Lebesgue space is introduced and studied in connection to a specific equilibrium problem.

Chapter 3

Radner Equilibrium and Preference Relations without Completeness: a Variational Approach

The aim of this Chapter is to study the Radner equilibrium problem, introduced in Chapter 1, by using a variational inequality approach to maximize the preferences of individuals. The consumers' preferences are described by a binary relation. Debreu in [19] proved that a continuous preference relation can be represented by means of a real function if and only if it is complete and transitive. The assumption of completeness means that an agent should be able to compare any two possible alternatives. One can imagine real-life situations in which this assumption does not hold, for example, when the consumer is not able to rank his preferences between two or more choices. This can occur when we are in uncertain conditions with both a large number of alternatives to choose from and partial, non-exhaustive, or of low quality information available.
This led us to consider an economic problem by dropping the assumptions of completeness and transitivity. However, the considered assumptions *are not sufficient to guarantee the existence of a utility function representing the preference relation*. Hence, we cannot consider the known results to characterize the maximum problem by means of a variational inequality. So, we reformulate the problem of maximizing preferences by means of a variational inequality problem without representation by a utility. A central role in this study is represented by the strictly upper counter set and normal cone associated with it.
The Chapter is organized as follows. Section 3.1 is devoted to the introduction of some preliminary definitions and tools to deal with the preference relations. Section 3.2 is dedicated to studying a maximization problem of a preference relation by using a variational approach. In particular, we prove that when the preference relation is lower semicontinuous and semistrictly convex a maximal element of a

compact and convex set can be characterized by means of a suitable variational inequality. Additionally, a parametric variational inequality is introduced and some regularity properties on the map of solutions are proved. Finally, in Section 3.3, we make use of the theoretical results obtained in the previous section on a Radner equilibrium model of plans, price, and price expectation.

3.1 Preference Relations

Let $X \subseteq \mathbb{R}^n$, with $X \neq \emptyset$, be the set of alternatives of an individual and $\succ$ be the binary relation which describes the preferences of him over the set X. We write $x \succ y$ and we read x *is strictly preferred to* y. When neither $x \succ y$ or $y \succ x$ we say that x *is incomparable to* y and we denote by $x \bowtie y$. We recall some basic properties of preference relations. For further details, we refer to [28, 46] and the references therein.

Definition 15. *Let $\succ$ be a preference relation over X. We say that $\succ$ is*

1. irreflexive: *$x \succ x$ never;*
2. transitive*: if $x \succ y$ and $y \succ z$, then $x \succ z$;*
3. complete: *for all x, $y \in X$, $x \succ y$ or $y \succ x$;*
4. negatively transitive: *if $x \succ y$, then for any $z \in X$, either $x \succ z$, or $z \succ y$, or both;*
5. non-satiated: *for all $x \in X$ there exists $y \in X$ s.t. $y \succ x$;*
6. semistrictly convex: *if $x \succ y$ then $\lambda x + (1-\lambda)y \succ y$, for all $\lambda \in (0,1)$.*

Moreover, $\succ$ is said to be a weak-order *or a* rational preference relation *if and only if it is complete and transitive.*

The assumption that $\succ$ is complete means that the individual has well-defined preferences between any two possible alternatives.

Definition 16. *Let $\succ$ be a preference relation over X. We say that $\succ$ is*

- lower semicontinuous: *if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to x with $x \succ y$, then there exists $\nu \in \mathbb{N}$ such that $x_n \succ y$ for all $n > \nu$;*
- upper semicontinuous: *if $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to x with $y \succ x$, then there exists $\nu \in \mathbb{N}$ such that $y \succ x_n$ for all $n > \nu$;*
- continuous: *it is lower and upper semicontinuous.*

Definition 17. *Given a preference relation $\succ$ on a set X, for all $x \in X$, the* strict upper contour set $U(x)$ *is the set of all elements of X strictly preferred to x, that is*

$$U(x) := \{y \in X : \ y \succ x\}.$$

The set $U(x)$ is open if and only if $\succ$ is lower semicontinuous.

Definition 18. *Let $\succ$ be a preference relation over X. We say that $\succ$ is*

- locally non-satiated: *for all $x \in X$ and every $\epsilon > 0$ there exists $y \in X$ such that $\|y - x\| \leq \epsilon$ and $y \succ x$;*
- non-satiated: *for all $x \in X$ there exists $y \in X$ such that $y \succ x$;*
- strictly increasing respect to component-1: *for all x, $y \in X$ such that $x_k \geq y_k$ for all $k = 1, \ldots, n$ and $x_1 > y_1$ one has $x \succ y$.*

If the preference $\succ$ is locally non satiated, then it is non-satiated.

Desirability is an important assumption in economic theory. Locally non-satiation and non-satiation assumptions can be reformulated by means of the set $U(x)$.

Proposition 7. *Let $\succ$ be a preference relation over X. Then, $\succ$ is*

- locally non-satiated: *for all $x \in X$ and every $\epsilon > 0$ one has $U(x) \cap B(x, \epsilon) \neq \emptyset$;*
- non-satiated: *for all $x \in X$ there exists $y \in X$ one has $U(x) \neq \emptyset$.*

Definition 19. *A function $u : X \to \mathbb{R}$ is a* utility function representing preference relation $\succ$ *if, for every $x, y \in X$,*

$$x \succ y \text{ if and only if } u(x) > u(y).$$

If there exists u, we say that $\succ$ is representable.

Debreu in [19] gave necessary and sufficient conditions to ensure that a preference relation is representable.

Theorem 15. *Let X finite or countably infinite and $\succ$ be a preference relation over X. It is representable if and only if $\succ$ is a rational preference relation on X.*

Theorem 16. *Let $X = \mathbb{R}^n$ and $\succ$ be a preference relation over X. It is representable if and only if $\succ$ is a continuous rational preference relation on X.*

Furthermore, we say that $\succ$ is *semistrictly convex* if

$$x \succ y \text{ then } \lambda x + (1-\lambda)y \succ y, \text{ for all } \lambda \in (0,1).$$

This definition will play a central role in the following sections. Moreover, when $\succ$ is representable, semistrictly convexity of preference corresponds to the semistrictly quasi-concavity of utility function u.

Proposition 8 (See [49], Th.2.1)**.** *Let $\succ$ be a preference relation over X. If $\succ$ is non-satiated and semistrictly convex, then $\succ$ is locally non-satiated.*

3.2 A Variational Approach for Preference Relations

The aim of this Section is to study a problem of maximization of preferences by means of a variational formulation. We introduce the following problem:

Problem 1 (Maximization of preference)**.** *Let $K \subseteq X$, $K \neq \emptyset$. Find $\bar{x} \in K$ such that if $x \succ \bar{x} \quad \Rightarrow \quad x \notin K$.*

A solution of Problem 1 is called a $\succ$*maximal element* of K.
We observe that the definition of maximal element is more general than the definition of greatest element of K, which requires that $\bar{x} \succ x$ for all $x \in K$. To find a maximal element it makes sense also for no complete relations.

If $\succ$ satisfies assumptions of Theorem 15 or 16 the preference relation is representable by means of the utility $u : X \to \mathbb{R}$, and then Problem 1 can be reformulated as the following:

$$\max_{x \in K} u(x)\,. \tag{3.2.1}$$

Moreover, if $\succ$ is also semistrictly convex and continuous, problems 1 and (3.2.1) are equivalent to a suitable generalized variational inequality, see Proposition 5. Here, we want to operate in a setting where completeness and transitivity assumptions are not satisfied. In this way, since the preference relation $\succ$ is not representable, we use a variational approach to study Problem 1 *without using the representation of utility.*
To our aim, we introduce the map $N : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that for all $x \in X$

$$N(x) := \{h \in \mathbb{R}^n : \ \langle h, y - x\rangle_n \leq 0 \quad \forall y \in U(x)\}$$

and $N(x) := \emptyset$ for all $x \notin X$. Let $\overline{B}(0,1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S(0,1) = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Let $G : X \rightrightarrows \mathbb{R}^n$ be the map such that for all $x \in X$

$$G(x) := \begin{cases} conv\,(N(x) \cap S(0,1)) & if\;\; U(x) \neq \emptyset\,, \\ \\ \overline{B}(0,1) & if\;\; U(x) = \emptyset\,. \end{cases}$$

We introduce the following generalized variational inequality, $GVI(G,K)$:

Problem 2 (GVI)**.** *Find* $\overline{x} \in K$ *s.t. there exists* $h \in G(\overline{x})$ *and* $\langle h, x - \bar{x}\rangle_n \geq 0 \quad \forall x \in K$.

Theorem 17. *Let* $\succ$ *be a lower semicontinuous preference relation over* X*.*

(a) If $\succ$ *is semistrictly convex every solution to Problem 1 is a solution to Problem 2.*

(b) Every solution to Problem 2 is a solution to Problem 1.

Proof. **(a)** Let $\bar{x}$ be a solution to Problem 1.
If $U(\bar{x}) = \emptyset$, then $h = 0 \in G(\bar{x})$ and $\langle h, x - \bar{x}\rangle_n = 0 \quad \forall x \in K$.
Let $U(\bar{x}) \neq \emptyset$, for any $x \in U(\bar{x})$ one has $x \notin K$. Hence, $U(\bar{x}) \cap K = \emptyset$ and since $U(\bar{x})$ is an open set from lower semicontinuity, it follows that $\mathrm{int}U(\bar{x}) \cap K = \emptyset$. From Separation Theorem there exists $h \in \mathbb{R}^n \setminus \{0\}$ such that

$$\langle h, r - s\rangle_n \geq 0 \qquad \forall s \in U(\bar{x})\;,\; \forall r \in K\,. \tag{3.2.2}$$

If we replace $r = \bar{x}$ in (3.2.2), it follows that $\langle h, s - \bar{x}\rangle_n \leq 0 \;\; \forall s \in U(\bar{x})$, hence $h \in N(\bar{x}) \setminus \{0\}$ and $\bar{h} = \frac{h}{\|h\|} \in G(\bar{x})$. From (3.2.2), it follows

$$\langle \bar{h}, r - s\rangle \geq 0 \qquad \forall s \in U(\bar{x})\;,\; \forall r \in K\,. \tag{3.2.3}$$

Being $U(\bar{x}) \neq \emptyset$ there exists $x' \in X$ such that $x' \succ \bar{x}$. For all $n \in \mathbb{N}$, we pose

$$y_n = \lambda_n x' + (1 - \lambda_n)\bar{x} \quad with \qquad 0 \leq \lambda_n \leq \min\left\{ \frac{1}{n\,\|x' - \bar{x}\|}\;, 1 \right\}$$

Being $\succ$ semistrictly convex, $y_n \succ \bar{x}$, that is $y_n \in U(\bar{x})$, and from (3.2.3), one has $\langle h, r - y_n\rangle_n \geq 0$ for all $r \in K$. Passing to the limit, we get

$$\langle \bar{h}, r - \bar{x}\rangle_n \geq 0 \quad \forall\, r \in K\,. \tag{3.2.4}$$

Hence, from (3.2.4) and being $\bar{h} \in G(\bar{x})$, we can conclude that $\bar{x}$ is a solution to Problem 2.
(b) Let $\bar{x}$ be a solution to Problem 2.
Clearly, if $U(\bar{x}) = \emptyset$, $\overline{x}$ is a solution to Problem 1. If $U(\bar{x}) \neq \emptyset$, we suppose that there exists $x' \in K$ such that $x' \succ \bar{x}$. Since $x' \in K$ and $\bar{x}$ is a solution to Problem

2 one has $\langle h, x' - \bar{x}\rangle_n \geq 0$. Moreover, being $h \in G(\bar{x}) = conv\,(N(\bar{x}) \cap S(0,1)) \subseteq conv\,N(\bar{x}) = N(\bar{x})$, one has $h \in N(\bar{x})$, with $h \neq 0$, and from definition of map N, it follows that $\langle h, x' - \bar{x}\rangle_n \leq 0$. Hence $\langle h, x' - \bar{x}\rangle_n = 0$.
Now, for all $n \in \mathbb{N}$, we pose $x_n := x' + \frac{1}{n}h$; since $x_n \to \bar{x}$ from lower semicontinuity of $\succ$ there exists $\nu \in \mathbb{N}$ such that $x_n \succ \bar{x}$ for all $n \geq \nu$. Hence, since $x_n \in U(\bar{x})$, one has $\langle h, x_n - \bar{x}\rangle_n \leq 0$. Then

$$0 \geq \langle h, x_n - \bar{x}\rangle_n = \langle h, x' - \bar{x}\rangle_n + \frac{1}{n}\|h\|^2 = \frac{1}{n}\|h\|^2 \geq 0$$

this contradicts the fact that $h \neq 0$. □

Theorem 18. *Let $\succ$ be upper semicontinuous and let K be a compact and convex set. Then, there exists $\bar{x}$ solution to Problem 2.*

Proof. For all $x \in X$ there exists $h \in N(x) \setminus \{0\}$ (see, e.g., [8]), and being $h' = \dfrac{h}{\|h\|} \in G(x)$, it follows that G is with nonempty values for all $x \in X$. Moreover, from definition, G is compact and with convex values. We prove that G is a closed map. Let $\{x_n\} \subseteq X$, $\{h_n\} \subseteq \mathbb{R}^n$ be such that $h_n \in G(x_n)$ and $h_n \to h$ and $x_n \to x$. We have to verify that $h \in G(x)$.
If $U(x) = \emptyset$, being $h_n \in G(x_n) \subseteq \overline{B}(0,1)$, one has $h \in \overline{B}(0,1) = G(x)$.
Let $U(x) \neq \emptyset$, there exists $x' \in X$ such that $x' \succ x$ and, from upper semicontinuity, there exists $\nu \in \mathbb{N}$ such that $x' \succ x_n$ for all $n > \nu$ and then $U(x_n) \neq \emptyset$ and $h_n \in conv(N(x_n) \cap S(0,1))$. Since $h_n \in conv(N(x_n) \cap S(0,1))$ there exists $g_n^k \in N(x_n) \cap S(0,1)$ with $k = 1, \ldots, n+1$, and $\lambda_n^k \geq 0$ such that $\displaystyle\sum_{k=1}^{n+1} \lambda_n^k = 1$ and $h_n = \displaystyle\sum_{k=1}^{n+1} \lambda_n^k g_n^k$. Since for all $k = 1, \ldots, n+1$, $\{g_n^k\} \subseteq S(0,1)$ one has $g_n^k \to g^k \in S(0,1)$.
Moreover $g^k \in N(x)$; indeed, for all $y \in U(x)$ from upper semicontinuity, there exists $\nu \in \mathbb{N}$ such that $y \succ x_n$, hence $y \in U(x_n)$ and, since $g_n^k \in N(x_n)$ one has $\langle g_n^k, y - x_n\rangle_n \leq 0$. Passing to the limit it follows $\langle g^k, y - x\rangle_n \leq 0$, that is $g^k \in N(x)$. Then, since $h = \sum_{k=1}^{n+1} \lambda^k g^k$ with $g^k \in N(x) \cap S(0,1)$, one has $h \in G(x)$; this proves that G is a closed map.
Then, being K a compact set and G a map closed and with nonempty, compact, and convex values, from Theorem 7 there exists at least a solution to Problem 2. □

Theorem 19. *Let $\succ$ be continuous and semistrictly convex. Then, there exists $\bar{x}$ solution to Problem 1.*

Proof. Thesis follows from Theorems 17 and 18. □

Remark 3. *If $\succ$ is a non-satiated preference relation on X, for all $x \in X$ one has that $U(x) \neq \emptyset$ and then $G(x) := conv\,(N(x) \cap S(0,1))$.*

Let $L \subset \mathbb{R}^m$ with $L \neq \emptyset$ and $K : L \rightrightarrows \mathbb{R}^n$ be a set-valued map. Let us introduce the following parametric variational inequality problem.

Problem 3 (Parametric VI)**.** *Fixed $l \in L$, find $\bar{x} \in K(l)$ such that there exists $h \in G(\bar{x})$ and $\langle h, x - \bar{x}\rangle_n \geq 0 \quad \forall x \in K(l)$.*

We introduce the set-valued map of solutions. Let $S : L \rightrightarrows \mathbb{R}^n$ be such that for all $l \in L$

$$S(l) := \{\bar{x} \in K(l) : \ \bar{x} \textit{ is a solution to Problem } 3\}.$$

In the same way, we can introduce the parametric maximization problem

Problem 4 (Parametric)**.** *Fixed $l \in L$. Find $\bar{x} \in K(l)$ such that if $x \succ \bar{x} \quad \Rightarrow x \notin K(l)$.*

Clearly, if $\succ$ is lower semicontinuous and semistrictly convex, one has

$$S(l) = \{\bar{x} \in K(l) : \ \bar{x} \textit{ is a solution to Problem } 4\}.$$

Theorem 20. *Let $\succ$ be irreflexive, negatively transitive, continuous, and semistrictly convex and let K be a closed, lower semicontinuous map with nonempty, compact, and convex values and such that $K(L)$ is a bounded set. Then the map of solutions S is upper semicontinuous and with nonempty, convex, and compact values.*

Proof. From Theorem 18, for all $l \in L$ it follows that $S(l) \neq \emptyset$. We prove that S is with convex values. For all $l \in L$, let $\bar{x}_1, \bar{x}_2 \in S(l)$ and, for all $\lambda \in (0,1)$, let $y = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2$. Being K with convex values, one has $y \in K(l)$. Firstly, we observe that from Theorem 17 $\bar{x}_1$ and $\bar{x}_2$ are solutions to Problem 1, hence $\bar{x}_1 \bowtie \bar{x}_2$. We suppose that there exists $x \in K(l)$ such that $x \succ y$; if we consider x, y and $\bar{x}_1$, since $\succ$ is negatively transitive one has $x \succ \bar{x}_1$ or $\bar{x}_1 \succ y$. Being $\bar{x}_1$ a maximal element of $K(l)$, one has $\bar{x}_1 \succ y$. Analogously one has $\bar{x}_2 \succ y$. Hence, $\bar{x}_1$, $\bar{x}_2 \in U(y)$ and from semistrict convexity of $\succ$, we have $y = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2 \succ y$, which contradicts the assumption of irreflexivity. Hence, y is a maximal element of $K(l)$ and from Theorem 17 $y \in S(l)$; then it follows that S is with convex values. We prove that S is with closed values. For all $l \in L$, let $\{\bar{x}_n\} \subseteq S(l)$ be a sequence converging to $\bar{x}$. For all $n \in \mathbb{N}$, there exists $h_n \in G(\bar{x}_n)$ such that $\langle h_n, x - \bar{x}_n\rangle_n \geq 0$ for all $x \in K(l)$; the sequence $\{h_n\}$ converges to h and, being G a closed map (as proved in Theorem 18), $h \in G(x)$. Hence, passing to the limit

one has $\langle h, x - \bar{x}\rangle_n \geq 0$ for all $x \in K(l)$, that is $\bar{x} \in S(l)$.
Since for all $l \in L$, $S(l)$ is a closed set and $S(l) \subseteq K(l)$ which is a bounded set, it follows that $S(l)$ is compact.
We prove that S is closed. Let $\{l_n\} \subseteq L$ and $\{\bar{x}_n\} \subseteq \mathbb{R}^n$ be two sequences with $\bar{x}_n \in S(l_n)$ and such that $l_n \to l$ and $\bar{x}_n \to \bar{x}$. Being K a closed map, $\bar{x} \in K(l)$. From lower semicontinuity of K, for all $x \in K(l)$ there exists a sequence $\{x_n\}$ converging to x such that $x_n \in K(l_n)$. Since for all $n \in \mathbb{N}$, $\bar{x}_n \in S(l_n)$, there exists $h_n \in G(\bar{x}_n)$ such that $\langle h_n, x_n - \bar{x}_n\rangle_n \geq 0$ and moreover, since $\{h_n\} \subseteq \overline{B}(0,1)$, one has $h_n \to h$ with $h \in G(\bar{x})$, being G a closed map. Hence, passing to the limit, we get $\langle h, x - \bar{x}\rangle_n \geq 0$, that is $\bar{x} \in S(l)$.
Finally, since S is closed and compact, S is upper semicontinuous. □

3.3 Radner Equilibrium

The aim of this Section is to apply the theoretical results of the previous Section to the Radner equilibrium model, introduced in Chapter 1, without the representation of the agent's preferences by a utility function. Let $\mathcal{E} := \Big(\mathcal{G}, (\succ_i, e_i)_{i\in\mathcal{I}}\Big)$ be an economy where for each agent $\succ_i$ is the preference relation over her consumption set, *without completeness and transitivity assumptions.*
In the market the aim of each agent i is to choose a maximal element of $M_i(p,q)$ respect to the preference relation $\succ_i$. Hence, in this case, we have the following mathematical formulation of equilibrium.

Definition 20. *An* equilibrium *of plans, prices, and price expectations for the economy $\mathcal{E}$ is a vector $\Big((\bar{x}_i, \bar{z}_i)_{i\in\mathcal{I}}, \bar{p}, \bar{q}\Big) \in \prod_{i\in\mathcal{I}} M_i(\bar{p},\bar{q}) \times \mathbb{R}^{HN}_+ \times \mathbb{R}^{N-1}_+$, such that*

(i) for any $i \in \mathcal{I}$, if $x_i \succ_i \bar{x}_i$, then for all $z_i \in \mathbb{R}^{N-1}$, $(x_i, z_i) \notin M_i(\bar{p},\bar{q})$;

(ii) for all $t \in \mathcal{T}_0$:
$$\sum_{i\in\mathcal{I}} \bar{x}_i(\xi_t^j) \leq \sum_{i\in\mathcal{I}} e_i(\xi_t^j) \qquad \forall \xi_t^j \in \Xi_t;$$

(iii) for all $t \in \mathcal{T}$:
$$\sum_{i\in\mathcal{I}} \bar{z}_i(\xi_t^j) = 0 \qquad \forall \xi_t^j \in \Xi_t.$$

Thanks to Proposition 1, without loss of generality, we can consider the prices in the simplex-set $\Delta := \Delta_{\xi_0} \times \prod_{\xi_t^j\in\Xi} \Delta_{\xi_t^j}$, defined as in (1.3.5).

Our aim is to study the equilibrium by means of a suitable variational inequality problem without completeness or transitivity assumptions on the preference relations of agents. For all $i \in \mathcal{I}$, we denote by $G_i : \mathbb{R}^{HN} \rightrightarrows \mathbb{R}^{HN}$ the map introduced in Section 3.2 deduced from the preference $\succ_i$:

$$G_i(x) := conv(N_i(x) \cap S(0,1)) \qquad \forall x \in \mathbb{R}^{HN}_+ ,$$

$G(x) := \prod_{i\in\mathcal{I}} G_i(x_i)$ for all $x = (x_i)_{i\in\mathcal{I}} \in \mathbb{R}^{HNI}$ and $\widetilde{M}(p,q) := M(p,q) \cap C$, where

$$C := \prod_{\xi_t^j \in \Xi_0} \left[0, \sum_{h\in\mathcal{H}} \sum_{i\in\mathcal{I}} e_i^h(\xi_t^j)\right] \times \prod_{\xi_t^j \in \Xi} \Big[- \sum_{i\in\mathcal{I}} e_i^1(\xi_t^j), \sum_{i\in\mathcal{I}} e_i^1(\xi_t^j)\Big].$$

We introduce the following generalized quasi-variational inequality:

Problem 5 (GQVI). *Find $(\bar{x},\bar{z},\bar{p},\bar{q}) \in \widetilde{M}(\bar{p},\bar{q}) \times \Delta$ such that there exists $h := (h_i)_{i\in I} \in G(\bar{x})$ and*

$$\sum_{i\in\mathcal{I}} \langle h_i, x_i - \bar{x}_i\rangle_{HN} + \langle (\sum_{i\in\mathcal{I}}(e_i - \bar{x}_i), -\sum_{i\in\mathcal{I}} \bar{z}_i), (p,q) - (\bar{p},\bar{q})\rangle_{HN+N-1} \geq 0 \quad (3.3.1)$$

$$\forall\, (x,z,p,q) \in \widetilde{M}\,(\bar{p},\bar{q}) \times \Delta\,.$$

Remark 4. *The vector $(\bar{x},\bar{z},\bar{p},\bar{q})$ is a solution to (3.3.1) if and only if the following inequalities simultaneously hold:*

(i) for each $i \in \mathcal{I}$, $(\bar{x}_i,\bar{z}_i)$ is a solution to

$$\langle h_i, x_i - \bar{x}_i\rangle_{HN} \geq 0 \qquad \forall\, (x_i,z_i) \in \widetilde{M}_i\,(\bar{p},\bar{q})\,, \quad (3.3.2)$$

(ii) $(\bar{p}(\xi_0),\bar{q})$ is a solution to

$$-\langle (\sum_{i\in\mathcal{I}}(\bar{x}_i(\xi_0) - e_i(\xi_0)), \sum_{i\in\mathcal{I}} \bar{z}_i), (p(\xi_0),q) - (\bar{p}(\xi_0),\bar{q})\rangle_{H+N-1} \geq 0 \qquad \forall\, (p(\xi_0),q) \in \Delta_{\xi_0}\,, \quad (3.3.3)$$

(iii) for all $\xi_t^j \in \Xi$, $\bar{p}(\xi_t^j)$ is a solution to

$$-\langle \sum_{i\in\mathcal{I}} (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)), p(\xi_t^j) - \bar{p}(\xi_t^j)\rangle_H \geq 0 \qquad \forall p(\xi_t^j) \in \Delta_{\xi_t^j}\,. \quad (3.3.4)$$

Indeed, let $(\bar{x},\bar{z},\bar{p},\bar{q})$ be a solution to (3.3.1). Fixed $i^* \in \mathcal{I}$, we consider $(x,z,p,q) \in \widetilde{M}\,(\bar{p},\bar{q}) \times \Delta$ such that $(p,q) = (\bar{p},\bar{q})$, $(x_i,z_i) = (\bar{x}_i,\bar{z}_i)$ for all $i \neq i^*$ and (x_{i^*}, z_{i^*})

an element in $\widetilde{M}_{i^*}(\overline{p},\overline{q})$. *We can replace* (x,z,p,q) *in (3.3.1) and we obtain condition (3.3.2). Now, we consider* $(x,z,p,q)\in\widetilde{M}(\bar p,\bar q)\times\Delta$ *such that* $(x,z)=(\bar x,\bar z)$, $p(\xi_t^j)=\overline{p}(\xi_t^j)$ *for all* ξ_t^j, *with* $t\neq 0$ *and* $(p(\xi_0),q)\in\Delta_{\xi_0}$. *By replacing* (x,z,p,q) *in (3.3.1) we obtain (3.3.3). Finally, fixed* $\xi_{t^*}^j$, *we consider* $(x,z,p,q)\in\widetilde{M}(\bar p,\bar q)\times\Delta$ *such that* $(x,z)=(\bar x,\bar z)$, $(p(\xi_0),q)=(\bar p(\xi_0),\bar q)$, $p(\xi_t^j)=\overline{p}(\xi_t^j)$ *for all* $\xi_t^j\neq\xi_{t^*}^j$, *and* $p(\xi_{t^*}^j)\in\Delta_{\xi_{t^*}^j}$. *By replacing* (x,z,p,q) *in (3.3.1) we obtain (3.3.4). Viceversa, let* $(\bar x_i,\bar z_i)$, $(\bar p(\xi_0),\bar q)$ *and* $(\bar p(\xi_t^j))_{\xi_t^j\in\Xi}$ *satisfy (3.3.2), (3.3.3) and (3.3.4), then (3.3.1) is verified.*

Next result characterizes the equilibrium by means of the variational problem (3.3.1).

Theorem 21. *Let* $\mathcal{E}$ *be an economy such that for all* $i\in\mathcal{I}$ *the preference relation* $\succ_i$ *is semistrictly convex, lower semicontinuous, non-satiated, and strictly increasing in commodity-1. If* $(\bar x,\bar z,\bar p,\bar q)$ *is a solution to the generalized quasi-variational problem (5), then it is an equilibrium vector for the economy* $\mathcal{E}$.

Proof.
Claim 1: From Remark 4 and Theorem 17 one has that for all $i\in\mathcal{I}$, $\bar x_i$ is maximal for $\succ_i$ in $\widetilde{M}_i(\bar p,\bar q)$.

Claim 2: $\sum_{i\in\mathcal{I}}\bar z_i(\xi_t^j)\leq 0$ for all $\xi_t^j\in\Xi$ and $\sum_{i\in\mathcal{I}}(\bar x_i(\xi_t^j)-e_i(\xi_t^j))\leq 0$ for all $\xi_t^j\in\Xi_0$.
Since for all $i\in\mathcal{I}$, $(\bar x_i,\bar z_i)\in\widetilde{M}_i(\bar p,\bar q)$, one has

$$\langle\sum_{i\in\mathcal{I}}\left(\bar x_i\left(\xi_0\right)-e_i\left(\xi_0\right)\right),\bar p\left(\xi_0\right)\rangle_H+\langle\sum_{i\in\mathcal{I}}\bar z_i,\bar q\rangle_{N-1}\leq 0. \tag{3.3.5}$$

Hence, from (3.3.5) and (3.3.3), one has:

$$\langle\sum_{i\in\mathcal{I}}\left(\bar x_i\left(\xi_0\right)-e_i\left(\xi_0\right)\right),p_0\left(\xi_0\right)\rangle_H+\langle\sum_{i\in\mathcal{I}}\bar z_i,q\rangle_{N-1}\leq 0\quad\forall\left(p_0,q\right)\in\Delta_{\xi_0}. \tag{3.3.6}$$

Now, fixed $h^*\in\mathcal{H}$ we pose $(\tilde p_0,\tilde q)$ such that:

$$\tilde q=0_{N-1}\quad and\quad \tilde p_0=\begin{cases}\tilde p_0^{h^*}=1\\ \tilde p_0^{h}=0\qquad\forall h\neq h^*\end{cases}$$

Being $(\tilde p_0,\tilde q)\in\Delta_{\xi_0}$, by replacing it in (4.2.13) we obtain $\sum_{i\in\mathcal{I}}(\bar x_i^{h^*}(\xi_0)-e_i^{h^*}(\xi_0))\leq 0$.
Fixed $t^*\in\mathcal{T}$ and $j^*=1,\dots,k_t$, we pose $(\tilde p_0,\tilde q)$ such that:

$$\tilde p_0=0_H\quad and\quad \tilde q:=\begin{cases}\tilde q(\xi_{t^*}^{j^*})=1 & t^*=t\quad j^*=j\\ \tilde q(\xi_t^j)=0 & otherwise\end{cases}.$$

Being $(\tilde{p}_0, \tilde{q}) \in \Delta_{\xi_0}$, by replacing it in (4.2.13) we obtain $\sum_{i \in \mathcal{I}} \bar{z}_i(\xi_{t^*}^{j^*}) \leq 0$.

Moreover, from condition (3.3.4), from the second constraint of $\widetilde{M}_i(\bar{p}, \bar{q})$ and from the above inequality, for all $\xi_t^j \in \Xi$ one has:

$$\begin{aligned} \langle \sum_{i \in \mathcal{I}} (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)), p(\xi_t^j) \rangle_H &\leq \langle \sum_{i \in \mathcal{I}} (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)), \overline{p}(\xi_t^j) \rangle_H \\ &\leq \bar{p}^1(\xi_t^j) \Big(\sum_{i \in \mathcal{I}} \bar{z}_i(\xi_t^j) \Big) \leq 0 \quad \forall p(\xi_t^j) \in \Delta_{\xi_t^j}. \end{aligned} \tag{3.3.7}$$

Fixed $\xi_t^j \in \Xi$ and $h^* \in \mathcal{H}$, we pose $\widetilde{p}(\xi_t^j) \in \Delta_{\xi_t^j}$ such that $\widetilde{p}^{h^*}(\xi_t^j) = 1$ and $\widetilde{p}^h(\xi_t^j) = 0$ for all $h \neq h^*$; by replacing $\widetilde{p}(\xi_t^j)$ in (3.3.7) we get

$$\sum_{i \in \mathcal{I}} (\bar{x}_i^{h^*}(\xi_t^j) - e_i^{h^*}(\xi_t^j)) \leq 0\,.$$

Claim 3: For all $i \in \mathcal{I}$ one has

$$\langle \bar{p}(\xi_0), \bar{x}_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \bar{q}, \bar{z}_i \rangle_{N-1} = 0 \tag{3.3.8}$$

$$\left\langle \bar{p}(\xi_t^j), (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)) \right\rangle_H = \bar{p}^1(\xi_t^j) \bar{z}_i(\xi_t^j) \qquad \forall \xi_t^j \in \Xi_t, t \in \mathcal{T} \tag{3.3.9}$$

Indeed, if there exists $i \in \mathcal{I}$ such that $\langle \bar{p}(\xi_0), \bar{x}_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \bar{q}, \bar{z}_i \rangle_{N-1} < 0$, we pose $\tilde{x}_i \in \mathbb{R}_+^{HN}$ such that $\tilde{x}_i(\xi_t^j) = \bar{x}(\xi_t^j)$ for all $\xi_t^j \in \Xi$ and

$$\tilde{x}_i^h(\xi_0) := \begin{cases} \bar{x}_i^1(\xi_0) + K & \\ \bar{x}_i^h(\xi_0) & \forall h \neq 1 \end{cases}$$

with

$$0 < K \leq \min \Big\{ \Big(\sum_{h \in \mathcal{H}} \sum_{i \in \mathcal{I}} e_{i0}^1 \Big) - \overline{x}_{i0}^1, - \frac{\langle \bar{p}\,(\xi_0)\,, \bar{x}_i\,(\xi_0) - e_i\,(\xi_0) \rangle_H + \langle \bar{q}, \bar{z}_i \rangle_{N-1}}{\bar{p}^1(\xi_0)} \Big\}\,.$$

We observe that, from Claim 2 and being $\overline{x}_i \in \mathbb{R}_+^{HN}$ and $e_i \in \mathbb{R}_{++}^{HN}$, one has

$$\overline{x}_i^1(\xi_0) \leq \sum_{i \in \mathcal{I}} \overline{x}_i^1(\xi_0) \leq \sum_{i \in \mathcal{I}} e_i^1(\xi_0) < \sum_{h \in \mathcal{H}} \sum_{i \in \mathcal{I}} e_i^h(\xi_0)\,. \tag{3.3.10}$$

Hence $(\tilde{x}_i, \bar{z}_i) \in \widetilde{M}_i\,(\bar{p}, \bar{q})$ and, since $\succ_i$ is strictly increasing in commodity-1, $\tilde{x}_i \succ_i \bar{x}_i$ which contradicts Claim 1.
Claim 4: For all $i \in \mathcal{I}$ if $x_i \succ_i \overline{x}_i$ then $x_i \notin M_i(\overline{p}, \overline{q})$.
We suppose that there exists $x_i' \in M_i(\overline{p}, \overline{q})$ such that $x_i' \succ_i \overline{x}_i$. Since $\succ_i$ is semistrictly convex, for all $\lambda \in (0,1)$ one has $m = \lambda x_i' + (1-\lambda)\overline{x}_i \succ_i \overline{x}_i$. Since

$M_i(\overline{p},\overline{q})$ is a convex set $m \in M_i(\overline{p},\overline{q})$. Moreover, from (3.3.10) there exists $\varepsilon > 0$ such that $B(\overline{x},\varepsilon) \cap \mathbb{R}^{HN}_{+} \subseteq C$, hence, for all $\lambda \in \left(0, \dfrac{\varepsilon}{\|x'_i - \overline{x}_i\|}\right)$ one has $m \in C$. Hence, one has $m \in \widetilde{M}_i(\overline{p},\overline{q})$ and $m \succ_i \overline{x}_i$ which contradicts Claim 1.
Claim 5: $\displaystyle\sum_{i\in\mathcal{I}} \bar{z}_i(\xi^j_t) = 0$ for all $\xi^j_t \in \Xi$.
We suppose that there exists $\xi^j_t \in \Xi$ such that $\displaystyle\sum_{i\in\mathcal{I}} \bar{z}_i(\xi^j_t) < 0$; since, from Proposition 1, $\overline{q} \in \mathbb{R}^{N-1}_{++}$, from Claims 2 and 3, it follows that

$$\langle \bar{p}_0, \sum_{i\in\mathcal{I}} \bar{x}_{i0} - e_{i0}\rangle_H = -\langle \overline{q}, \sum_{i\in\mathcal{I}} \overline{z}_i\rangle > 0,$$

which contradicts Claim 2 since $\overline{p}_0 \in \Delta_0$.

From Claims 1-5 we can conclude that $(\bar{x},\bar{z},\bar{p},\bar{q})$ satisfies all equilibrium conditions for the economy $\mathcal{E}$. □

Lemma 1. *For all $i \in \mathcal{I}$, the map $\widetilde{M}_i : \Delta \rightrightarrows \mathbb{R}^{HN}_{+} \times \mathbb{R}^{N-1}$ is closed, lower semi-continuous and with nonempty, compact, and convex values.*

Proof.
$\widetilde{M}_i$ is with nonempty, closed, and convex values.
Firstly, we observe that for all $i \in \mathcal{I}$ and $(p,q) \in \Delta$, since $(e_i, 0_{N-1}) \in \widetilde{M}_i(p,q)$ one has $\widetilde{M}_i(p,q)$ is nonempty and, from definition, $\widetilde{M}_i(p,q)$ is a convex set.
Let $\{(x_{i,n}, z_{i,n})\}_{n\in\mathbb{N}} \in \widetilde{M}_i(p,q)$ such that $(x_{i,n}, z_{i,n}) \to (x_i, z_i)$. For each $n \in \mathbb{N}$, one has

$$\begin{aligned}
&0 \le \langle p(\xi_0), x_{i,n}(\xi_0)\rangle_H \le -\langle q(\omega), z_{i,n}(\omega)\rangle_{N-1} + \langle p(\xi_0), e_i(\xi_0)\rangle_H \\
&0 \le \left\langle p(\xi^j_t), x_{i,n}(\xi^j_t)\right\rangle_H \le \left\langle p(\xi^j_t), e_i(\xi^j_t)\right\rangle_H + p^1(\xi^j_t) z_{i,n}(\xi^j_t) \qquad \forall \xi^j_t \in \Xi.
\end{aligned} \tag{3.3.11}$$

Since $z_{i,n} \in \prod_{\xi^j_t\in\Xi}\left[-\sum_{i\in\mathcal{I}} e^1_i(\xi^j_t), \sum_{i\in\mathcal{I}} e^1_i(\xi^j_t)\right]$ it follows that $\{z_{i,n}\}_{n\in\mathbb{N}}$ converges to z_i. Hence, from (3.3.11), one has that the sequence $\{x_{i,n}\}_{n\in\mathbb{N}}$ is bounded and converges to x_i. Then, $(x_i, z_i) \in \widetilde{M}_i(p,q)$.

$\widetilde{M}_i$ is a closed map.
Firstly, we observe that since $\{(p_n, q_n)\}_{n\in\mathbb{N}} \subseteq \Delta$, one has that this sequence converges to (p,q). So, in similar way to the above Claim, one has $(x_i, z_i) \in \widetilde{M}_i(p,q)$, that is $\widetilde{M}_i$ is a closed map.

$\widetilde{M}_i$ *is lower semicontinuous.*
Let $\{(p_n, q_n)\}_{n\in\mathbb{N}} \subseteq \Delta$ be a sequence converging to (p, q); for all $(x_i, z_i) \in \widetilde{M}_i(p, q)$, we have to prove that there exists a sequence $\{(x_{i,n}, z_{i,n})\}_{n\in\mathbb{N}}$ converging to (x_i, z_i) and such that $(x_{i,n}, z_{i,n}) \in \widetilde{M}_i(p_n, q_n)$ for all $n \in \mathbb{N}$. If (x_i, z_i) is such that

$$\begin{aligned} &\langle p(\xi_0), x_i(\xi_0) - e_i(\xi_0)\rangle_H + \langle q, z_i\rangle_{N-1} < 0, \\ &\langle p(\xi_t^j), x_i(\xi_t^j) - e_i(\xi_t^j)\rangle_H < p^1(\xi_t^j) z_i(\xi_t^j) \qquad \forall \xi_t^j \in \Xi_t, t \in \mathcal{T}, \end{aligned} \tag{3.3.12}$$

we can choose $(x_{i,n}, z_{i,n}) = (x_i, z_i)$ for all n, and, form the Theorem of sign permanence $(x_i, z_i) \in \widetilde{M}_i(p_n, q_n)$. We suppose that at least one inequality of (4.2.21) is not satisfied. Let

$$Li\,\widetilde{M}_i(p_n, q_n) := \{(x_i, z_i) : \ (x_i, z_i) = \lim\,(x_{ik}, z_{ik}), \ \ (x_{ik}, z_{ik}) \in \widetilde{M}_i(p_k, q_k) \ \text{ eventually}\}\,.$$

From Proposition 8.2.1 by [45], one has that $Li\,\widetilde{M}_i(p_n, q_n)$ is a closed set; moreover, being $e_i(\xi_t^j) \in \mathbb{R}^H_{++}$ for all ξ_t^j, for all $(p, q) \in \Delta$, there exists x_i such that $(x_i, 0_{N-1}) \in int\,\widetilde{M}_i(p, q)$, then $\widetilde{M}_i(p, q) = cl\ int\ \widetilde{M}_i(p, q)$. Hence, one has:

$$\widetilde{M}_i(p, q) = cl\,int\,\widetilde{M}_i(p, q) \subset cl\ \ Li\,\widetilde{M}_i(p_n, q_n) = Li\,\widetilde{M}_i(p_n, q_n).$$

Then, we can conclude that $\widetilde{M}_i$ is lower semicontinuous. □
In the next Theorem we give the existence of equilibrium by means of the $GQVI$ (3.3.1). The characteristics of our problem allow us to consider two variational inequalities, instead of a single quasi-variational inequality. Firstly, we fix prices (p, q) and we study the first part of inequality (3.3.1): this is a parametric variational problem and we can introduce the map of solutions. This map represents the operator of the second part of the inequality and we can solve this variational problem thanks to the properties of the solution map proven in Theorem 20. The pair of solutions given by the two variational problems represent the solution to (3.3.1).

Theorem 22. *Let $\mathcal{E}$ be an economy such that for all $i \in \mathcal{I}$ the preference relation $\succ_i$ is irreflexive, negatively transitive, semistrictly convex, continuous and strictly increasing in commodity-1. Then there exists a equilibrium of plans, price and price expectations for $\mathcal{E}$.*

Proof. For each $i \in \mathcal{I}$ and $(p, q) \in \Delta$, we consider the parametric GVI:

Find $(\bar{x}_i, \bar{z}_i) \in \widetilde{M}_i(p, q)$ *such that there exists* $h_i \in G_i(\bar{x}_i)$ *with*

$$\langle h_i, x_i - \bar{x}_i\rangle_G \geq 0 \qquad \forall (x_i, z_i) \in \widetilde{M}_i(p, q)\,. \tag{3.3.13}$$

We introduce the map of solutions $S_i : \Delta \rightrightarrows \mathbb{R}^{HN+N-1}$ such that, for all $(p,q) \in \Delta$,

$$S_i(p,q) = \{(\bar{x}_i, \bar{z}_i) : (\bar{x}_i, \bar{z}_i) \ \ is \ \ solution \ \ of \ \ (3.3.13)\} \ .$$

From Lemma 1, being $M_i(\Delta) \subset C$ and from Theorem 20 it follows that S_i is upper semicontinuous and with nonempty, convex, and compact values, hence the map $S : \Delta \rightrightarrows \mathbb{R}^{HN+N-1}$ such that $S(p,q) := \prod_{i\in\mathcal{I}} S_i(p,q)$ for all $(p,q) \in \Delta$ has the same properties. Now, we consider the following GVI

$$\textit{Find } (\bar{p},\bar{q}) \in \Delta \textit{ such that there exists } (\bar{x}_i, \bar{z}_i) \in S(\bar{p},\bar{q}) \textit{ and}$$

$$\langle (\sum_{i\in\mathcal{I}} (e_i - \bar{x}_i), -\sum_{i\in\mathcal{I}} \bar{z}_i), (p,q) - (\bar{p},\bar{q}) \rangle_{HN+N-1} \geq 0 \qquad \forall (p,q) \in \Delta\,. \quad (3.3.14)$$

From properties of the map S and being Δ a compact set, from Theorem 7 there exists $(\bar{p},\bar{q}) \in \Delta$ and $(\bar{x}_i, \bar{z}_i) \in S(\bar{p},\bar{q})$ solution to (3.3.14). Then, from (3.3.13), with $(\bar{p},\bar{q})$, and (3.3.14) we get that $(\bar{x},\bar{z},\bar{p},\bar{q}) \in \widetilde{M}(\bar{p},\bar{q}) \times \Delta$ is a solution to the Problem 5. Hence, from Theorem 21, we can conclude that $(\bar{x},\bar{z},\bar{p},\bar{q}) \in \widetilde{M}(\bar{p},\bar{q}) \times \Delta$ is an equilibrium for $\mathcal{E}$. $\square$

Chapter 4

Radner Equilibrium: a Stochastic Variational Approach

The aim of this Chapter is to study the Radner equilibrium problem, introduced in Chapter 1, by means of a multistage stochastic variational formulation. This new variational approach, introduced in Chapter 2, allows us to encompass multistage models to capture the essential dynamics of stochastic decision processes in response to an increasing level of information. Indeed, in the equilibrium model, the market evolves in a finite sequence of time and, at each future date, different states of the world are possible. At the beginning, agents do not know the possible evolution of the market; the environment is progressively revealed, and, all information is revealed at the final time. Agents have to make their decisions under uncertain conditions. A central role in this study is represented by the concept of *scenario*. The key point is so to consider the uncertain quantities of the equilibrium problem as functions instead of vectors.
The Chapter is organized as follows. Section 4.1 is devoted to rewriting the Radner equilibrium problem in a probabilistic setting throughout a finite set of scenarios and information fields. This allows to investigate how the information influences the choices of agents and how these evolve over time. Subsequently, in Section 4.2 we provide a new formulation of the equilibrium problem in terms of a suitable stochastic quasi-variational inequality, both in basic and extensive form, and, by using variational tools, we give the existence of equilibrium. Finally, in Section 4.3 a procedure to compute the equilibrium solution is provided by using the Progressive Hedging Algorithm.

4.1 Scenarios Formulation

In this Section, we study the Radner economic equilibrium problem in a stochastic framework by means of a scenarios setting, following that introduced in Section 2.3. Indeed, we consider an economy which is characterized by the information-partitions $\mathcal{P}$ of the set of scenario Ω and by a probability measure on elements of Ω, $\mathbb{P} = (\pi(\omega))_{\omega\in\Omega}$. For each $i \in \mathcal{I}$, we suppose that $x_i, p, e_i \in \mathcal{L}_{H(T+1)}$ and $z_i, q \in \mathcal{L}_{(N-1)(T+1)}$. In particular, since z_i and q represent a decision at time 0, one has

$$z_{i0}(\omega) \in \mathbb{R}^{N-1}, \;\; z_{it}(\omega) = 0 \;\; \forall t \in \mathcal{T} \quad \text{and} \quad q_0(\omega) \in \mathbb{R}^{N-1}_+, \;\; q_t(\omega) = 0 \;\; \forall t \in \mathcal{T}$$

Hence, thanks to the above remark, we can consider $z_i, q \in \mathcal{L}_{N-1}$. Moreover, we require that all vectors x_i, p, e_i and z_i, q are measurable, that is for each $F_t^j \in \mathcal{P}$, $x_{it}(\omega)$ and $e_{it}(\omega)$ are constants for all $\omega \in F_t^j$. From an economic viewpoint, for all $\omega \in F_t^j$, $x_{it}(\omega)$ represents the bundle of spot consumption chosen by agent i at contingency ξ_t^j and $e_{it}(\omega)$ represents the initial endowment in contingency ξ_t^j. Moreover, for any $\omega \in F_t^j$, $p_t(\omega)$ is the spot price at time t and $\displaystyle\sum_{\omega\in F_t^j} p_t(\omega)$ represents the spot price vector at contingency $\xi_t^j = F_t^j$, see e.g. [11]. Hence, from F_t^j-measurability requirement, it follows that:

$$\forall\omega \in F_t^j \qquad x_{it}(\omega) = x_i(\xi_t^j)\;, \quad e_{it}(\omega) = e_i(\xi_t^j) \quad \textit{and} \quad \sum_{\omega\in F_t^j} p_t(\omega) = p(\xi_t^j). \tag{4.1.1}$$

Furthermore, for each $\omega \in \Omega$, $z_i(\omega)$ represents the $N-1$ quantities sold or bought at $t = 0$ of commodity-1 eventually to be delivered or received by agent i in all possible contingencies ξ_t^j, with $t \in \mathcal{T}$ and $k = 1, \dots, k_t$. Although we allow the decisions to depend on Ω, then the use of measurability constraints restricts the choice of z_i to the linear subspace of functions that are *constant* for each $\omega \in \Omega$. In this way, we pose that $z_i(\omega) = (z_{iF_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t}$ for each $\omega \in \Omega$. With similar comments, for each $\omega \in \Omega$, the vector $q(\omega) = (q_{F_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t}$ represents the forward prices at time 0 and it is such that, if we consider $\sum_{\omega\in\Omega} q(\omega) = |\Omega|\,(q_{F_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t}$, this sum represents the forward price vector as defined in (1.3.1).
Summarizing, from F_0-measurability requirement, it follows that:

$$\forall\omega \in \Omega \qquad z_i(\omega) = (z_{iF_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t} = z_i \quad \textit{and} \quad \sum_{\omega\in\Omega} q(\omega) = |\Omega|\,(q_{F_t^j})_{t\in\mathcal{T}}^{j=1,\dots,k_t} = q \tag{4.1.2}$$

We point out that z_i can't really depend on ω, but the requirement that $z_i \in \mathcal{N}$ allows us to study the problem by events.

We use following notations for the nonanticipativity sets: $\mathcal{N}^1 \subseteq \mathcal{L}_{H(T+1)}$ and $\mathcal{N}^2 \subseteq \mathcal{L}_{N-1}$ respectively the sets of commodities x and e and contracts z which satisfies the first conditions (4.1.1) and (4.1.2); $\widetilde{\mathcal{N}}^1 \subseteq \mathcal{L}_{H(T+1)}$ and $\widetilde{\mathcal{N}}^2 \subseteq \mathcal{L}_{N-1}$ the sets of prices p and q which satisfies the second conditions (4.1.1) and (4.1.2). Hence, for sake of simplicity, we pose $C = H(T+1)$, $D = H(T+1) + N - 1$ and

$$\mathcal{L} := \mathcal{L}_{H(T+1)} \times \mathcal{L}_{N-1}, \qquad \mathcal{N} := \mathcal{N}^1 \times \mathcal{N}^2, \qquad \widetilde{\mathcal{N}} := \widetilde{\mathcal{N}}^1 \times \widetilde{\mathcal{N}}^2 .$$

In this setting, we suppose that the utility function is represented by the expected utility

$$\mathcal{U}_i : \mathcal{L}_C \to \mathbb{R} \qquad \mathcal{U}_i(x_i) = \mathbb{E}\left[f_{i\omega}(x_i)\right] = \sum_{\omega \in \Omega} \pi(\omega)\, f_{i\omega}(x_i(\omega)) .$$

where, for each $\omega \in \Omega$, $f_{i\omega} : \mathbb{R}_+^C \to \mathbb{R}$. Hence the economy is characterized by the vector $\mathcal{E} := \left(\mathcal{P}, \mathbb{P}, (\mathcal{U}_i, e_i)_{i\in\mathcal{I}}\right)$. The budget constraint space, at the price system $(p, q) \in \widetilde{\mathcal{N}}$, can be rewritten in the following form:

$$B_i(p,q) = \{(x_i, z_i) \in \mathcal{L} : \ (x_i(\omega), z_i(\omega)) \in B_{i\omega}(p,q) \quad \forall \omega \in \Omega\}$$

where, let $R(\omega) = \prod_{t\in\mathcal{T}} \left[-\sum_{i\in\mathcal{I}} e^1_{it}(\omega), \sum_{i\in\mathcal{I}} e^1_{it}(\omega)\right]$, for all $\omega \in \Omega$ one has

$$\begin{aligned} B_{i\omega}(p,q) :=\{ & (x_i(\omega), z_i(\omega)) \in \mathbb{R}_+^C \times R(\omega) : \\ & \langle p_0(\omega), x_{i0}(\omega)\rangle_H + \langle q(\omega), z_i(\omega)\rangle_{N-1} \le \langle p_0(\omega), e_{i0}(\omega)\rangle_H \\ & \langle p_t(\omega), x_{it}(\omega)\rangle_H \le \langle p_t(\omega), e_{it}(\omega)\rangle_H + p^1_t(\omega)\, z_{it}(\omega) \quad \forall t \in \mathcal{T}\}. \end{aligned} \tag{4.1.3}$$

The aim of each consumer is to maximize the expected utility on the set $B_i(p,q) \cap \mathcal{N}$, which is a nonempty, closed, and convex set of $\mathcal{L}$. Finally, we can reformulate the equilibrium from a viewpoint of scenarios and, then, we can set the problem in the space of function $\mathcal{L}$.

Definition 21. *An* equilibrium of plans, prices, and price expectations *for the economy* $\mathcal{E} := \left(\mathcal{P}, \mathbb{P}, (\mathcal{U}_i, e_i)_{i\in\mathcal{I}}\right)$ *is a vector* $\left((\bar{x}_i, \bar{z}_i)_{i\in\mathcal{I}}, \bar{p}, \bar{q}\right) \in \prod_{i\in\mathcal{I}}(B_i(\bar{p}, \bar{q}) \cap \mathcal{N}) \times \widetilde{\mathcal{N}}$*, such that*

- *for any* $i \in \mathcal{I}$*:*
$$\max_{(x_i,z_i)\in B_i(\bar{p},\bar{q})\cap\mathcal{N}} \mathbb{E}\left[f_{i\omega}(x_i)\right] = \mathbb{E}\left[f_{i\omega}(\bar{x}_i)\right] \tag{4.1.4}$$

- *for any* $\omega \in \Omega$
$$\sum_{i\in\mathcal{I}} \bar{x}_i(\omega) \le \sum_{i\in\mathcal{I}} e_i(\omega) ; \tag{4.1.5}$$

- *for any* $\omega \in \Omega$
$$\sum_{i\in\mathcal{I}} \bar{z}_i(\omega) = 0. \tag{4.1.6}$$

Conditions (4.1.5) and (4.1.6) can be rewritten in terms of components of the vectors $\bar{x}_i(\omega)$, $e_i(\omega)$ and $\bar{z}_i(\omega)$:

$$\sum_{i\in\mathcal{I}} \bar{x}_{it}(\omega) \leq \sum_{i\in\mathcal{I}} e_{it}(\omega) \quad \forall t \in \mathcal{T}_0\ , \qquad\qquad \sum_{i\in\mathcal{I}} \bar{z}_{iF_t^j} = 0 \quad \forall F_t^j \in \mathcal{P}\setminus F_0$$

Remark 5. *We introduce, for all* $i \in \mathcal{I}$ *and* $\omega \in \Omega$*, the maximization problem*

$$\max_{(x_i(\omega),z_i(\omega))\in B_{i\omega}(\bar{p},\bar{q})} f_{i\omega}(x_i(\omega)) = f_{i\omega}(\bar{x}_i(\omega)). \tag{4.1.7}$$

We observe that if $\bar{x}_i \in \mathcal{L}_C$ *is such that, for all* $\omega \in \Omega$, $\bar{x}_i(\omega)$ *is a solution to (4.1.7) and* $\bar{x}_i \in \mathcal{N}^1$*, then* $\bar{x}_i$ *is a solution to (5.1.8).*

The following proposition shows that the definitions in terms of contingencies and in terms of scenarios are equivalent.

Proposition 9. *The vector* $(\bar{x},\bar{z},\bar{p},\bar{q}) \in \prod_{i\in\mathcal{I}}(B_i(\bar{p},\bar{q})\cap\mathcal{N})\times\widetilde{\mathcal{N}}$ *is an equilibrium according to Definition 21 if and only if it is an equilibrium according to Definition 4.*

Proof. Since each pair (ω, t) identifies the contingency ξ_t^j, it follows that conditions (20), (20) and (4.1.5), (4.1.6) are equivalent. Now, we have to prove that $B_i(\bar{p},\bar{q})\cap \mathcal{N} \cong M_i(\bar{p},\bar{q})$. Let $(x_i, z_i) \in B_i(\bar{p},\bar{q})\cap\mathcal{N}$. For all $\omega \in \Omega$ one has:

$$\langle p_0(\omega), x_{i0}(\omega)\rangle_H + \langle q(\omega), z_i(\omega)\rangle_{N-1} \leq \langle p_0(\omega), e_{i0}(\omega)\rangle_H .$$

Summing up $\omega \in \Omega$, it follows that:

$$\sum_{\omega\in\Omega}\langle p_0(\omega), x_{i0}(\omega)\rangle_H + \sum_{\omega\in\Omega}\langle q(\omega), z_i(\omega)\rangle_{N-1} \leq \sum_{\omega\in\Omega}\langle p_0(\omega), e_{i0}(\omega)\rangle_H .$$

Since $(x_i, z_i) \in \mathcal{N}$, $(p, q) \in \widetilde{\mathcal{N}}$ and e_i measurable, from (4.1.1) and (4.1.2) we get

$$\langle p(\xi_0), x_i(\xi_0)\rangle_H + \langle q, z_i\rangle_{N-1} \leq \langle p(\xi_0), e_i(\xi_0)\rangle_H$$

that is the first inequality of the constraint set $M_i(\bar{p},\bar{q})$. In similar way, we can prove that all constraints of $B_i(\bar{p},\bar{q})\cap\mathcal{N}$ hold if and only if constraints in $M_i(\bar{p},\bar{q})$ hold. We conclude that $(\bar{x},\bar{z},\bar{p},\bar{q}) \in \mathcal{L}$ is an equilibrium according to Definition 21 if and only if it is an equilibrium according to Definition 4. $\square$

We introduce, for all $i \in \mathcal{I}$, the following assumptions.

Assumptions F

(F.1) $f_{i\omega}$ is C^1 and concave.

(F.2) $f_{i\omega}$ is strictly increasing in commodity-1: $\forall \tilde{x}_i(\omega), \tilde{\tilde{x}}_i(\omega) \in \mathbb{R}^C_+$ with $\tilde{x}_i(\omega) \geq \tilde{\tilde{x}}_i(\omega)$, then

$$\tilde{x}^1_{it}(\omega) > \tilde{\tilde{x}}^1_{it}(\omega) \quad for \quad some \quad t \in \mathcal{T}_0 \quad \Rightarrow \quad f_{i\omega}(\tilde{x}_i) > f_{i\omega}\left(\tilde{\tilde{x}}_i\right).$$

(F.3) $f_{i\omega}$ is non-satiated: $\forall x_i(\omega) \in \mathbb{R}^C_+ \ \exists \tilde{x}_i(\omega) \in \mathbb{R}^C_+$ s.t. $f_{i\omega}(\tilde{x}_i) > f_{i\omega}(x_i)$.

Assumptions U

(U.1) $\mathcal{U}_i$ is C^1 and concave.

(U.2) $\mathcal{U}_i$ is strictly increasing in commodity-1: $\forall \tilde{x}_i, \tilde{\tilde{x}}_i \in \mathcal{L}_C$ with $\tilde{x}_i \geq \tilde{\tilde{x}}_i$, then

$$\tilde{x}^1_i(\omega) > \tilde{\tilde{x}}^1_i(\omega) \quad for \quad some \quad \omega \in \Omega \quad \Rightarrow \quad \mathcal{U}_i(\tilde{x}_i) > \mathcal{U}_i\left(\tilde{\tilde{x}}_i\right).$$

(U.3) $\mathcal{U}_i$ is non-satiated: $\forall x_i \in \mathcal{L}_C \ \exists \tilde{x}_i \in \mathcal{L}_C$ s.t. $\mathcal{U}_i(\tilde{x}_i) > \mathcal{U}_i(x_i)$.

Proposition 10. *Let $i \in \mathcal{I}$. If for each $\omega \in \Omega$, $f_{i\omega}$ satisfies Assumptions F, then the expected utility $\mathcal{U}_i$ satisfies Assumptions U. Moreover, the gradient of $\mathcal{U}_i$ is monotone.*

Proof. Firstly, we introduce the gradient operator $\nabla\mathcal{U}_i : \mathcal{L}_C \to \mathcal{L}_C$, such that for all $x_i \in \mathcal{L}_C$ associates the map $\nabla\mathcal{U}_i(x_i)$, with

$$\begin{aligned} \nabla\mathcal{U}_i(x_i) :& \Omega \to \mathbb{R}^C \\ & \omega \to \nabla f_{i\omega}(x_i(\omega)). \end{aligned}$$

It follows that $\mathcal{U}_i$ and $\nabla\mathcal{U}_i$ are continuous (see Section 4 in [59]). Moreover, since for each $\omega \in \Omega$ we have that $f_{i\omega}$ is concave, then

$$\begin{aligned} & f_{i\omega}(\lambda x_i + (1-\lambda)\tilde{x}_i) \geq \lambda f_{i\omega}(x_i) + (1-\lambda) f_{i\omega}(\tilde{x}_i) \\ & \pi(\omega) f_{i\omega}(\lambda x_i + (1-\lambda)\tilde{x}_i) \geq \pi(\omega)(\lambda f_{i\omega}(x_i) + (1-\lambda) f_{i\omega}(\tilde{x}_i)) = \\ =& \lambda(\pi(\omega) f_{i\omega}(x_i)) + (1-\lambda)(\pi(\omega) f_{i\omega}(\tilde{x}_i)). \end{aligned}$$

For each $x_i, \tilde{x}_i \in \mathcal{L}_C$, it results that

$$\begin{aligned} \mathcal{U}_i(\lambda x_i + (1-\lambda)\tilde{x}_i) &= \sum_{\omega\in\Omega} \pi(\omega) f_{i\omega}(\lambda x_i + (1-\lambda)\tilde{x}_i) \geq \sum_{\omega\in\Omega} [\lambda f_{i\omega}(x_i) + (1-\lambda) f_{i\omega}(\tilde{x}_i)] = \\ =& \lambda(\sum_{\omega\in\Omega} \pi(\omega) f_{i\omega}(x_i)) + (1-\lambda)(\sum_{\omega\in\Omega} \pi(\omega) f_{i\omega}(\tilde{x}_i)) = \lambda\mathcal{U}_i(x_i) + (1-\lambda)\mathcal{U}_i(\tilde{x}_i) \end{aligned}$$

The strict increase in commodity-1 of $\mathcal{U}_i$ are immediate consequences of Assumptions (F.1) and (F.2). Furthermore, for all $x_i, \tilde{x}_i \in \mathcal{L}_C$, since from Assumption (F.1) $f_{i\omega}$ is concave, so $\nabla f_{i\omega}$ is monotonic decreasing. For all $\omega \in \Omega$ one has:

$$\langle \nabla f_{i\omega}(x_i) - \nabla f_{i\omega}(\tilde{x}_i), x_i(\omega) - \tilde{x}_i(\omega) \rangle_C \geq 0 \qquad \forall x_i(\omega), \tilde{x}_i(\omega) \in \mathbb{R}^C_+ .$$

Hence:

$$\sum_{\omega \in \Omega} \pi(\omega) \langle \nabla f_{i\omega}(x_i) - \nabla f_{i\omega}(\tilde{x}_i), x_i(\omega) - \tilde{x}_i(\omega) \rangle_C = \langle\langle \nabla \mathcal{U}_i(x_i) - \nabla \mathcal{U}_i(\tilde{x}_i), x_i - \tilde{x}_i \rangle\rangle_C \geq 0$$

that is $\nabla \mathcal{U}_i$ is a monotone operator. Now, we prove that $\mathcal{U}_i$ is non-satiated. Let $x_i \in \mathcal{L}_C$, $\tilde{\omega} \in \Omega$ and $x_i(\tilde{\omega}) \in \mathbb{R}^C_+$. From Assumption (F.3), there exists $\tilde{x}_i(\tilde{\omega}) \in \mathbb{R}^C_+$ such that $f_{i\tilde{\omega}}(\tilde{x}_i) > f_{i\tilde{\omega}}(x_i)$. Let $\tilde{\tilde{x}}_i \in \mathcal{L}_C$ be such that $\tilde{\tilde{x}}_i(\omega) = x_i(\omega)$ for all $\omega \neq \tilde{\omega}$ and $\tilde{\tilde{x}}_i(\tilde{\omega}) = \tilde{x}_i(\tilde{\omega})$. One has:

$$\begin{aligned} \mathcal{U}_i(\tilde{\tilde{x}}_i) = \sum_{\omega \in \Omega} \pi(\omega) f_{i\omega}\left(\tilde{\tilde{x}}_i\right) &= \sum_{\omega \neq \tilde{\omega}} \pi(\omega) f_{i\omega}(x_i) + \pi(\tilde{\omega}) f_{i\tilde{\omega}}(\tilde{x}_i) > \\ > \sum_{\omega \neq \tilde{\omega}} \pi(\omega) f_{i\omega}(x_i) + \pi(\tilde{\omega}) f_{i\tilde{\omega}}(x_i) &= \sum_{\omega \in \Omega} \pi(\omega) f_{i\omega}(x_i) = \mathcal{U}_i(x_i) \end{aligned}$$

that is, $\mathcal{U}_i(\tilde{\tilde{x}}_i) > \mathcal{U}_i(x_i)$. $\square$

We observe that, in order to have the non-satiation assumption of $\mathcal{U}_i$, it is sufficient that there exists at least one ω such that $f_{i\omega}$ satisfies Assumption (F.3).
Thanks to the Proposition 1, without loss of generality, we can opportunely rewrite the simplex set (1.3.5) in this probabilistic setting. For all $\omega \in \Omega$ and $t \in \mathcal{T}_0$, we pose:

- $\Delta^0_\omega := \left\{ (p_0(\omega), q(\omega)) \in \mathbb{R}^H_+ \times \mathbb{R}^{N-1}_+ : \sum_{h \in \mathcal{H}} p^h_0(\omega) + \sum_{F^j_t \in \mathcal{P} \setminus F_0} q_{F^j_t} = \frac{1}{|\Omega|} \right\}$
  and $\Delta_{F_0} := \{(p_0, q) \in \mathcal{L} : \quad (p_0(\omega), q(\omega)) \in \Delta^0_\omega \quad \forall \omega \in \Omega\}$;
- $\Delta^t_\omega := \left\{ p_t(\omega) \in \mathbb{R}^H_+ : \sum_{h \in \mathcal{H}} p^h_t(\omega) = \frac{1}{\left|F^j_t\right|} \;\; \textit{with} \;\; F^j_t \subseteq \Omega \;\; \textit{s.t.} \;\; \omega \in F^j_t \right\}$
  and $\Delta_{F^j_t} := \left\{ p_t \in \mathcal{L}_H : \quad p_t(\omega) \in \Delta^t_\omega \quad \forall \omega \in F^j_t \right\}$.

Therefore, by considering $\Delta_\omega := \prod_{t \in \mathcal{T}_0} \Delta^t_\omega$, the following simplex subspace is obtained

$$\Delta := \{(p, q) \in \widetilde{\mathcal{N}} : \quad \left((p_0(\omega), q(\omega)), (p_t(\omega))_{t \in \mathcal{T}}\right) \in \Delta_\omega \quad \forall \omega \in \Omega\}. \tag{4.1.8}$$

4.2 A Stochastic Variational Formulation

In this Section, our aim is to reformulate the equilibrium problem as a suitable *stochastic quasi-variational problem* (SQVI). We introduce the following problem:

$$\text{Find } (\bar{x},\bar{z},\bar{p},\bar{q}) \in (B(\bar{p},\bar{q}) \cap \mathcal{N}) \times \Delta \text{ such that}$$

$$\sum_{i\in\mathcal{I}} \langle\langle \nabla\mathcal{U}_i(\bar{x}_i), x_i - \bar{x}_i\rangle\rangle_C + \langle\langle (\sum_{i\in\mathcal{I}}(\bar{x}_i - e_i), \sum_{i\in\mathcal{I}} \bar{z}_i), (p,q) - (\bar{p},\bar{q})\rangle\rangle_D \le 0 \quad (4.2.1)$$

$$\forall (x,z,p,q) \in (B(\bar{p},\bar{q}) \cap \mathcal{N}) \times \Delta$$

Remark 6. *The vector* $(\bar{x},\bar{z},\bar{p},\bar{q})$ *is a solution of the* $SQVI$ *(4.2.1) if and only if following inequalities simultaneously hold:*

(i) for each $i \in \mathcal{I}$*,* $(\bar{x}_i,\bar{z}_i)$ *is a solution to*

$$\langle\langle \nabla\mathcal{U}_i(\bar{x}_i), x_i - \bar{x}_i\rangle\rangle_C \le 0 \qquad \forall (x_i,z_i) \in B_i(\bar{p},\bar{q}) \cap \mathcal{N}. \quad (4.2.2)$$

(ii) $(\bar{p},\bar{q})$ *is a solution to*

$$\langle\langle (\sum_{i\in\mathcal{I}}(\bar{x}_i - e_i), \sum_{i\in\mathcal{I}} \bar{z}_i), (p,q) - (\bar{p},\bar{q})\rangle\rangle_D \le 0 \qquad \forall (p,q) \in \Delta \quad (4.2.3)$$

The following proposition will be useful to obtain the characterization.

Proposition 11. *Let* $(\bar{x}_i,\bar{z}_i) \in B_i(\bar{p},\bar{q}) \cap \mathcal{N}$ *be a solution to (4.1.4). Then, for each* $\omega \in \Omega$ *one has:*

$$\langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega)\rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega)\rangle_{N-1} = 0, \quad (4.2.4)$$

$$\langle \bar{p}_t(\omega), \bar{x}_{it}(\omega)\rangle_H = \langle \bar{p}_{it}(\omega), e_{it}(\omega)\rangle_H + \bar{p}^1_t(\omega)\,\bar{z}_{iF^j_t} \qquad \forall t \in \mathcal{T}. \quad (4.2.5)$$

Proof. If there exists $\tilde{\omega} \in \Omega$ such that $\langle \bar{p}_0(\tilde{\omega}), \bar{x}_{i0}(\tilde{\omega}) - e_{i0}(\tilde{\omega})\rangle_H + \langle \bar{q}(\tilde{\omega}), \bar{z}_i(\tilde{\omega})\rangle_{N-1} < 0$, from F_0-measurability the strict inequality holds for each $\omega \in \Omega$. We define $\hat{x}_i \in \mathcal{L}_C$ such that, for all $\omega \in \Omega$:

$$\hat{x}_{it}(\omega) := \begin{cases} \bar{x}_{i0}(\omega) + K\underline{e}_1 \\ \bar{x}_{it}(\omega) \quad \forall t \in \mathcal{T} \end{cases} \quad (4.2.6)$$

with

$$0 < K \le -\frac{\langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega)\rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega)\rangle_{N-1}}{\bar{p}^1_0(\omega)}.$$

Since

$$\begin{aligned}
&\langle \bar{p}_0(\omega), \hat{x}_{i0}(\omega) - e_{i0}(\omega)\rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega)\rangle_{N-1} = \\
&= \langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega)\rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega)\rangle_{N-1} + K\bar{p}_0^1(\omega) \leq \\
&\leq \langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega)\rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega)\rangle_{N-1} + \\
&\quad + \left(-\frac{\langle \bar{p}_0(\omega), \bar{x}_{i0}(\omega) - e_{i0}(\omega)\rangle_H + \langle \bar{q}(\omega), \bar{z}_i(\omega)\rangle_{N-1}}{\bar{p}_0^1(\omega)}\right)\bar{p}_0^1(\omega) = 0,
\end{aligned}$$

one has that $(\hat{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$. Since $\mathcal{U}_i$ is strictly increasing in commodity-1 and $\hat{x}_i > \bar{x}_i$ we have that $\mathcal{U}_i(\hat{x}_i) > \mathcal{U}_i(\bar{x}_i)$, contradicting the fact that $\bar{x}_i$ is a maximum point of $\mathcal{U}_i$ in $B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$.
In similar way, we get relation (4.2.5). □

Remark 7. *Let $(\overline{x}, \overline{z}) = (\overline{x}_i, \overline{z}_i)_{i\in\mathcal{I}}$ be such that $(\overline{x}_i, \overline{z}_i)$ is a solution to (4.1.4); then from Proposition 11, summing up to i inequalities (4.2.4) and (4.2.5) one has*

$$\langle \bar{p}_0(\omega), \sum_{i\in\mathcal{I}} (\bar{x}_{i0}(\omega) - e_{i0}(\omega))\rangle\rangle_H + \langle\langle \bar{q}(\omega), \sum_{i\in\mathcal{I}} \bar{z}_i(\omega)\rangle_{N-1} = 0, \tag{4.2.7}$$

$$\langle \bar{p}_{it}(\omega), \sum_{i\in\mathcal{I}} (\bar{x}_{it}(\omega) - e_{it}(\omega))\rangle_H = \bar{p}_t^1(\omega) \sum_{i\in\mathcal{I}} \bar{z}_{iF_t^j}. \tag{4.2.8}$$

Theorem 23. *For all $i \in \mathcal{I}$, let $\mathcal{E}$ be an economy which satisfies the Assumptions U. Then, $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a solution to the SQVI (4.2.1) if and only if it is an equilibrium vector of plans, prices, and price expectations for $\mathcal{E}$.*

Proof. **Claim 1** *For all $i \in \mathcal{I}$, $(\bar{x}_i, \bar{z}_i)$ is a solution of the maximization problem (4.1.4) if and only if it is a solution of (4.2.2).*
It follows from Proposition 10 and from Example 1 of Section 4 in [59], where $\mathcal{G} = -\mathcal{U}$ and $\mathcal{C} = B(\bar{p}, \bar{q})$.
Claim 2 *For all $\omega \in \Omega$, $\sum_{i\in\mathcal{I}} \bar{z}_i(\omega) \leq 0$ and $\sum_{i\in\mathcal{I}} (\bar{x}_i(\omega) - e_i(\omega)) \leq 0$.*
Let $G_0 = H + N - 1$ and $G_t = H$ for each $t \in \mathcal{T}$, from Remark 6, it follows that the following inequalities simultaneously hold

$$\langle\langle(\sum_{i\in\mathcal{I}} (\bar{x}_{i0} - e_{i0}), \sum_{i\in\mathcal{I}} \bar{z}_i), (p_0, q) - (\bar{p}_0, \bar{q})\rangle\rangle_{H+N-1} \leq 0 \quad \forall (p_0, q) \in \Delta_{F_0} \tag{4.2.9}$$

and for all $t \in \mathcal{T}$

$$\langle\langle \sum_{i\in\mathcal{I}} (\bar{x}_{it} - e_{it}), p_t - \bar{p}_t\rangle\rangle_H \leq 0 \quad \forall p_t \in \Delta_{F_t^j}. \tag{4.2.10}$$

Since, for all $i \in \mathcal{I}$, $(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$, summing up i the inequalities of (5.1.7), one has:

(i) $\left\langle \bar{q}\left(\omega\right), \sum_{i\in\mathcal{I}} \bar{z}_i\left(\omega\right)\right\rangle_{N-1} + \left\langle \bar{p}_0\left(\omega\right), \sum_{i\in\mathcal{I}} \left(\bar{x}_{i0}\left(\omega\right) - e_{i0}\left(\omega\right)\right)\right\rangle_H \leq 0$ for all $\omega \in \Omega$, that is

$$\langle\langle \bar{q}, \sum_{i\in\mathcal{I}} \bar{z}_i\rangle\rangle_{N-1} + \langle\langle \bar{p}_0, \sum_{i\in\mathcal{I}} \left(\bar{x}_{i0} - e_{i0}\right)\rangle\rangle_H \leq 0 \tag{4.2.11}$$

(ii) $\left\langle \bar{p}_t\left(\omega\right), \sum_{i\in\mathcal{I}} \left(\bar{x}_{it}\left(\omega\right) - e_{it}\left(\omega\right)\right)\right\rangle_H - \bar{p}_t^1\left(\omega\right)\left(\sum_{i\in\mathcal{I}} \bar{z}_{iF_t^j}\right) \leq 0$ for all $t \in \mathcal{T}$ and $\omega \in \Omega$, that is

$$\langle\langle \bar{p}_t, \sum_{i\in\mathcal{I}} \left(\bar{x}_{it} - e_{it}\right)\rangle\rangle_H - \langle\langle \bar{p}_t^1, \sum_{i\in\mathcal{I}} \bar{z}_{iF_t^j}\rangle\rangle_1 \leq 0 \tag{4.2.12}$$

From (4.2.9) and (4.2.11), we get

$$\langle\langle q, \sum_{i\in\mathcal{I}} \bar{z}_i\rangle\rangle_{N-1} + \langle\langle p_0, \sum_{i\in\mathcal{I}} \left(\bar{x}_{i0} - e_{i0}\right)\rangle\rangle_H \leq 0 \quad \forall\left(p_0, q\right) \in \Delta_{F_0}. \tag{4.2.13}$$

For all $h^* \in \mathcal{H}$, we pose $(\tilde{p}_0, \tilde{q}) \in \mathcal{L}_{H+N-1}$ such that

$$\forall \omega \in \Omega : \qquad \tilde{p}_0^h\left(\omega\right) = \begin{cases} \frac{1}{|\Omega|} & if\ h = h^* \\ 0 & \forall h \neq h^* \end{cases} \quad , \quad \tilde{q}(\omega) = 0_{N-1}$$

Being $(\tilde{p}_0, \tilde{q}) \in \Delta_{F_0}$, we can replace it in (4.2.13) and since x_i and e_i are F_0-measurable one has:

$$\sum_{\omega\in\Omega}\left(\pi(\omega)\tilde{p}_0^{h^*}\left(\omega\right)\sum_{i\in\mathcal{I}}\left(\bar{x}_{i0}^{h^*}(\omega) - e_{i0}^{h^*}(\omega)\right)\right) = \left(\sum_{\omega\in\Omega}\pi(\omega)\frac{1}{|\Omega|}\right)\sum_{i\in\mathcal{I}}\left(\bar{x}_{i0}^{h^*}(\omega) - e_{i0}^{h^*}(\omega)\right) =$$
$$= \sum_{i\in\mathcal{I}}\left(\bar{x}_{i0}^{h^*}\left(\omega\right) - e_{i0}^{h^*}\left(\omega\right)\right) \leq 0.$$

Hence, it follows that

$$\sum_{i\in\mathcal{I}}\left(\bar{x}_{i0}^{h}\left(\omega\right) - e_{i0}^{h}\left(\omega\right)\right) \leq 0 \quad \forall \omega \in \Omega \ and\ \forall h \in \mathcal{H}.$$

Further, fixed $F_{t^*}^{j^*}$, we pose $(\tilde{p}_0, \tilde{q}) \in \mathcal{L}_{H+N-1}$ such that

$$\forall \omega \in \Omega : \qquad \tilde{p}_0(\omega) = 0_H \ , \quad \tilde{q}\left(\omega\right) = \begin{cases} \frac{1}{|\Omega|} & if\ F_t^j = F_{t^*}^{j^*} \\ 0 & \forall F_t^j \neq F_{t^*}^{j^*} \end{cases}$$

Being $(\tilde{p}_0, \tilde{q}) \in \Delta_{F_0}$, we can replace it in (4.2.13) and from measurability of z_i one has:

$$\sum_{\omega\in\Omega}\left(\pi(\omega)\tilde{q}(\omega)\sum_{i\in\mathcal{I}}\bar{z}_i\left(\omega\right)\right) = \left(\sum_{\omega\in\Omega}\pi(\omega)\frac{1}{|\Omega|}\right)\sum_{i\in\mathcal{I}}\bar{z}_{iF_t^j} = \sum_{i\in\mathcal{I}}\bar{z}_{iF_t^j} \leq 0$$

Moreover, from the previous result and from (4.2.12), we have for all $t \in \mathcal{T}$

$$\langle\langle \bar{p}_t, \sum_{i\in\mathcal{I}} (\bar{x}_{it} - e_{it})\rangle\rangle_H \leq \langle\langle \bar{p}_t^1, \sum_{i\in\mathcal{I}} \bar{z}_{iF_t^j}\rangle\rangle_1 \leq 0$$

so that by (4.2.10) we get for all $t \in \mathcal{T}$

$$\langle\langle p_t, \sum_{i\in\mathcal{I}} (\bar{x}_{it} - e_{it})\rangle\rangle_H \leq 0 \qquad \forall p_t \in \Delta_{F_t^j} \tag{4.2.14}$$

Fixed a F_t^j, we pose $\tilde{p}_t \in \mathcal{L}_H$ such that

$$\forall \omega \in F_t^j, \qquad \tilde{p}_t^h(\omega) = \begin{cases} \frac{1}{|F_t^j|} & if\ h = h^* \\ 0 & \forall h \neq h^* \end{cases} .$$

Being $\tilde{p}_t \in \Delta_{F_t^j}$, we can replace it in (4.2.14) and since x_i and e_i are F_t-measurable one has:

$$\sum_{\omega\in F_t^j}\left(\pi(\omega)\tilde{p}_t(\omega)\sum_{i\in\mathcal{I}}(\bar{x}_{it}^{h^*}(\omega) - e_{it}^{h^*}(\omega))\right) = \left(\sum_{\omega\in F_t^j}\pi(\omega)\frac{1}{\left|F_t^j\right|}\right)\sum_{i\in\mathcal{I}}\left(\bar{x}_{it}^{h^*}(\omega) - e_{it}^{h^*}(\omega)\right) =$$
$$= \sum_{i\in\mathcal{I}}\left(\bar{x}_{it}^{h^*}(\omega) - e_{it}^{h^*}(\omega)\right) \leq 0.$$

Hence, it follows that for all $t \in \mathcal{T}$ and for all $h \in \mathcal{H}$

$$\sum_{i\in\mathcal{I}}\left(\bar{x}_{it}^{h}(\omega) - e_{it}^{h}(\omega)\right) \leq 0 \quad \forall \omega \in \Omega\,.$$

Claim 3 *For all $\omega \in \Omega$, $\sum_{i\in\mathcal{I}} \bar{z}_i(\omega) = 0$.*
From Proposition 1 and Claim 2, for all $\omega \in \Omega$, we have $\bar{q}(\omega) > 0$ and $\sum_{i\in\mathcal{I}} \bar{z}_i(\omega) \leq 0$ for all , hence $\langle\langle \bar{q}, \sum_{i\in\mathcal{I}} \bar{z}_i\rangle\rangle_{N-1} \leq 0$. If we suppose that $\left\langle \bar{q}(\omega), \sum_{i\in\mathcal{I}} \bar{z}_i(\omega)\right\rangle_{N-1} < 0$ for some $\omega \in \Omega$, from Proposition 11 one has

$$\langle \bar{p}_0(\omega), \sum_{i\in\mathcal{I}} (\bar{x}_{i0}(\omega) - e_{i0}(\omega))\rangle_H > 0$$

which, being $p_0 \in \Delta_{F_0}$, contradicts Claim 1.
Then, one has $\langle\langle \bar{q}, \sum_{i\in\mathcal{I}} \bar{z}_i\rangle\rangle_{N-1} = 0$ and since $\bar{q}(\omega) > 0$ for all $\omega \in \Omega$, we get $\sum_{i\in\mathcal{I}} \bar{z}_i(\omega) = 0$ for all $\omega \in \Omega$.

Then, thanks to Claim 1, 2 and 3, and Remark 6 if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a solution to SQVI (4.2.1), then it is an equilibrium solution. Moreover, if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium solution, from (4.2.7) and (4.2.8), condition (4.2.3) hold, and from Claim 1 (4.2.2) is satisfied. Then $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a solution to (4.2.1). $\square$

From theoretical and computational viewpoints, sometimes it will be useful to relax the nonanticipativity constraints of the decision variables. In doing this, we get the tools to formulate an equivalent problem allowing for point-wise optimization. We pose $\mathcal{M}^1 = (\mathcal{N}^1)^{\perp}$ and $\mathcal{M}^2 = (\mathcal{N}^2)^{\perp}$ respectively the subspaces of the nonanticipativity multipliers relative to x and z and we pose $\mathcal{M} := \mathcal{M}^1 \times \mathcal{M}^2$, so that $\rho = (\rho^1, \rho^2) \in \mathcal{M}$.
Hence, for the Riesz orthogonal decomposition, one has $\mathcal{L}_C = \mathcal{N}^1 + (\mathcal{N}^1)^{\perp}$ and $\mathcal{L}_{N-1} = \mathcal{N}^2 + (\mathcal{N}^2)^{\perp}$, that is

$$\mathcal{L}_C = \mathcal{N}^1 + \mathcal{M}^1 \qquad \mathcal{L}_{N-1} = \mathcal{N}^2 + \mathcal{M}^2. \tag{4.2.15}$$

We fix $(p,q) \in \Delta$ and we introduce the following stochastic Variational inequality in *Extensive Form*

$$\text{Find } (\bar{x}_i, \bar{z}_i) \in \mathcal{N} \text{ such that exists } \bar{\rho}_i \in \mathcal{M} \text{ and for all } \omega \in \Omega \text{ one has}$$
$$\left\langle \left(\nabla f_{i\omega}(\bar{x}_i) + \bar{\rho}_i^1(\omega), \bar{\rho}_i^2(\omega)\right), (x_i(\omega), z_i(\omega)) - (\bar{x}_i(\omega), \bar{z}_i(\omega)) \right\rangle_D \leq 0 \tag{4.2.16}$$
$$\forall\, (x_i(\omega), z_i(\omega)) \in B_{i\omega}(\bar{p}, \bar{q}).$$

Proposition 12. *The stochastic variational problems (4.2.16) and (4.2.2) are equivalent.*

Proof. We suppose that $(\bar{x}_i, \bar{z}_i)$ is a solution to (4.2.16); for each $\omega \in \Omega$, it follows that

$$\begin{aligned}
&\langle \left(\nabla f_{i\omega}(\bar{x}_i) + \bar{\rho}_i^1(\omega), \bar{\rho}_i^2(\omega)\right), (x_i(\omega), z_i(\omega)) - (\bar{x}_i(\omega), \bar{z}_i(\omega)) \rangle_D = \\
=&\langle \nabla f_{i\omega}(\bar{x}_i), x_i(\omega) - \bar{x}_i(\omega) \rangle\rangle_C + \langle\langle \bar{\rho}_i^1(\omega), x_i(\omega) - \bar{x}_i(\omega) \rangle_C + \\
&+ \langle \bar{\rho}_i^2(\omega), z_i(\omega) - \bar{z}_i(\omega) \rangle_{N-1} \leq 0 \quad \forall\, (x_i(\omega), z_i(\omega)) \in B_{i\omega}(\bar{p}, \bar{q}).
\end{aligned}$$

We multiply for $\pi(\omega)$ and we sum up to ω; one has

$$\langle\langle \nabla \mathcal{U}(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C + \langle\langle \bar{\rho}_i^1, x_i - \bar{x}_i \rangle\rangle_C + \langle\langle \bar{\rho}_i^2, z_i - \bar{z}_i \rangle\rangle_{N-1} \leq 0 \quad \forall\, (x_i, z_i) \in B_i(\bar{p}, \bar{q}). \tag{4.2.17}$$

Moreover, since $\bar{\rho}_i^1 \in \mathcal{M}^1 = (\mathcal{N}^1)^{\perp}$ and $\bar{\rho}_i^2 \in \mathcal{M}^2 = (\mathcal{N}^2)^{\perp}$, from (4.2.17) one has

$$\langle\langle \nabla \mathcal{U}(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C \leq 0 \quad \forall\, (x_i, z_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N} \tag{4.2.18}$$

Hence, $(\bar{x}_i, \bar{z}_i)$ is a solution to (4.2.2).
Being $B_{i\omega}(\bar{p}, \bar{q})$ a polyhedron for each $\omega \in \Omega$, from Theorem 13, the converse still holds. $\square$

Thanks to Proposition 12 we can characterize the equilibrium vector as a solution to a variational problem in extensive form.

Corollary 2. *For all $i \in \mathcal{I}$ and $\omega \in \Omega$, let Assumptions F be satisfied. Then, $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (B(\bar{p}, \bar{q}) \cap \mathcal{N}) \times \Delta$ is a solution of the stochastic variational problem*

$$\sum_{i\in\mathcal{I}} \left\langle \left(\nabla f_{i\omega}\left(\bar{x}_i\right) + \bar{\rho}_i^1\left(\omega\right), \bar{\rho}_i^2\left(\omega\right)\right), \left(x_i\left(\omega\right), z_i\left(\omega\right)\right) - \left(\bar{x}_i\left(\omega\right), \bar{z}_i\left(\omega\right)\right)\right\rangle_D +$$
$$+ \langle (\sum_{i\in\mathcal{I}} \left(\bar{x}_i(\omega) - e_i(\omega)\right), \sum_{i\in\mathcal{I}} \bar{z}_i(\omega)), (p(\omega), q(\omega)) - (\bar{p}(\omega), \bar{q}(\omega))\rangle_D \leq 0 \tag{4.2.19}$$
$$\forall \left(x_i\left(\omega\right), z_i\left(\omega\right), p(\omega), q(\omega)\right) \in B_{i\omega}\left(\bar{p}, \bar{q}\right) \times \Delta_\omega$$

for all $\omega \in \Omega$ and for some $(\bar{\rho}^1, \bar{\rho}^2) \in \mathcal{M}$ if and only if it is an equilibrium vector of plans, prices, and price expectations for $\mathcal{E}$.

Proof. From Proposition 12 condition (4.2.16) is equivalent to the variational problem (4.2.2) which is equivalent to the equilibrium conditions. □

Proposition 13. *For each $i \in \mathcal{I}$, the set-valued map $B_i : \Delta \rightrightarrows \mathcal{L}$ is lower semicontinuous, closed and with nonempty, closed, and convex values.*

Proof. *B_i is with nonempty, closed, and convex values.*
We fix $(p, q) \in \Delta$. Since $(e_i, 0_{\mathcal{L}}) \in B_i(p, q)$, it follows that $B_i(p, q)$ is nonempty and, from definition, $B_i(p, q)$ is a convex set.
Let $\{(x_{i,n}, z_{i,n})\}_{n\in\mathbb{N}} \in B_i(p, q)$ such that $(x_{i,n}, z_{i,n}) \xrightarrow{\mathcal{L}} (x_i, z_i)$. For each $n \in \mathbb{N}$, when $(x_{i,n}, z_{i,n}) \in B_i(p, q)$ one has $(x_{i,n}(\omega), z_{i,n}(\omega)) \in B_{i\omega}(p, q)$ for each $\omega \in \Omega$, that is

$$\begin{aligned} &0 \leq \left\langle p_0\left(\omega\right), x_{i0,n}\left(\omega\right)\right\rangle_H \leq -\left\langle q\left(\omega\right), z_{i,n}\left(\omega\right)\right\rangle_{N-1} + \left\langle p_0\left(\omega\right), e_{i0}\left(\omega\right)\right\rangle_H \\ &0 \leq \left\langle p_t\left(\omega\right), x_{it,n}\left(\omega\right)\right\rangle_H \leq \left\langle p_t\left(\omega\right), e_{it}\left(\omega\right)\right\rangle_H + p_t^1\left(\omega\right) z_{it,n}(\omega) \qquad \forall t \in \mathcal{T}. \end{aligned} \tag{4.2.20}$$

Since $z_{i,n}\left(\omega\right) \in R(\omega)$ one has that, for all $\omega \in \Omega$, $\{z_{i,n}(\omega)\}_{n\in\mathbb{N}}$ converges to $z_i\left(\omega\right)$. Hence, from (4.2.20), one has that the sequence $\{x_{i,n}\}_{n\in\mathbb{N}}$ is bounded and converges to x_i. Then $(x_i(\omega), z_i(\omega)) \in B_{i\omega}(p, q)$, for all $\omega \in \Omega$, and $(x_i, z_i) \in B_i(p, q)$.

B_i is a closed map.
Firstly, we observe that since $\{(p_n, q_n)\}_{n\in\mathbb{N}} \subseteq \Delta$, one has, for each $\omega \in \Omega$, $\{(p_{0,n}(\omega), q_n(\omega))\}_{n\in\mathbb{N}} \subseteq \Delta^0_\omega$ and $\{p_{t,n}(\omega)\}_{n\in\mathbb{N}} \subseteq \Delta^t_\omega$ for each $t \in \mathcal{T}$; hence this sequence converges to $(p(\omega), q(\omega))$. So, in similar way to the above Claim, one has $(x_i, z_i) \in B_i(p, q)$, that is B_i is a closed map.

B_i is lower semicontinuous.
Let $\{(p_n, q_n)\}_{n\in\mathbb{N}} \subseteq \Delta$ be converging to (p, q); for all $(x_i, z_i) \in B_i(p, q)$ we have to prove that there exists $\{(x_{i,n}, z_{i,n})\}_{n\in\mathbb{N}} \in \mathcal{L}$ such that $(x_{i,n}, z_{i,n}) \in B_i(p_n, q_n)$ and $(x_{i,n}, z_{i,n}) \xrightarrow{\mathcal{L}} (x_i, z_i)$. It is clear that, for all $\omega \in \Omega$, we can consider $(x_i(\omega), z_i(\omega))$

and it is sufficient to find $(x_{i,n}(\omega), z_{i,n}(\omega)) \in B_{i\omega}(p_n, q_n)$ such that $(x_{i,n}(\omega), z_{i,n}(\omega)) \to (x_i(\omega), z_i(\omega))$. Fixed $\omega \in \Omega$, if

$$\begin{aligned} &\langle p_0(\omega), x_{i0}(\omega)\rangle_H + \langle q(\omega), z_i(\omega)\rangle_{N-1} < \langle p_0(\omega), e_{i0}(\omega)\rangle_H \\ &\langle p_t(\omega), x_{it}(\omega)\rangle_H < \langle p_t(\omega), e_{it}(\omega)\rangle_H + p_t^1(\omega) z_{it}(\omega) \qquad t \in \mathcal{T} \end{aligned} \tag{4.2.21}$$

then

$$\begin{aligned} &\langle p_{0,n}(\omega), x_{i0}(\omega)\rangle_H + \langle q_n(\omega), z_i(\omega)\rangle_{N-1} < \langle p_{0,n}(\omega), e_{i0}(\omega)\rangle_H \\ &\langle p_{t,n}(\omega), x_{it}(\omega)\rangle_H < \langle p_{t,n}(\omega), e_{it}(\omega)\rangle_H + p_{t,n}^1(\omega) z_{it}(\omega) \qquad t \in \mathcal{T}. \end{aligned}$$

Hence $(x_i(\omega), z_i(\omega)) \in B_{i\omega}(p_n, q_n)$ and then $(x_i(\omega), z_i(\omega)) \in LiB_{i\omega}(p_n, q_n)$, where we identify with $LiB_{i\omega}(p_n, q_n)$ the lower limit, in Kuratowski sense, of the sequence $B_{i\omega}(p_n, q_n)$. If $(x_i(\omega), z_i(\omega))$ is such that at least one inequality (4.2.21) is not satisfied, being $e_i(\omega) \in \mathbb{R}^C_{++}$, there exists $x_i(\omega) \in \mathbb{R}^C_{++}$ such that $(x_i(\omega), 0_{N-1})$ belong to the relative interior in $\mathbb{R}^C_{++} \times R(\omega)$, then $B_{i\omega}(p, q) = cl\ int\ B_{i\omega}(p, q)$. It follows that

$$B_{i\omega}(p, q) = cl\ int\ B_{i\omega}(p, q) \subset cl\ LiB_{i\omega}(p_n, q_n) = LiB_{i\omega}(p_n, q_n).$$

since the set $LiB_{i\omega}(p_n, q_n)$ is a closed set (see [45], Prop.8.2.1). Hence, we can conclude that B_i is lower semicontinuous. □

Theorem 24. *For each $\omega \in \Omega$ and $i \in \mathcal{I}$, let Assumptions F be satisfied. There exists an equilibrium vector of plans, prices, and price expectations for $\mathcal{E}$.*

Proof. In order to prove the existence of equilibrium, thank to Theorem 23, we prove that the SQVI (4.2.1) admits at least one solution. For each $\omega \in \Omega$ and $(p(\omega), q(\omega)) \in \Delta_\omega$, we introduce the bounded set

$$\widetilde{B}_{i\omega}(p, q) := \prod_{i\in\mathcal{I}} \left[B_{i\omega}(p, q) \cap \left(\left[0, \sum_{i\in\mathcal{I}} e_i(\omega) + \widetilde{M} \right] \times \mathbb{R}^{N-1} \right) \right] \tag{4.2.22}$$

where $\widetilde{M} \in \mathbb{R}_+$. We observe that from properties of map $B_{i\omega}$, proved in Proposition 15, the map $\widetilde{B}_{i\omega}$ is lower semicontinuous, closed, and with nonempty, closed, and convex values. We denote by $SQVI(\widetilde{B})$ the variational problem (4.2.1) in the convex set $\widetilde{B}(p, q)$.
There exists the solution of $SQVI(\widetilde{B})$.
For each $i \in \mathcal{I}$ and $(p, q) \in \Delta$, we consider the parametric stochastic Variational inequality SQVI(p, q):

$$\text{Find } (\bar{x}_i, \bar{z}_i) \in \widetilde{B}_i(p, q) \cap \mathcal{N} \text{ such that}$$

$$\langle\langle \nabla \mathcal{U}_i(\bar{x}_i), x_i - \bar{x}_i \rangle\rangle_C \leq 0 \qquad \forall (x_i, z_i) \in \widetilde{B}_i(p, q) \cap \mathcal{N}. \tag{4.2.23}$$

We introduce the map of solutions $\Phi_i : \Delta \rightrightarrows \mathcal{L}$ such that, for all $(p,q) \in \Delta$,

$$\Phi_i(p,q) := \{(\bar{x}_i, \bar{z}_i) : (\bar{x}_i, \bar{z}_i) \;\; is \;\; solution \;\; of \;\; SVI(p,q) \; (4.2.23)\}$$

and we pose $\Phi(p,q) := \prod_{i\in\mathcal{I}} \Phi_i(p,q)$. From Proposition 10, it follows that operator $\nabla\mathcal{U}_i$ is continuous and monotone; moreover, since $(e_i, 0) \in \widetilde{B}_i(p,q)$, which is measurable, we get $\widetilde{B}_i(p,q) \cap \mathcal{N} \neq \emptyset$. Thanks to Theorem 14, it follows that for all $(p,q) \in \Delta$ $\Phi_i(p,q)$ is nonempty, bounded, closed, and convex. We prove that Φ_i is closed. Let $\{(p_n, q_n)\} \subseteq \Delta$ and $\{(\bar{x}_{in}, \bar{z}_{in})\} \subseteq \mathcal{L}$ be two sequences with $(\bar{x}_{in}, \bar{z}_{in}) \in \Phi_i(p_n, q_n)$, and such that $(p_n, q_n) \xrightarrow{\mathcal{L}} (p,q)$ and $(\bar{x}_{in}, \bar{z}_{in}) \xrightarrow{\mathcal{L}} (\bar{x}, \bar{z})$, we have to prove that $(\bar{x}_i, \bar{z}_i) \in \Phi_i(p,q)$. Since $\widetilde{B}_i$ is a closed map then $(\bar{x}_i, \bar{z}_i) \in \widetilde{B}_i(p,q)$. Being $\widetilde{B}_i$ is lower semicontinuous, it follows that for each $(x_i, z_i) \in \widetilde{B}_i(p,q)$ there exists a sequence $\{(x_{in}, z_{in})\}$ converging to $\{(x_i, z_i)\}$ such that $\{(x_{in}, z_{in})\} \in \widetilde{B}_i(p_n, q_n)$. Since $(\bar{x}_{in}, \bar{z}_{in}) \in \Phi_i(p_n, q_n)$, then

$$\langle\langle \nabla\mathcal{U}_i(\bar{x}_{in}), x_{in} - \bar{x}_{in}\rangle\rangle_C \leq 0 \text{ and passing to the limit } \langle\langle \nabla\mathcal{U}_i(\bar{x}_i), x_i - \bar{x}_i\rangle\rangle_C \leq 0.$$

Hence $(\bar{x}_i, \bar{z}_i) \in \Phi_i(p,q)$. Hence, for each $(p,q) \in \Delta$, since $\Phi(p,q) \subseteq \left(\left[0, \sum_{i\in\mathcal{I}} e_i(\omega) + \widetilde{M}\right]\right) \times R$, it follows that $\Phi_i(p,q)$ is also a compact map. Being $\Phi_i(p,q)$ a closed and compact map, it is upper semicontinuous, and then $\Phi(p,q)$ is upper semicontinuous. Now, we consider the following stochastic generalized variational inequality:

Find $(\bar{p}, \bar{q}) \in \Delta$ such that there exists $(\bar{x}, \bar{z}) \in \Phi(\bar{p}, \bar{q})$ and

$$\langle\langle (\sum_{i\in\mathcal{I}} (\bar{x}_i - e_i), \sum_{i\in\mathcal{I}} \bar{z}_i), (p,q) - (\bar{p}, \bar{q})\rangle\rangle_D \leq 0 \qquad \forall\, (p,q) \in \Delta. \tag{4.2.24}$$

From properties of Δ and Φ and thanks to Theorem 7, there exists $(\bar{p}, \bar{q}) \in \Delta$ and $(\bar{x}, \bar{z}) \in \Phi(\bar{p}, \bar{q})$ solutions to (4.2.24). So, $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \left(\widetilde{B}\,(\bar{p}, \bar{q}) \cap \mathcal{N}\right) \times \Delta$, with $(\overline{p}, \overline{q})$ solution to (4.2.24) and $(\overline{x}, \overline{z}) \in \Phi(\overline{p}, \overline{q})$, is a solution to $SQVI(\widetilde{B})$.

Any solution of the $SQVI(\widetilde{B})$ is a solution of $SQVI$ (4.2.1).
Let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be a solution of $SQVI(\widetilde{B})$. Thanks to Remark 6, it is sufficient to prove that $(\bar{x}_i, \bar{z}_i)$ is a solution to (4.2.2). We suppose that there exists $(\hat{x}_i, \hat{z}_i) \in B_i\,(\bar{p}, \bar{q}) \cap \mathcal{N}$ such that

$$\langle\langle \nabla\mathcal{U}_i(\bar{x}_i), \hat{x}_i - \bar{x}_i\rangle\rangle_C > 0. \tag{4.2.25}$$

Let $\lambda \in [0,1]$ be such that

$$0 < \lambda < \min\left\{1; \frac{\sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \bar{x}_i^h(\omega)}{\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega)} \;\; , \;\; with \;\; h \in \mathcal{H} \;\; s.t. \;\; \hat{x}_i^h(\omega) - \bar{x}_i^h(\omega) > 0\right\}. \tag{4.2.26}$$

and we pose $(\tilde{x}_i, \tilde{z}_i) = \lambda(\hat{x}_i, \hat{z}_i) + (1-\lambda)(\bar{x}_i, \bar{z}_i)$. From convexity of $B(\bar{p}, \bar{q}) \cap \mathcal{N}$ one has $(\tilde{x}_i, \tilde{z}_i) \in B_i(\bar{p}, \bar{q}) \cap \mathcal{N}$ and it results that $(\tilde{x}_i, \tilde{z}_i)$ is still in (4.2.22). Indeed, for each $\omega \in \Omega$ and $h \in \mathcal{H}$, one has:

$$\sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \tilde{x}_i^h(\omega) = \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \lambda\hat{x}_i^h(\omega) - (1-\lambda)\bar{x}_i^h(\omega) =$$
$$= \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \lambda\left[\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega)\right] - \bar{x}_i^h(\omega).$$

Hence, for all $h \in \mathcal{H}$, one has:

(i) if $\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega) = 0 \Rightarrow \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \tilde{x}_i^h(\omega) = \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \bar{x}_i^h(\omega) \geq 0$;

(ii) if $\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega) < 0 \Rightarrow \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \tilde{x}_i^h(\omega) > \sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \bar{x}_i^h(\omega) \geq 0$;

(iii) if $\hat{x}_i^h(\omega) - \bar{x}_i^h(\omega) > 0 \Rightarrow$ from (4.2.26) one has $\sum_{i\in\mathcal{I}} e_i^h(\omega) + \widetilde{M} - \tilde{x}_i^h(\omega) > 0$.

Hence $(\tilde{x}_i, \tilde{z}_i) \in \widetilde{B}(\bar{p}, \bar{q}) \cap \mathcal{N}$ and moreover, from inequality (4.2.25)

$$\langle\langle \nabla\mathcal{U}_i(\bar{x}_i), \tilde{x}_i - \bar{x}_i \rangle\rangle_C = \langle\langle \nabla\mathcal{U}_i(\bar{x}_i), \lambda\hat{x}_i + (1-\lambda)\bar{x}_i - \bar{x}_i \rangle\rangle_C = \lambda\langle\langle \nabla\mathcal{U}_i(\bar{x}_i), \hat{x}_i - \bar{x}_i \rangle\rangle_C > 0\,.$$

This contradicts the fact that $(\bar{x}_i, \bar{z}_i)$ is a solution to SVI (4.2.23). Thus, we can conclude that $(\bar{x}_i, \bar{z}_i, \bar{p}, \bar{q})$ is still a solution of $SQVI$ (4.2.1). □

4.3 Computation Procedure

In this Section, we present a computational procedure to find the equilibrium solution by solving the $SQVI$ (4.2.19). To this aim, we use the same procedure used to prove Theorem 24. Under Assumptions F, for all $i \in \mathcal{I}$ and $\omega \in \Omega$, we build two sequences $\{(\hat{x}^\nu, \hat{z}^\nu)\}_{\nu\in\mathbb{N}} \subseteq \mathcal{L}$ and $\{(\hat{p}^n, \hat{q}^n)\}_{n\in\mathbb{N}} \subseteq \Delta$ which converge to a solution of (4.2.19).
The procedure is structured in two sequential phases. At each phase, we split the stochastic variational problem into a finite number of deterministic ones and we solve them in *parallel*. This allows us to deal efficiently with large-scale problems arising from real-world applications in a dynamic-stochastic framework.

Proposition 14. *For each $(x, z) \in \mathcal{L}$ it follows that $(x, z) = (P_{\mathcal{N}}(x, z)) + (P_{\mathcal{M}}(x, z))$.*

Proof. Firstly, we recall that if V is a closed subspace of an Hilbert space H, fixed $x \in H$ one has

$$\langle y - x, v \rangle = 0 \qquad \forall v \in V \quad iff \quad y = P_V(x) \tag{4.3.1}$$

where $P_V(x)$ is the orthogonal projection of $x \in H$ on V. Indeed, if y is a solution of (4.3.1), then let $v = r - y$, for all $r \in V$, it follows the (b) of Th.1.5.5. in [25], that is $P_V(x)$ is the unique vector $y \in V$ such that $\langle y - x, r - y \rangle \geq 0$ for all $r \in V$. Conversely, by taking $r = y + \lambda v$, with $\lambda \in \mathbb{R}$ and $v \in V$, we get

$$\langle y - x, y + \lambda v - y \rangle \geq 0 \to \lambda \langle y - x, v \rangle \geq 0 \to \langle y - x, v \rangle = 0 \quad \forall v \in V$$

being that λ can be any real number.
For all $(x, z) \in \mathcal{L}$, since $\mathcal{L} = \mathcal{N} + \mathcal{M}$ where $\mathcal{M} = (\mathcal{N})^{\perp}$ one has there exists $(\hat{x}, \hat{z}) \in \mathcal{N}$ and $(\rho^1, \rho^2) \in \mathcal{M}$ such that

$$(x, z) = (\hat{x}, \hat{z}) + (\rho^1, \rho^2). \tag{4.3.2}$$

Since $\mathcal{M} = (\mathcal{N})^{\perp}$, one has

(i) for all $(v_1, v_2) \in \mathcal{M}$ and from (4.3.1)

$$\langle\langle (\rho^1, \rho^2) - (x, z), (v_1, v_2) \rangle\rangle = -\langle\langle (\hat{x}, \hat{z}), (v_1, v_2) \rangle\rangle = 0 \quad \Rightarrow \quad (\rho^1, \rho^2) = P_{\mathcal{M}}(x, z)\ ;$$

(ii) for all $(v_1, v_2) \in \mathcal{N}$ and from (4.3.1)

$$\langle\langle (\hat{x}, \hat{z}) - (x, z), (v_1, v_2) \rangle\rangle = -\langle\langle (\rho^1, \rho^2), (v_1, v_2) \rangle\rangle = 0 \quad \Rightarrow \quad (\hat{x}, \hat{z}) = P_{\mathcal{N}}(x, z)\ .$$

Hence, from (i), (ii) and (4.3.2) it follows that: $(x, z) = (P_{\mathcal{N}}(x, z)) + (P_{\mathcal{M}}(x, z))$. □

Procedure: Phase 1

In the first phase, we fix $(p, q) \in \Delta$ and we solve the parametric stochastic variational inequality (4.2.23). We use the procedure known in the literature as Progressive Hedging Algorithm, which allows us to split the variational problem, which is set in the space of functions $\mathcal{L}$, into $|\mathcal{I}| \cdot |\Omega| = IS$ variational problems $[SVI\,(i, \omega)]$ in $\mathbb{R}^D$.

Progressive Hedging Algorithm
We introduce two sequences $\{(\hat{x}^\nu, \hat{z}^\nu)\}_{\nu \in \mathbb{N}} \subseteq \mathcal{L}$ and $\{\rho^\nu\}_{\nu \in \mathbb{N}} \subseteq \mathcal{M}$:
let $\rho^0 = 0$ as starting point, $r > 0$ a fixed parameter and $\nu \in \mathbb{N}$ an iteration index.

$$\boxed{\nu = 1}$$

(i) `Choice of` $(\hat{x}^1, \hat{z}^1) \in \mathcal{L}$. For all $i \in \mathcal{I}$ and $\omega \in \Omega$, we consider the $[SVI\,(i,\omega)]$:

$$\left\langle \nabla f_{i\omega}\left(\hat{x}_i^1\right), x_i(\omega) - \hat{x}_i^1(\omega)\right\rangle_C \leq 0 \qquad \forall\,(x_i\,(\omega)\,, z_i\,(\omega)) \in \widetilde{B}_{i\omega}(p,q) \tag{4.3.3}$$

Since the operator is continuous and $\widetilde{B}_{i\omega}(p,q)$ is a bounded set, there exists at least one solution of (4.3.3). We choose $(\hat{x}_i^1\,(\omega)\,, \hat{z}_i^1\,(\omega))$ arbitrarily, among the solution set of (4.3.3).

(ii) We pose $(\tilde{x}_i^1, \tilde{z}_i^1) = P_{\mathcal{N}}\,(\hat{x}_i^1, \hat{z}_i^1)$ and $\rho_i^1 = rP_{\mathcal{M}}\,(\hat{x}_i^1, \hat{z}_i^1)$. We denote by $P_{\mathcal{N}}\,(\hat{x}_i^1, \hat{z}_i^1)$ and $P_{\mathcal{M}}\,(\hat{x}_i^1, \hat{z}_i^1)$ the projection of $(\hat{x}_i^1, \hat{z}_i^1)$ to sets, respectively, $\mathcal{N}$ and $\mathcal{M}$.

$$\boxed{\forall\ \nu \in \mathbb{N}}$$

(i) `Choice of` $(\hat{x}^\nu, \hat{z}^\nu) \in \mathcal{L}$. For all $i \in \mathcal{I}$ and $\omega \in \Omega$, we consider the stochastic variational problem

$$\begin{gathered}\langle \nabla f_{i\omega}\,(\hat{x}_i^\nu, \hat{z}_i^\nu) + \rho_i^{\nu-1}\,(\omega) \\ +r\left[(\hat{x}_i^\nu\,(\omega)\,, \hat{z}_i^\nu\,(\omega)) - \left(\tilde{x}_i^{\nu-1}\,(\omega)\,, \tilde{z}_i^{\nu-1}\,(\omega)\right)\right], (x_i\,(\omega)\,, z_i\,(\omega)) - (\hat{x}_i\,(\omega)\,, \hat{z}_i\,(\omega))\rangle_D \leq 0 \\ \forall (x_i\,(\omega)\,, z_i\,(\omega)) \in \widetilde{B}_{i\omega}(p,q)\,.\end{gathered} \tag{4.3.4}$$

The operator is strongly monotone, then there exists a unique solution $(\hat{x}_i^\nu\,(\omega)\,, \hat{z}_i^\nu\,(\omega))$. Hence, we set $(\hat{x}_i^\nu, \hat{z}_i^\nu) \in \mathcal{L}$ such that, for all $\omega \in \Omega$, $(\hat{x}_i^\nu\,(\omega)\,, \hat{z}_i^\nu\,(\omega))$ is the unique solution to (4.3.4).

(ii) We pose $(\tilde{x}_i^\nu, \tilde{z}_i^\nu) = P_{\mathcal{N}}\,(\hat{x}_i^\nu, \hat{z}_i^\nu)$ and $\rho_i^\nu = \rho_i^{\nu-1} + rP_{\mathcal{M}}\left(\hat{x}_i^{\nu-1}, \hat{z}_i^{\nu-1}\right)$.

Convergence
From Theorem 2 of [60] it follows that $\{(\hat{x}^\nu, \hat{z}^\nu)\} \xrightarrow{\mathcal{L}} (\bar{x}, \bar{z}) \in \mathcal{N}$ *and* $\hat{\rho}^\nu \xrightarrow{\mathcal{L}} \bar{\rho} \in \mathcal{M}$. Moreover, $(\bar{x}, \bar{z})$ is a solution to the parametric SVI in extensive form (4.2.16) and, thanks to Proposition 12, $(\bar{x}, \bar{z}) \in \widetilde{B}(p,q) \cap \mathcal{N}$ is a solution to (4.2.23). We call $(\bar{x}, \bar{z})$ as *optimal strategy solution.*

Procedure: Phase 2

In this phase we use the Projected Subgradient Algorithm to solve the SVI (4.2.24), where for all $(p,q) \in \Delta$, $(\bar{x}(p,q), \bar{z}(p,q))$ is the optimal strategy solution obtained in Phase 1. We pose

$$\varphi(p,q) := -(\varphi_1(p,q), \varphi_2(p,q)) \quad \varphi_1(p,q) := \sum_{i\in\mathcal{I}} (\bar{x}_i(p,q) - e_i)\ \ ,\ \ \varphi_2(p,q) := \sum_{i\in\mathcal{I}} \bar{z}_i(p,q)$$

and, for each $\omega \in \Omega$, we consider the problem

$$\text{Find } (\bar{p}(\omega), \bar{q}(\omega)) \in \Delta_\omega \text{ such that}$$
$$\langle \varphi_\omega(\bar{p}, \bar{q}), (p(\omega), q(\omega)) - (\bar{p}(\omega), \bar{q}(\omega)) \rangle_D \geq 0 \qquad \forall (p(\omega), q(\omega)) \in \Delta_\omega \,. \tag{4.3.5}$$

Thanks to the structure of Δ and the measurability of $(\bar{x}(p,q), \bar{z}(p,q))$, we can consider the S deterministic variational problems (4.3.5) in $\mathbb{R}^D$ and solving them in parallel. We introduce the Auslender's gap function (see, e.g. [3]):

$$\begin{array}{rcl} \Psi_\omega : & \Delta_\omega \rightarrow & \mathbb{R} \\ (\tilde{p}(\omega), \tilde{q}(\omega)) \rightarrow & \Psi_\omega(\tilde{p}, \tilde{q}) = & \max\limits_{(p(\omega), q(\omega)) \in \Delta_\omega} \langle \varphi_\omega(\tilde{p}, \tilde{q}), (\tilde{p}(\omega), \tilde{q}(\omega)) - (p(\omega), q(\omega)) \rangle_D \end{array} \tag{4.3.6}$$

For this map following properties hold. From Theorem 24, since φ_ω is a single-valued map, it follows that φ_ω is continuous; hence, from compactness of Δ_ω, one has that Ψ_ω is well posed. Moreover, from Theorem 8.3. in [57], it follows that operator Ψ_ω is proper, convex and lower semicontinuity being the maximum of a family of affine continue functions and $\Psi_\omega(\tilde{p}, \tilde{q}) \geq 0$ for all $(\tilde{p}, \tilde{q})$. We pose $\partial\, \Psi_\omega$ the subdifferential of Ψ_ω:

$$\begin{array}{l} \partial \Psi_\omega(\tilde{p}, \tilde{q}) = \{ \tau \in \mathbb{R}^D \; : \\ \qquad \Psi_\omega(p,q) - \Psi_\omega(\tilde{p}, \tilde{q}) \geq \langle \tau_\omega, (p(\omega), q(\omega)) - (\tilde{p}(\omega), \tilde{q}(\omega)) \rangle_D \; \forall (p(\omega), q(\omega)) \in \Delta_\omega \} \,. \end{array}$$

From Theorem 3.2.15 in [45], $\partial \Psi_\omega(p,q) \neq \emptyset$ for all $(p,q) \in$ *ri dom* Ψ_ω. Moreover, one has:

$$\Psi_\omega(\tilde{p}, \tilde{q}) = 0 \text{ if and only if } (\tilde{p}(\omega), \tilde{q}(\omega)) \text{ is a solution to (4.3.5).}$$

Indeed, from definition of Ψ_ω, if $\Psi_\omega(\tilde{p}, \tilde{q}) = 0$ one has

$$\begin{array}{l} \max\limits_{(p(\omega), q(\omega)) \in \Delta_\omega} \langle \varphi_\omega(\tilde{p}, \tilde{q}), (\tilde{p}(\omega), \tilde{q}(\omega)) - (p(\omega), q(\omega)) \rangle_D = 0 \quad \Rightarrow \\ \langle \varphi_\omega(\tilde{p}, \tilde{q}), (\tilde{p}(\omega), \tilde{q}(\omega)) - (p(\omega), q(\omega)) \rangle_D \leq 0 \;\; \forall\, (p(\omega), q(\omega)) \in \Delta_\omega \quad \Rightarrow \quad \textit{solution} \quad \textit{to} \quad (4.3.5) \end{array}$$

Conversely, if $(\tilde{p}(\omega), \tilde{q}(\omega))$ is solution to (4.3.5) it follows that

$$\begin{array}{l} \langle \varphi_\omega(\tilde{p}, \tilde{q}), (\tilde{p}(\omega), \tilde{q}(\omega)) - (p(\omega), q(\omega)) \rangle_D \leq 0 \;\; \forall\, (p(\omega), q(\omega)) \in \Delta_\omega \;\; \Rightarrow \\ \max\limits_{(p(\omega), q(\omega)) \in \Delta_\omega} \langle \varphi_\omega(\tilde{p}, \tilde{q}), (\tilde{p}(\omega), \tilde{q}(\omega)) - (p(\omega), q(\omega)) \rangle_D \leq 0 \;\; \Rightarrow \;\; \Psi_\omega(\tilde{p}, \tilde{q}) = 0. \end{array}$$

and, being that $\Psi_\omega(\tilde{p}, \tilde{q}) \geq 0$ for all $(\tilde{p}(\omega), \tilde{q}(\omega)) \in \Delta_\omega$, the equality must holds. Thus, $(\tilde{p}(\omega), \tilde{q}(\omega)) = (p(\omega), q(\omega))$ and $\Psi_\omega(\tilde{p}, \tilde{q}) = 0$. Thus, the same holds for Ψ relatively to stochastic variational problem (4.2.24).

In this way, thanks to the property of (4.3.6), we can transform the variational problem in the following optimization problem

$$\min_{(\tilde{p}(\omega),\tilde{q}(\omega))\in\Delta_\omega} \Psi_\omega(\tilde{p},\tilde{q}) = \Psi_\omega(\bar{p},\bar{q}) = 0$$

so that one can use methods for solving optimization problems to find the solution of (4.3.5). For all $\omega \in \Omega$ and $(\tilde{p}(\omega),\tilde{q}(\omega)) \in \Delta_\omega$, we consider the subdifferential of $\Psi_\omega(\tilde{p},\tilde{q})$

Projected Subgradient Algorithm
We introduce the sequence $\{(\hat{p}^n(\omega),\hat{q}^n(\omega))\} \subseteq \Delta_\omega$. We fix a starting point $(\hat{p}^1(\omega),\hat{q}^1(\omega)) \in \Delta_\omega$; it is usual to consider the centroid of Δ_ω. Clearly, if $\Psi_\omega(\hat{p}^1,\hat{q}^1) = 0$, one has that $(\hat{p}^1(\omega),\hat{q}^1(\omega))$ is a solution to (4.3.5). We suppose that $\Psi_\omega(\hat{p}^1,\hat{q}^1) > 0$.

$$\boxed{n \in \mathbb{N}}$$

`Choice of` $(\hat{p}^{n+1}(\omega),\hat{q}^{n+1}(\omega)) \in \Delta_\omega$. For all $n \in \mathbb{N}$:

$$\left(\hat{p}^{n+1}(\omega),\hat{q}^{n+1}(\omega)\right) = P_{\Delta_\omega}\left((\hat{p}^n(\omega),\hat{q}^n(\omega)) - \tau_\omega^n\rho_\omega^n\right) \tag{4.3.7}$$

where

$$\tau_\omega^n \in \partial\Psi_\omega(\hat{p}^n,\hat{q}^n) \quad \textit{and} \quad \rho_\omega^n = \frac{\Psi_\omega(\hat{p}^n,\hat{q}^n)}{\|\tau_\omega^n\|^2}$$

Also in this case, at each iteration $n \in \mathbb{N}$, the variational sub-problems are solved in *parallel* through a *warm start* procedure, until a suitable solution of (4.2.24) is obtained, that is up to we get for each $\omega \in \Omega$ a limit point $(\hat{p}(\omega),\hat{q}(\omega))$, of the approximating sequence $\{(\hat{p}^n(\omega),\hat{q}^n(\omega))\}_{n\in\mathbb{N}}$, such that $\Psi_\omega(\hat{p},\hat{q}) = 0$.

Convergence
Let $\{(\hat{p}^n,\hat{q}^n)\}_{n\in\mathbb{N}} \subseteq \Delta$ be the sequence such that for all $\omega \in \Omega$, $\{(\hat{p}^n(\omega),\hat{q}^n(\omega))\}_{n\in\mathbb{N}} \subseteq \Delta_\omega$ is given by (4.3.7) with $\{\tau_\omega^n\}_{n\in\mathbb{N}}$ bounded. We prove that the sequence converges to the solution to SVI (4.2.24).
Let $(\bar{p}(\omega),\bar{q}(\omega))$ be a solution to (4.3.5); it is sufficient to prove that for all $\omega \in \Omega$, $\{(\hat{p}^n(\omega),\hat{q}^n(\omega))\}_{n\in\mathbb{N}}$ converges to $(\bar{p}(\omega),\bar{q}(\omega))$. Firstly, we observe that

$$\langle\tau_\omega^n,(\bar{p}(\omega),\bar{q}(\omega)) - (\hat{p}^n(\omega),\hat{q}^n(\omega))\rangle_D \le \Psi_\omega(\bar{p},\bar{q}) - \Psi_\omega(\hat{p}^n,\hat{q}^n) = -\Psi_\omega(\hat{p}^n,\hat{q}^n)\,. \tag{4.3.8}$$

Hence, from (4.3.8), (4.3.7) and from nonexpansivity of projection mapping, it follows that

$$
\begin{aligned}
&\left\|(\hat{p}^{n+1}(\omega),\hat{q}^{n+1}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2=\\
=&\left\|P_{\Delta_\omega}\left((\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-\tau^n_\omega\rho^n_\omega\right)-P_{\Delta_\omega}(\bar{p}(\omega),\bar{q}(\omega))\right\|^2\leq\\
&\leq\left\|(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-\tau^n_\omega\rho^n_\omega-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2=\\
=&\left\|(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2+(\rho^n_\omega)^2\left\|\tau^n_\omega\right\|^2+2\rho^n_\omega\langle\langle\tau^n_\omega,(\bar{p}(\omega),\bar{q}(\omega))-(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))\rangle\rangle\leq\\
\leq&\left\|(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2+(\rho^n_\omega)^2\left\|\tau^n_\omega\right\|^2-2\rho^n_\omega\Psi_\omega(\hat{p}^n,\hat{q}^n)=\\
=&\left\|(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2+\frac{\Psi_\omega(\hat{p}^n,\hat{q}^n)^2}{\left\|\tau^n_\omega\right\|^4}\left\|\tau^n_\omega\right\|^2-2\frac{\Psi_\omega(\hat{p}^n,\hat{q}^n)}{\left\|\tau^n_\omega\right\|^2}\Psi_\omega(\hat{p}^n,\hat{q}^n)=\\
=&\left\|(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2-\frac{\Psi_\omega(\hat{p}^n,\hat{q}^n)^2}{\left\|\tau^n_\omega\right\|^2}\leq\left\|(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2.
\end{aligned}
$$

Hence, the sequence $\left\{\left\|(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2\right\}_{n\in\mathbb{N}}$ is decreasing, and we get

$$0\leq\Psi_\omega(\hat{p}^n,\hat{q}^n)^2\leq\left\|\tau^n_\omega\right\|^2 l_\omega$$

with

$$l_\omega=\Big(\left\|(\hat{p}^{n}(\omega),\hat{q}^{n}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2-\left\|(\hat{p}^{n+1}(\omega),\hat{q}^{n+1}(\omega))-(\bar{p}(\omega),\bar{q}(\omega))\right\|^2\Big).$$

Since $\{\tau^n_\omega\}_{n\in\mathbb{N}}$ is bounded, it follows that $\lim\limits_{n\to+\infty}\Psi_\omega(\hat{p}^n,\hat{q}^n)=\Psi_\omega(\hat{p},\hat{q})=0$, hence $(\hat{p}(\omega),\hat{q}(\omega))$ is a solution to (4.3.5). Then we can conclude that $(\hat{p},\hat{q})$ is a solution to (4.2.24).

So, when $\nu\to\infty$ and $n\to\infty$, we get that the sequences converge to $(\bar{x},\bar{z},\bar{p},\bar{q})\in(\widetilde{B}(\bar{p},\bar{q})\cap\mathcal{N})\times\Delta$. This limit point is still a solution of (4.2.1) and thanks to Theorem 23, it is an equilibrium of plans, prices, and price expectation for economy $\mathcal{E}$.

Chapter 5

A Deregulated Electricity Market with a Continuum of States

The aim of this Chapter is to study, by means of a stochastic variational approach, an electricity market model with multiple trading dates and a continuum of states, motivated by the increasing interest in this topic in the last two decades. To deal with the complications that this case under study involves, we need to generalize and formalize the structure of time-uncertainty-information considered in Chapter 4 by introducing a suitable functional setting relative to an opportunely built filtered probability space. We point out that these complications are introduced only to be as much as close to the realistic case where most real-world phenomena vary with continuity and influence, in a relevant manner, the decision process of the decision-makers. For instance, one could imagine the variability of weather conditions that influence the power generation from renewable resources. In this setting, decisions are so classified on the basis of when they are made and, then, of the information available: we distinguish between *here-and-now* and *wait-and-see* decisions. We pose our attention to the energy procurement problem face by large consumers: they consume, have the opportunity to produce, and the capability of signing contracts. The problem has a strategic nature. Indeed, each agent is allowed to sign opportune contracts, before the uncertainty is resolved and for each future times, as tools to reduce her exposure to risk associated with the volatility of market prices that open at each time after the uncertainty is revealed; however, at the same time, the reduction of the risks associated with the volatility of procurement cost usually comes at the cost of high average prices for the signed contracts. Hence, an optimal mix among the different sources is needed.
The Chapter is organized as follows. Section 5.1 is devoted to the introduction of the electricity market model, and the relative time-uncertainty-information structure. As in Chapter 4, the chosen probabilistic setting allows us to investigate how the information influences the decision processes of the agents and how these

choices evolve over time. In addition, opportune equilibrium conditions for the considered economy are involved, following the philosophy of the Radner scheme. On this basis, in Section 5.2, the resulting equilibrium for the electricity market is studied by means of a suitable stochastic quasi-variational problem. This formulation paves the way to other interesting subsequent studies, closely linked to the computational procedure introduced in Chapter 4.

5.1 Electricity Market Model

In this Section, we introduce an economic equilibrium problem for a deregulated electricity market. In Chapter 9 of [15], it is presented a multistage energy procurement problem, from the decision-making framework of large-consumers, under uncertainty in terms of a finite set of scenarios. By considering a simplified version of this market model, we frame it into an equilibrium problem of plans, prices, and price expectations. All are studied into a general stochastic framework by following Mas-Colell [47], who introduced a Radner equilibrium problem with a continuum of the states of nature and multiple trading dates.
By using the same notation of Chapters 3 and 4 regarding time, now we introduce an economy that, at each time $t \in \mathcal{T}$, is characterized by a continuous set of alternatives $\Omega_t \subset \mathbb{R}$. Let ω_0 be the initial situation at $t = 0$, we pose:

$$\Omega := \{\omega_0\} \times \Omega_1 \times \Omega_2 \times \ldots \times \Omega_T \qquad \text{s.t.} \qquad \omega_s := (\omega_0, \omega_s^1, \omega_s^2, \ldots, \omega_s^T) \in \Omega$$

where each ω_s describes all evolution of the market. In order to model the uncertainty we suppose that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given, where Ω represents the sample space[1], $\mathcal{F}$ is a Borel σ-algebra of events on Ω and $\mathbb{P}$ is a probability measure. We introduce a filtration on $\mathcal{F}$ to describe how the level of information available evolves over time: we pose $\mathcal{F}_0$ the σ-algebra of null events and for all $t \in \mathcal{T}$, $\mathcal{F}_t$ is the sub-σ algebra of $\mathcal{F}$ generated by $\mathcal{F}_0$ and the Borel subsets of Ω that are independent of the last $T - t$ coordinates. The family of sub-σ-algebras $\{\mathcal{F}_t\}_{t \in \mathcal{T}_0}$ represents an increasing filtration:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}\,.$$

The filtration specifies the partial information available at each time: at $t = 0$ the set of events $\mathcal{F}_0$ is essentially deterministic; $\mathcal{F}_t$ represents the family of events observable till t and $\mathcal{F}_T = \mathcal{F}$ means that full information is available at the end of the planning horizon. Furthermore, from the probability measure $\mathbb{P}$ on the continuum set of complete histories ω, for each time t it is possible to derive the

[1]See Appendix for concepts related to probability theory.

probability measure on all partial histories, conditioned by $\mathcal{F}_t$. We assume that, for each $t \in \mathcal{T}_0$, this conditioned probability measure has to be non-atomic (see, e.g. [47]).
All variables of the market are represented by means of a measurable function $f : \Omega \to \mathbb{R}^{G(T+1)}$ where $f_t : \Omega \to \mathbb{R}^G$ represents the variable observed at time t. Let $\mathcal{L}^2((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{R}^{G(T+1)}) = \mathcal{L}_G$ be the Lebesgue space of 2-summable functions and we denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the *expectational inner product* on $\mathcal{L}_G$:

$$\forall f,\, g \in \mathcal{L}_G \qquad \langle\langle f, g \rangle\rangle_{G(T+1)} := \mathbb{E}[\langle f, g \rangle] = \int_\Omega \langle f(\omega), g(\omega) \rangle_{G(T+1)} \; d\,\mathbb{P}(\omega)\,. \quad (5.1.1)$$

When f represents a decision variable, then it depends on the information available time by time; this motivates the introduction of the concept of *progressively measurable function*. We say that a function $f \in \mathcal{L}_G$ is progressively measurable if, for all $\omega = (\omega_0, \omega_1, \omega_2, \ldots, \omega_T) \in \Omega$, f has the form:

$$f(\omega) := (f_0, f_1(\omega_1), f_2(\omega_1, \omega_2), \ldots, f_t(\omega_1, \ldots, \omega_t), \ldots, f_T(\omega))\,. \quad (5.1.2)$$

We denote by $\mathcal{N}_G$ the set of progressively measurable functions, also known as *nonanticipativity subspace*. Equivalently, a function is progressively measurable if for all $t \in \mathcal{T}$ the component f_t is measurable with respect to $\mathcal{F}_t$, hence one has:

$$\mathcal{N}_G := \mathcal{L}^2((\Omega, \mathcal{F}_0, \mathbb{P}), \mathbb{R}^G) \times \cdots \times \mathcal{L}^2((\Omega, \mathcal{F}_t, \mathbb{P}), \mathbb{R}^G) \times \cdots \times \mathcal{L}^2((\Omega, \mathcal{F}_T, \mathbb{P}), \mathbb{R}^G)\,. \quad (5.1.3)$$

Remark 8. *We stress out that in (5.1.2) we do not write the dependence by ω_0. This is due to the fact that it is a deterministic event, introduced in the set up of the model only for mathematical convenience (see, e.g. [67]), and it is the same for all $\omega \in \Omega$. This latter requirement is guaranteed by the structure given to $\mathcal{N}_G$ when it is posed $(\Omega, \mathcal{F}_0, \mathbb{P})$. Indeed, under (5.1.3), one has that $f_0(\omega) = f_0$ for each $\omega \in \Omega$.*

Through this time-uncertain structure, I agents participate in the market: $\mathcal{I} = \{1, \ldots, i, \ldots, I\}$ is the set of consumers. These agents are large consumers: they have significant electricity consumption, the opportunity to produce, and the capability of signing contracts. Decisions of each agent i are distinguished on the basis of when they are made, and, then, of the available information.

Wait-and-see decisions are made at each time when the information is known. This means that, at each $t \in \mathcal{T}_0$, wait-and-see decisions depend only on the partial history of the event ω known up to that time, that is, depends only on $(\omega_0, \omega_1, \cdots, \omega_t)$ and not on $(\omega_{t+1}, \omega_{t+2}, \cdots, \omega_T)$.

- *Production*: For all $t \in \mathcal{T}_0$ the agent produces a strictly positive quantity of electricity $s_{it}(\omega)$.

  $$s_i : \Omega \to \mathbb{R}^{T+1}_{++} \qquad s_i(\omega) = (s_{i0}, s_{i1}(\omega_1), \ldots, s_{it}(\omega_1, \cdot, \omega_t), \ldots, s_{iT}(\omega)) \in \mathbb{R}^{T+1}_{++}\,.$$

  Let $\underline{s}_{it}$ and $\overline{s}_{it}$, with $0 < \underline{s}_{it} < \overline{s}_{it}$, be the minimum and the maximum quantities that agent can produce. We denote by S_i the set of feasible production for agent i:

  $$S_i = \widehat{S}_i \cap \mathcal{N}_1 \qquad \text{where} \qquad \widehat{S}_i := \{s_i \in \mathcal{L}_1 :\ \underline{s}_{it} \le s_{it}(\omega) \le \overline{s}_{it} \ \ \forall \omega \in \Omega\}\,.$$

- *Spot trade*: for all $t \in \mathcal{T}_0$ the agent trades and consumes a nonnegative amount of energy $x_{it}(\omega)$ produced in the market.

  $$x_i : \Omega \to \mathbb{R}^{T+1}_{+} \qquad x_i(\omega) = (x_{i0}, x_{i1}(\omega_1), \ldots, x_{it}(\omega_1, \cdot, \omega_t), \ldots, x_{iT}(\omega)) \in \mathbb{R}^{T+1}_{+}\,.$$

  One has that $x_i \in \mathcal{L}_1 \cap \mathcal{N}_1$.

Furthermore, at each time and in each possible uncertain occurrence, the total spot consumption have not exceed the total energy available in the market from the production:

$$\sum_{i \in \mathcal{I}} x_i(\omega) \le \sum_{i \in \mathcal{I}} s_i(\omega) \qquad a.e.\ \omega \in \Omega\,. \tag{5.1.4}$$

Here-and-now decisions are made at the initial time, before knowing the information on the future occurrences. Since a here-and-now decision $f : \Omega \to \mathbb{R}^{GT(T+1)}$ is taken only in t=0, we have that $f_t \equiv 0$ for all $t \in \mathcal{T}$ and we pose $f_0(\omega) = f_0 = (f_0^0, f_0^1, \ldots, f_0^t, \ldots, f_0^T)$ where $f_0^t \in \mathbb{R}^G$ represents the decision taken at 0 about the time t.

- *Forward contracts*: At the beginning of the planning horizon agent i signs a contract of selling or buying electricity produced in the market for the future time; we denote by $\mathcal{J}$ the set of J forward contracts available at each time t.

  $$z_i : \Omega \to \mathbb{R}^{(JT)(T+1)},\ z_{i0}(\omega) = z_{i0} = (z_{i0}^1, \ldots, z_{i0}^t, \ldots, z_{i0}^T) \in \mathbb{R}^{JT},\ \ z_{it} \equiv 0\ \ \forall t \in \mathcal{T}\,,$$

  where $z_{i0}^t := (z_{i0}^{tj})_{j \in \mathcal{J}} \in \mathbb{R}^J$ represents the quantities of electricity that will be traded at t and z_{i0} is the total forward contracts vector. If $z_{i0}^{tj} < 0$, the quantity $|z_{i0}^{tj}|$ represents the sold energy and if $z_{i0}^{tj} > 0$, it is the amount of energy bought. Let $R > 0$ be a fixed bound (see [51]), we denote by Z_i the set of forward contracts of agent i:

  $$Z_i = \widehat{Z}_i \cap \mathcal{N}_{JT} \qquad \text{where}$$

$$\widehat{Z}_i := \{z_i \in \mathcal{L}_{JT} : z_{it} \equiv 0 \; \forall t \in \mathcal{T}, \; -R \le z_{i0}^{jt}(\omega) \le R \;\; \forall j \in \mathcal{J}, \, t \in \mathcal{T}\} \, .$$

Moreover, since the forward contracts are signed among the agents participating in the market, the total quantity bought must be equal to the total quantity sold, for each time and possible event:

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} z_{i0}^{tj} = 0 \qquad \forall t \in \mathcal{T}. \tag{5.1.5}$$

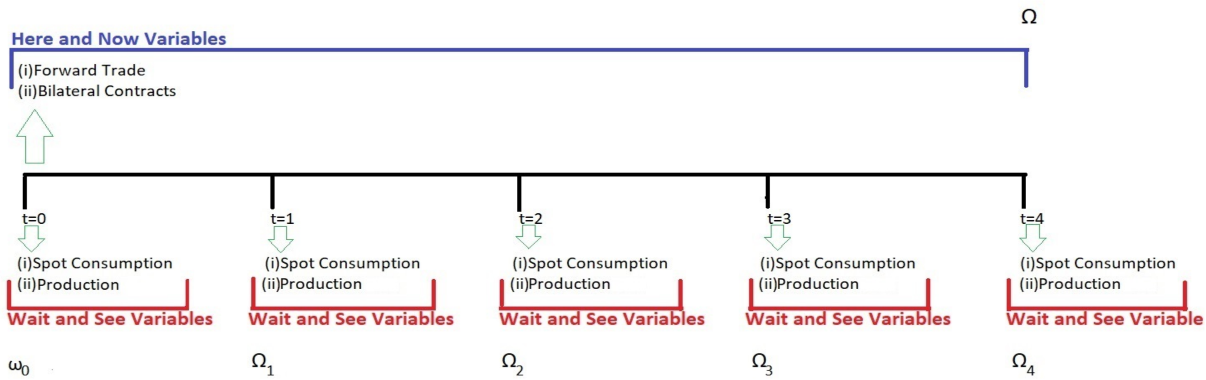
- *Bilateral contracts*: In order to satisfy part of its electricity demand, at the initial time the agent can stipulate private arrangements with suppliers outside the market for all possible future times. Such suppliers can be producers or retailers from which consumers can buy electrical energy prior to its physical delivery. Let $\mathcal{H}$ be the set of H suppliers, we pose

$$b_i : \Omega \to \mathbb{R}_+^{(HT)(T+1)}, \; b_{i0}(\omega) = b_{i0} = (b_{i0}^1, \ldots, b_{i0}^t, \ldots, b_{i0}^T) \in \mathbb{R}_+^{HT}, \;\; b_{it} \equiv 0 \;\; \forall t \in \mathcal{T} \, ,$$

where b_{i0}^{th} is the quantity bought by consumer i from supplier h by means of the bilateral contracts and $b_{i0}^t := (b_{i0}^{th})_{h \in \mathcal{H}} \in \mathbb{R}_+^H$ the total energy from bilateral contract at time t. When $b_{i0}^{th} = 0$ it means that, for time t, consumer i does not stipulate any contract with supplier h. Let $\overline{b}_{i0}^{th}$ be the maximum quantity that agent i can purchase at the beginning from supplier h about stage t. We denote by C_i the set of bilateral contracts of agent i:

$$C_i = \widehat{C}_i \cap \mathcal{N}_{HT} \qquad \text{where}$$

$$\widehat{C}_i := \{b_i \in \mathcal{L}_{HT} : b_{it} \equiv 0 \; \forall t \in \mathcal{T}, \; 0 \le b_{i0}^{th}(\omega) \le \overline{b}_{i0}^{th} \;\; \forall h \in \mathcal{H}, \, t \in \mathcal{T}\} \, .$$



Prices. We denote by $p : \Omega \to \mathbb{R}_{++}^{T+1}$ the function of the spot price of the energy, and $\underline{p}_t$, $\overline{p}_t$ a minimum and a maximum price imposed in the market, with $0 < \underline{p}_t < \overline{p}_t$. We set P the feasible set of spot prices:

$$P := \{p \in \mathcal{N}_1 : \; \underline{p}_t \le p_t(\omega) \le \overline{p}_t \;\; \forall \omega \in \Omega, \; t \in \mathcal{T}_0\} \, .$$

Thanks to relation (5.1.3) one has

$$P = \prod_{t \in \mathcal{T}_0} P_t \qquad where \qquad P_t := \{p_t \in \mathcal{L}^2((\Omega, \mathcal{F}_t, \mathbb{P}), \mathbb{R}) : \ \underline{p}_t \leq p_t(\omega) \leq \overline{p}_t \ \ \forall \omega \in \Omega\} \, .$$

Let $\lambda_i : \Omega \to \mathbb{R}_+^{T+1}$, with $\lambda_i \in \mathcal{N}$, be a given function which describes the production cost to produce the commodity. Furthermore, we denote by $q : \Omega \to \mathbb{R}_+^{JT(T+1)}$ the price of the forward contracts where $q_t \equiv 0$ and $q_0^t := (q_0^{tj})_{j \in \mathcal{J}} \in \mathbb{R}_+^J$ represents the price to sell or buy forward contracts at the initial time about t. We pose

$$Q := \{q \in \mathcal{N}_{JT} : \ q_t \equiv 0, \forall t \in \mathcal{T}, \ 0 \leq q_0^{tj} \leq \overline{q}_0^{tj} \ \ \forall \, t \in \mathcal{T}_0\} \, .$$

Thanks relation (5.1.3), one has

$$Q = \prod_{t \in \mathcal{T}_0} Q_t \qquad where \qquad Q_0 := \{q \in \mathbb{R}_+^{JT} : \ 0 \leq q_0^{tj} \leq \overline{q}_0^{tj}\},$$

$$Q_t = \{q_t \in \mathcal{L}_J : \ q_t \equiv 0\} \quad \forall \, t \in \mathcal{T} \, .$$

Finally, we suppose that the given function $\beta : \Omega \to \mathbb{R}_+^{HT(T+1)}$ describes the prices of the bilateral contracts, with $\beta_t \equiv 0, \forall t \in \mathcal{T}, \ \beta_0 \in \mathbb{R}_+^{HT}$ and $\beta_0^t := (\beta_0^{th})_{h \in \mathcal{H}} \in \mathbb{R}_+^H$ represents the price to buy bilateral contracts at the initial time about t.
For all $i \in \mathcal{I}$ and $t \in \mathcal{T}$, the product $\lambda_{it}(\omega) s_{it}(\omega)$ is the costs of production sustained by agent i to produce the quantity $s_{it}(\omega)$; $p_t(\omega) s_{it}(\omega)$ is the income obtained by selling the quantity $s_{it}(\omega)$ and $p_t(\omega) \sum_{j \in \mathcal{J}} z_{i0}^{tj}$ is the revenue that finance the consumption-trade. At initial time the agent has an additional outcome of the forward and bilateral contracts: $\langle q, z_{i0} \rangle_{JT}$ and $\langle \beta, b_{i0} \rangle_{HT}$. However, if $z_{i0}^{th} < 0$, then $q_0^{th} z_{i0}^{th}$ is an *income* and, if $z_{i0}^{th} > 0$, then $q_0^{th} z_{i0}^{th}$ is an *outcome*. The amount that each agent i can spend is bounded from his income; hence, let $\mathcal{N} := \mathcal{N}_1 \times \mathcal{N}_{JT} \times \mathcal{N}_1 \times \mathcal{N}_{HT}$, at the current price $(p, q) \in P \times Q$ each agent has the budget constraints set

$$\begin{aligned} B_i(p,q) \cap \mathcal{N} := \{(x_i, z_i, s_i, b_i) \in (\mathcal{L}_1 \times \widehat{Z}_i \times \widehat{S}_i \times \widehat{C}_i) \cap \mathcal{N} : \\ (x_i(\omega), z_i(\omega), s_i(\omega), b_i(\omega)) \in B_i(\omega, p, q)\} \end{aligned} \tag{5.1.6}$$

where, it results

$$\begin{aligned} B_i(\omega, p, q) := &\{(x_i(\omega), z_i(\omega), s_i(\omega), b_i(\omega)) \in \mathbb{R}_+^{T+1} \times \mathbb{R}^{JT(T+1)} \times \mathbb{R}_{++}^{T+1} \times \mathbb{R}_+^{HT(T+1)} : \\ &p_0 x_{i0} + \langle q, z_{i0} \rangle_{JT} + \langle \beta, b_{i0} \rangle_{HT} + \lambda_{i0} s_{i0} \leq p_0 s_{i0} \\ &p_t(\omega) x_{it}(\omega) + \lambda_{it}(\omega) s_{it}(\omega) \leq p_t(\omega) s_{it}(\omega) + p_t(\omega) \sum_{j \in \mathcal{J}} z_{i0}^{tj} \ \ \forall t \in \mathcal{T}\} \, . \end{aligned}$$

We pose $\widetilde{B}(p,q) := \prod_{i\in\mathcal{I}} B_i(p,q) \cap \mathcal{N}$. For each $\omega \in \Omega$ and $t \in \mathcal{T}_0$, we consider the function λ_i such that $0 \le \lambda_{it}(\omega) < \underline{p}_t(\omega)$; by setting $s_{it}(\omega) > 0$ the agent always produces a quantity of energy. These assumptions ensure us that the agent i has a profit $(p_t(\omega) - \lambda_{it}(\omega))s_{it}(\omega) > 0$ and then she can participate in the market. Clearly, for each $i \in \mathcal{I}$, one has that $B_i(p,q) \cap \mathcal{N}$ is convex and nonempty. Indeed, one could always select $(\hat{x}_i, 0, \underline{s}_i, 0) \in B_i(p,q) \cap \mathcal{N}$, with $\hat{x}_i = \frac{s_i(p-\lambda_i)}{p}$. Furthermore, we observe that, since $(x_i, z_i, s_i, b_i) \in (\mathcal{L}_1 \times Z_i \times S_i \times C_i) \cap \mathcal{N}$ the constraints in the set $B_i(p,q) \cap \mathcal{N}$ can be written in the unified way for all $t \in \mathcal{T}_0$:

$$p_t(\omega)(x_{it}(\omega) - s_{it}(\omega)) + \langle q_t(\omega), z_{it}(\omega)\rangle_{JT} + \langle \beta_t(\omega), b_{it}(\omega)\rangle_{HT} + \lambda_i(\omega)s_{it}(\omega) \le p_t(\omega) \sum_{j\in\mathcal{J}} z_{i0}^{tj} .$$

Each agent i has utility from the electricity usage, obtained by spot trade and bilateral contracts; this utility is represented by a $u_i : \Omega \times \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{HT(T+1)} \to \mathbb{R}$ such that $u_i(\omega, x_i, b_i)$ is measurable in ω for each fixed (x_i, b_i). The objective of each agent i is to find the decision rules $(x_i, z_i, s_i, b_i) \in B_i(p,q) \cap \mathcal{N}$ which are nonanticipative and which maximize the expected value of utility. Let

$$\mathcal{U}_i : \mathcal{L}_1 \times \mathcal{L}_{HT} \to \mathbb{R} \qquad \text{s.t.} \qquad \mathcal{U}_i(x_i, b_i) = \mathbb{E}[u_i(x_i, b_i)] = \int_\Omega u_i(\omega, x_i, b_i)\, d\mathbb{P}(\omega)$$

then, for each $i \in \mathcal{I}$, it results the following maximization problem with recourse

$$\max_{(x_i,z_i,s_i,b_i)\in B_i(p,q)\cap\mathcal{N}} \mathcal{U}_i(x_i, b_i) = \mathbb{E}[u_i(\tilde{x}_i, \tilde{b}_i)] \; . \tag{5.1.7}$$

Finally, we can give the mathematical formulation of equilibrium.

Definition 22. *An* equilibrium of plans, prices, and price expectations *for the Electricity Market* $\mathcal{E} := ((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{t\in\mathcal{T}_0}, (\mathcal{U}_i)_{i\in\mathcal{I}})$ *is a vector* $\left(\left(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i\right)_{i\in\mathcal{I}}, \tilde{p}, \tilde{q}\right) \in \widetilde{B}(\tilde{p}, \tilde{q}) \times P \times Q$ *such that*

1. *for any* $i \in \mathcal{I}$
$$\max_{(x_i,z_i,s_i,b_i)\in B_i(\tilde{p},\tilde{q})\cap\mathcal{N}} \mathcal{U}_i(x_i, b_i) = \mathbb{E}[u_i(\tilde{x}_i, \tilde{b}_i)] \; ; \tag{5.1.8}$$

2. *a.e.* $\omega \in \Omega$
$$\sum_{i\in\mathcal{I}} \tilde{x}_i(\omega) \le \sum_{i\in\mathcal{I}} \tilde{s}_i(\omega) \; ; \tag{5.1.9}$$

3. *for any* $t \in \mathcal{T}$
$$\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \tilde{z}_i^{tj} = 0. \tag{5.1.10}$$

5.2 A Stochastic Variational Formulation

This Section deals with the connection between an equilibrium vector for $\mathcal{E}$ and the solution of a suitable stochastic quasi-variational problem. To this end, we follow the approaches used in [58, 59, 55].
Firstly, for all $i \in \mathcal{I}$, we made the followings hold.

Assumptions A

(A.1.) $u_i(\cdot, x_i, b_i)$ is measurable for each $\omega \in \Omega$;

(A.2.) $u_i(\omega, \cdot, \cdot)$ is continuously differentiable and concave $a.e.\ \omega \in \Omega$;

(A.3.) for each $(x_i, b_i) \in \mathcal{L}_1 \times \mathcal{L}_{HT}$ there exists $g \in \mathcal{L}_1 \times \mathcal{L}_{HT}$ such that for all $(\hat{x}_i, \hat{b}_i)$, $(\hat{\hat{x}}_i, \hat{\hat{b}}_i) \in \mathcal{L}_1 \times \mathcal{L}_{HT}$ in a neighborhood of (x_i, b_i)

$$\left| u_i(\omega, \hat{x}_i, \hat{b}_i) - u_i(\omega, \hat{\hat{x}}_i, \hat{b}_i) \right| \leq g(\omega) \left\| (\hat{x}_i, \hat{b}_i) - (\hat{\hat{x}}_i, \hat{\hat{b}}_i) \right\| \qquad a.e.\ \omega \in \Omega\ ;$$

(A.4.) u_i is strictly-increasing in x_i.

Remark 9. *Thanks to Th.7.43 and Th.7.44. in [22], Assumptions (A.1.), (A.2.), and (A.3.) ensure the continuity of* $\mathcal{U}_i$*, and the interchangeability of the expectation and differential operator. Indeed, for each* $i \in \mathcal{I}$*, let*

$$\nabla \mathcal{U}_i : \mathcal{L}_1 \times \mathcal{L}_{HT} \to \mathcal{L}_1 \times \mathcal{L}_{HT} \quad s.t. \quad \nabla \mathcal{U}_i(x_i, b_i) = (\nabla_{x_i} \mathcal{U}_i(x_i, b_i), \nabla_{b_i} \mathcal{U}_i(x_i, b_i))$$

one has

$$\nabla \mathcal{U}_i(x_i, b_i) : \Omega \to \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{HT(T+1)} \ \ s.t\ \nabla \mathcal{U}_i(\omega, x_i, b_i) = (\nabla_{x_i} \mathcal{U}_i(\omega, x_i, b_i), \nabla_{b_i} \mathcal{U}_i(\omega, x_i, b_i))\ .$$

So, it results that $\nabla \mathcal{U}_i(x_i, b_i) = \mathbb{E}\left[\nabla u_i(\omega, x_i, b_i)\right]$*, where* $\nabla u_i(x_i, b_i) : \Omega \to \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{HT(T+1)}$*. Furthermore, from Assumptions (A.1.), (A.2.), and (A.4.) it follows the concavity, and the strictly-increasing in* x_i *of* $\mathcal{U}_i$*.*

We introduce the following stochastic quasi-variational inequality:

$$\text{Find } (\tilde{x}, \tilde{z}, \tilde{s}, \tilde{b}, \tilde{p}, \tilde{q}) \in \widetilde{B}(\tilde{p}, \tilde{q}) \times P \times Q \text{ such that}$$

$$\begin{aligned} &\sum_{i \in \mathcal{I}} \langle\langle \nabla u_i(\tilde{x}_i, \tilde{b}_i), (x_i, b_i) - (\tilde{x}_i, \tilde{b}_i) \rangle\rangle_{(HT+1)(T+1)} + \\ &+ \langle\langle (\sum_{i \in \mathcal{I}} (\tilde{x}_i - \tilde{s}_i), \sum_{i \in \mathcal{I}} \tilde{z}_i), (p, q) - (\tilde{p}, \tilde{q}) \rangle\rangle_{(JT+1)(T+1)} \leq 0 \end{aligned} \tag{5.2.1}$$

$$\forall (x, z, s, b, p, q) \in \widetilde{B}(\tilde{p}, \tilde{q}) \times P \times Q.$$

Remark 10. *The vector $(\tilde{x}, \tilde{z}, \tilde{s}, \tilde{b}, \tilde{p}, \tilde{q})$ is a solution to SGQVI (5.2.1) if and only if the following inequalities simultaneously hold:*

(i) for each $i \in \mathcal{I}$, $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i)$ is a solution to

$$\langle\langle \nabla u_i(\tilde{x}_i, \tilde{b}_i), (x_i, b_i) - (\tilde{x}_i, \tilde{b}_i)\rangle\rangle_{(HT+1)(T+1)} \leq 0 \quad \forall (x_i, z_i, s_i, b_i) \in B_i(\tilde{p}, \tilde{q}) \cap \mathcal{N} \; ; \tag{5.2.2}$$

(ii) $\tilde{p}$ is a solution to

$$\langle\langle \sum_{i \in \mathcal{I}} (\tilde{x}_i - \tilde{s}_i), p - \tilde{p}\rangle\rangle_{T+1} \leq 0 \qquad \forall p \in P \; ; \tag{5.2.3}$$

(iii) $\tilde{q}$ is a solution to

$$\langle\langle \sum_{i \in \mathcal{I}} \tilde{z}_i, q - \tilde{q}\rangle\rangle_{JT(T+1)} \leq 0 \qquad \forall q \in Q \; . \tag{5.2.4}$$

Proposition 15. *Fixed $i \in \mathcal{I}$ and $(p, q) \in P \times Q$. Let u_i strictly-increasing in x_i and let $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) \in B_i(p, q) \cap \mathcal{N}$ be a solution to the maximization problem (5.1.8). Then a.e. $\omega \in \Omega$ one has:*

$$p_0 \tilde{x}_{i0} + \langle q, \tilde{z}_{i0}\rangle_{JT} + \langle \beta, \tilde{b}_{i0}\rangle_{HT} + \lambda_{i0} \tilde{s}_{i0} = p_0 \tilde{s}_{i0} \tag{5.2.5}$$

$$p_t(\omega) \tilde{x}_{it}(\omega) + \lambda_{it}(\omega) \tilde{s}_{it}(\omega) = p_t(\omega) \tilde{s}_{it}(\omega) + p_t(\omega) \sum_{j \in \mathcal{J}} \tilde{z}_{i0}^{tj} \qquad \forall t \in \mathcal{T} \; . \tag{5.2.6}$$

Proof. For all $t \in \mathcal{T}_0$ and $\omega \in \Omega$ we pose:

$$\begin{aligned} M_t(\omega) : &= p_t(\omega) \tilde{x}_{it}(\omega) + \langle q_t(\omega), \tilde{z}_{it}(\omega)\rangle_{JT} + \langle \beta_t(\omega), \tilde{b}_{it}(\omega)\rangle_{HT} \\ &+ \lambda_i(\omega) \tilde{s}_{it}(\omega) - p_t(\omega) \tilde{s}_{it}(\omega) - p_t(\omega) \sum_{j \in \mathcal{J}} \tilde{z}_{i0}^{tj} \; . \end{aligned}$$

We have to prove that $M_t(\omega) = 0$ for all $t \in \mathcal{T}_0$ and for all $\omega \in \Omega$. We suppose that there exists $t^* \in \mathcal{T}_0$ such that $M_{t*}(\omega^*) < 0$ for some $\omega^* \in \Omega$, with $\omega^* \in E_{t*} \subset \mathcal{F}_{t*}$ and $\mathbb{P}(E_{t*}) > 0$. Let $\hat{x} : \Omega \to \mathbb{R}_+^{T+1}$, with $\hat{x}_i = \{\hat{x}_{it}\}_{t \in \mathcal{T}_0}$, be such that $\hat{x}_{it} \equiv \tilde{x}_{it}$ for all $t \neq t^*$ and

$$\hat{x}_{it^*}(\omega) := \begin{cases} \tilde{x}_{it^*}(\omega) + K & \forall \omega \in \Omega \text{ s.t. } \omega \in E_{t*} \\ \tilde{x}_{it^*}(\omega) & \textit{otherwise} \end{cases}$$

with $0 < K < -\dfrac{M_{t^*}(\omega^*)}{p_{t^*}(\omega^*)}$. It results $(\hat{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) \in B_i(p, q) \cap \mathcal{N}$. Being u_i strictly-increasing in x_i, from Remark 9, it follows that $\mathcal{U}_i(\hat{x}_i \tilde{b}_i) > \mathcal{U}_i(\tilde{x}_i, \tilde{b}_i)$, contradicting the fact that $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i)$ is a maximum point of $\mathcal{U}_i$ in $B_i(p, q) \cap \mathcal{N}$. $\square$

Proposition 16. *For all $i \in \mathcal{I}$, let Assumptions (A.1.), (A.2.), and (A.3.) be satisfied. For all $(p,q) \in P \times Q$, $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) \in B_i(p,q) \cap \mathcal{N}$ is a solution to (5.1.7) if and only if it is a solution to the following stochastic variational inequality*

$$\langle\langle \nabla u_i(\tilde{x}_i, \tilde{b}_i), (x_i, b_i) - (\tilde{x}_i, \tilde{b}_i) \rangle\rangle_{(HT+1)(T+1)} \qquad \forall (x_i, z_i, s_i, b_i) \in B_i(p,q) \cap \mathcal{N} \, .$$

Proof. Firstly, for all $i \in \mathcal{I}$, let us assume that $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) \in B_i(p,q) \cap \mathcal{N}$ is a solution to (5.1.7); for all $\mu \in [0,1]$, let $(\hat{x}_i, \hat{z}_i, \hat{s}_i, \hat{b}_i) = \mu(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) + (1-\mu)(x_i, z_i, s_i, b_i) \in B_i(p,q) \cap \mathcal{N}$. We define the following functional

$$F(\mu) := \int_\Omega u_i(\omega, \hat{x}_i, \hat{b}_i) \, d\mathbb{P}(\omega)$$

$$\Downarrow$$

$$\forall \mu \in [0,1] \qquad F(\mu) \leq \max_{(x_i,z_i,s_i,b_i) \in B_i(p,q) \cap \mathcal{N}} \int_\Omega u_i(\omega, x_i, b_i) \, d\mathbb{P}(\omega) = \mathbb{E}[u_i(\tilde{x}_i, \tilde{b}_i)] = F(1).$$

It follows that $F'(1) > 0$, and since

$$\begin{aligned} F'(\mu) &= \frac{\partial}{\partial \mu} \int_\Omega u_i(\omega, \hat{x}_i, \hat{b}_i) \, d\mathbb{P}(\omega) = \\ &= \int_\Omega \sum_{t \in \mathcal{T}_0} \frac{\partial u_i(\omega, \hat{x}_i, \hat{b}_i)}{\partial x_{it}} (\tilde{x}_{it}(\omega) - x_{it}(\omega)) + \sum_{t \in \mathcal{T}_0, h \in \mathcal{H}} \frac{\partial u_i(\omega, \hat{x}_i, \hat{b}_i)}{\partial b_{i0}^{th}} (\tilde{b}_{i0}^{th} - b_{i0}^{th}) \, d\mathbb{P}(\omega) \end{aligned} \tag{5.2.7}$$

it results that

$$\begin{aligned} F'(1) &= \int_\Omega \sum_{t \in \mathcal{T}_0} \frac{\partial u_i(\omega, \tilde{x}_i, \tilde{b}_i)}{\partial x_{it}} (\tilde{x}_{it}(\omega) - x_{it}(\omega)) + \sum_{t \in \mathcal{T}_0, h \in \mathcal{H}} \frac{\partial u_i(\omega, \tilde{x}_i, \tilde{b}_i)}{\partial b_{i0}^{th}} (\tilde{b}_{i0}^{th} - b_{i0}^{th}) \, d\mathbb{P}(\omega) = \\ &= \langle\langle \nabla u_i(\tilde{x}_i, \tilde{b}_i), (\tilde{x}_i, \tilde{b}_i) - (x_i, b_i) \rangle\rangle_{(HT+1)(T+1)} \geq 0 \quad \forall (x_i, z_i, s_i, b_i) \in B_i(p,q) \cap \mathcal{N} \end{aligned} \tag{5.2.8}$$

Conversely, for all $i \in \mathcal{I}$, let us assume that $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) \in B_i(p,q) \cap \mathcal{N}$ is a solution to (5.2.2); for all $\mu \in [0,1]$ and $(x_i, z_i, s_i, b_i) \in B_i(p,q) \cap \mathcal{N}$, let $\mu(x_i, z_i, s_i, b_i) + (1-\mu)(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) \in B_i(p,q) \cap \mathcal{N}$. Thanks to Remark 9, from the concavity of $\mathcal{U}_i$, we get:

$$\mathcal{U}_i(\mu(x_i, b_i) + (1-\mu)(\tilde{x}_i, \tilde{b}_i)) \geq \mu \, \mathcal{U}_i(x_i, b_i) + (1-\mu) \, \mathcal{U}_i(\tilde{x}_i, \tilde{b}_i) \qquad \forall \mu \in [0,1]$$

$$\Downarrow$$

$$\frac{\mathcal{U}_i((\tilde{x}_i + \mu(x_i - \tilde{x}_i)), (\tilde{b}_i + \mu(b_i - \tilde{b}_i))) - \mathcal{U}_i(\tilde{x}_i, \tilde{b}_i)}{\mu} \geq \mathcal{U}_i(x_i, b_i) - \mathcal{U}_i(\tilde{x}_i, \tilde{b}_i) \qquad \forall \mu \in (0,1] \tag{5.2.9}$$

When $\mu \to 0^+$, the left-hand side of (5.2.9) converges to $\frac{\partial}{\partial\mu}\mathcal{U}_i((\tilde{x}_i + \mu(x_i - \tilde{x}_i)), (\tilde{b}_i + \mu(b_i - \tilde{b}_i)))$. By posing $\left[\frac{\partial}{\partial\mu}\mathcal{U}_i((\tilde{x}_i + \mu(x_i - \tilde{x}_i)), (\tilde{b}_i + \mu(b_i - \tilde{b}_i)))\right]_{\mu=0}$, we get $\langle\langle\nabla u_i(\tilde{x}_i, \tilde{b}_i), (x_i, b_i) - (\tilde{x}_i, \tilde{b}_i)\rangle\rangle_{(HT+1)(T+1)}$. Since $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) \in B_i(p,q) \cap \mathcal{N}$ is a solution to (5.2.2), then, from (5.2.9), for all $(x_i, z_i, s_i, b_i) \in B_i(p,q) \cap \mathcal{N}$ one has

$$0 \geq \langle\langle\nabla u_i(\tilde{x}_i, \tilde{b}_i), (x_i, b_i) - (\tilde{x}_i, \tilde{b}_i)\rangle\rangle_{(HT+1)(T+1)} \geq \mathcal{U}_i(x_i, b_i) - \mathcal{U}_i(\tilde{x}_i, \tilde{b}_i)\ .$$

Hence, $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i) \in B_i(p,q) \cap \mathcal{N}$ is a solution to (5.1.7). □

Theorem 25. *For all $i \in \mathcal{I}$, let Assumptions A be satisfied. Then, $(\tilde{x}, \tilde{z}, \tilde{s}, \tilde{b}, \tilde{p}, \tilde{q}) \in \widetilde{B}(\tilde{p}, \tilde{q}) \times P \times Q$ is a solution to SGQVI (5.2.1), then it is an equilibrium vector of plans, prices, and price expectations for the Electricity Market $\mathcal{E}$.*

Proof. **Claim 1** *For all $i \in \mathcal{I}$, $(\tilde{x}_i, \tilde{z}_i, \tilde{s}_i, \tilde{b}_i)$ is a solution of the maximization problem (5.1.8) if and only if it is a solution of (5.2.2).*
It follows from Proposition 16.

Claim 2 $\sum_{i\in\mathcal{I}} \tilde{x}_{i0} \leq \sum_{i\in\mathcal{I}} \tilde{s}_{i0}$ *and* $\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{tj} \leq 0$ *for all* $t \in \mathcal{T}$.
Let $(\tilde{x}, \tilde{z}, \tilde{s}, \tilde{b}) \in \widetilde{B}(\tilde{p}, \tilde{q})$. Firstly, we observe that, from Proposition 15, it follows that:

$$\tilde{p}_0 \sum_{i\in\mathcal{I}} (\tilde{x}_{i0} - \tilde{s}_{i0}) + \langle \tilde{q}, \sum_{i\in\mathcal{I}} \tilde{z}_{i0}\rangle_{JT} + \langle \beta, \sum_{i\in\mathcal{I}} \tilde{b}_{i0}\rangle_{HT} + \sum_{i\in\mathcal{I}} \lambda_{i0}\tilde{s}_{i0} = 0$$

and, since $\langle \beta, \tilde{b}_{i0}\rangle_{HT} \geq 0$ and $\lambda_{i0}\tilde{s}_{i0} \geq 0$ for all $i \in \mathcal{I}$, one has:

$$\tilde{p}_0 \sum_{i\in\mathcal{I}} (\tilde{x}_{i0} - \tilde{s}_{i0}) + \langle \tilde{q}, \sum_{i\in\mathcal{I}} \tilde{z}_{i0}\rangle_{JT} \leq 0\,. \tag{5.2.10}$$

In this way, one can consider $\langle\langle \tilde{p}_0, \sum_{i\in\mathcal{I}} (\tilde{x}_{i0} - \tilde{s}_{i0})\rangle\rangle_1 + \langle\langle \tilde{q}, \sum_{i\in\mathcal{I}} \tilde{z}_i\rangle\rangle_{JT} \leq 0$ so that, from (5.2.3) and (5.2.4), one gets:

$$\langle\langle p_0, \sum_{i\in\mathcal{I}} (\tilde{x}_{i0} - \tilde{s}_{i0})\rangle\rangle_1 + \langle\langle q, \sum_{i\in\mathcal{I}} \tilde{z}_i\rangle\rangle_{JT} \leq 0 \qquad \forall (p_0, q) \in P_0 \times Q_0 \tag{5.2.11}$$

We select $(p_0, 0_{JT}) \in P_0 \times Q_0$. Since $\tilde{x}_{i0}(\omega) = \tilde{x}_{i0}$, $\tilde{s}_{i0}(\omega) = \tilde{s}_{i0}$, it results:

$$\sum_{i\in\mathcal{I}} (\tilde{x}_{i0} - \tilde{s}_{i0}) \leq 0.$$

Further, we suppose that there exists a $t^* \in \mathcal{T}$ and $j^* \in \mathcal{J}$ such that $\sum_{i\in\mathcal{I}} \tilde{z}_{i0}^{t^*j^*} > 0$. We select $\hat{q} \in Q$ such that

$$\hat{q} := \begin{cases} \hat{q}_0^{t^*j^*} = K > 0 \\ \hat{q}_0^{tj} = 0 \quad \forall t \neq t^* \quad and \quad j \neq j^* \end{cases}$$

so that, from (5.2.11), one has

$$p_0 \sum_{i\in\mathcal{I}} (\tilde{x}_{i0} - \tilde{s}_{i0}) + K \sum_{i\in\mathcal{I}} \tilde{z}_{i0}^{t^*j^*} \leq 0 \qquad \forall p_0 \in P_0. \tag{5.2.12}$$

For sake of simplicity, we pose:

$$C := \sum_{i\in\mathcal{I}} (\tilde{x}_{i0} - \tilde{s}_{i0}) \qquad \text{and} \qquad D := \sum_{i\in\mathcal{I}} \tilde{z}_{i0}^{t^*j^*}$$

with $C \leq 0$ and $D > 0$. Naturally, if $C = 0$, then it follows that relation (5.2.12) is always strictly positive for each $p_0 \in P_0$, contradicting (5.2.11). Instead, if $C < 0$ we firstly observe the following

$$\underline{p}_0 < \overline{p}_0 \ \Rightarrow \ \overline{p}_0 C < \underline{p}_0 C \ \Rightarrow \ \overline{p}_0 C + KD < \underline{p}_0 C + KD \tag{5.2.13}$$

so that there exists $\epsilon > 0$, with $\epsilon \in [\max\{0, \overline{p}_0 C + KD\}, \underline{p}_0 C + KD]$, for which we can select $\hat{p}_0 \in P_0$:

$$\hat{p}_0 = \frac{KD - \epsilon}{-C} \tag{5.2.14}$$

Indeed, one can show that

$$\overline{p}_0 C + KD \leq \epsilon \leq \underline{p}_0 C + KD \quad \Rightarrow \quad \overline{p}_0 C \leq \epsilon - KD \leq \underline{p}_0 C \quad \Rightarrow \quad \underline{p}_0 \leq \frac{\epsilon - KD}{C} \leq \overline{p}_0 \,,$$

hence $\hat{p}_0 \in P_0$. Furthermore:

$$\hat{p}_0 \sum_{i\in\mathcal{I}} (\tilde{x}_{i0} - \tilde{s}_{i0}) + K \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{t^*j} = \frac{\epsilon - KD}{C} C + KD = \epsilon > 0$$

which contradicts inequality (5.2.12). Hence, for all $t \in \mathcal{T}$ one has

$$\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \tilde{z}_i^{tj} \leq 0.$$

Claim 3 *For all $t \in \mathcal{T}$, $\sum_{i\in\mathcal{I}} \tilde{x}_{it}(\omega) \leq \sum_{i\in\mathcal{I}} \tilde{s}_{it}(\omega)$ a.e. $\omega \in \Omega$.*

Firstly, we observe that, from Proposition 15, *a.e.* $\omega \in \Omega$ one has:

$$\tilde{p}_t(\omega) \sum_{i\in\mathcal{I}} (\tilde{x}_{it}(\omega) - \tilde{s}_{it}(\omega)) + \sum_{i\in\mathcal{I}} \lambda_{it}(\omega) \tilde{s}_{it}(\omega) - \tilde{p}_t(\omega) \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{tj} = 0 \tag{5.2.15}$$

and, since $\lambda_{it}(\omega)\tilde{s}_{it}(\omega) \geq 0$ for all $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{tj} \leq 0$, it follows that:

$$\tilde{p}_t(\omega) \sum_{i\in\mathcal{I}} (\tilde{x}_{it}(\omega) - \tilde{s}_{it}(\omega)) \leq 0 \qquad \Rightarrow \qquad \langle\langle \tilde{p}_t, \sum_{i\in\mathcal{I}} (\tilde{x}_{it} - \tilde{s}_{it}) \rangle\rangle_1 \leq 0. \tag{5.2.16}$$

In this way, thanks to (5.2.3) and (5.2.16), for each $t \in \mathcal{T}$ one has

$$\langle\langle p_t, \sum_{i\in\mathcal{I}}(\tilde{x}_{it} - \tilde{s}_{it})\rangle\rangle_1 = \int_\Omega p_t(\omega)\sum_{i\in\mathcal{I}}(\tilde{x}_{it}(\omega) - \tilde{s}_{it}(\omega))\, d\,\mathbb{P}(\omega) \le 0 \qquad \forall p_t \in P_t \tag{5.2.17}$$

Let us suppose that there exists $t^* \in \mathcal{T}$ and $E_{t^*} \in \mathcal{F}_{t^*}$, with $\mathbb{P}(E_{t^*}) > 0$, such that

$$\sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega)) > 0 \qquad \forall \omega \in E_{t^*}.$$

We pose:

$$C := \int_{\Omega\setminus E_{t^*}} \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega))\, d\,\mathbb{P}(\omega) \quad \textit{and} \quad D := \int_{E_{t^*}} \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega))\, d\,\mathbb{P}(\omega)$$

with $C \le 0$ and $D > 0$. If $C = 0$, the condition $D > 0$ contradicts inequality (5.2.17). If $C < 0$ let $\hat{p}_{t^*} : \Omega \to \mathbb{R}$ be such that

$$\hat{p}_{t^*}(\omega) := \begin{cases} \dfrac{\epsilon - KD}{C} & \forall \omega \in \Omega \setminus E_{t^*} \\ K & \forall \omega \in E_{t^*}. \end{cases} \quad \text{with } K \in [\underline{p}_{t^*}, \overline{p}_{t^*}]\ ;$$

$$\epsilon \in [\max\{0, \overline{p}_{t^*}C + KD\}, \underline{p}_{t^*}C + KD]\ .$$

Since $C < 0$, it follows that $\hat{p}_{t^*} \in P_{t^*}$. One has:

$$\begin{aligned}\langle\langle \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*} - \tilde{s}_{it^*}), \hat{p}_{t^*}\rangle\rangle_1 &= \int_{\Omega\setminus E_{t^*}} \hat{p}_{t^*}(\omega)\sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega))\, d\,\mathbb{P}(\omega) + \\ &+ \int_{E_{t^*}} \hat{p}_{t^*}(\omega)\sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega))\, d\,\mathbb{P}(\omega) = \\ &= \frac{\epsilon - KD}{C}C + KD = \epsilon > 0\end{aligned}$$

contradicting (5.2.17). Hence, for all $t \in \mathcal{T}$, one has

$$\sum_{i\in\mathcal{I}}(\tilde{x}_{it}(\omega) - \tilde{s}_{it}(\omega)) \le 0 \qquad a.e.\ \omega \in \Omega\ .$$

Claim 4 *For all* $t \in \mathcal{T}$ $\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{tj} = 0$.

We suppose that there exists $t^* \in \mathcal{T}$ such that $\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{t^*j} < 0$. Thanks to Proposition 15 and the fact that $\lambda_{it^*}(\omega)\tilde{s}_{it^*}(\omega) \ge 0$ $a.e.\ \omega \in \Omega$ for all $i \in \mathcal{I}$, one has that

$$\tilde{p}_{t^*}(\omega)\sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega)) \le \tilde{p}_{t^*}(\omega)\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{t^*j}$$

$$\Downarrow$$

$$\langle\langle \tilde{p}_{t^*}, \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*} - \tilde{s}_{it^*})\rangle\rangle_1 \le \langle\langle \tilde{p}_{t^*}, \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \tilde{z}_{i0}^{t^*j}\rangle\rangle_1\ .$$

From (5.2.3), it follows that

$$\langle\langle p_{t^*}, \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*} - \tilde{s}_{it^*})\rangle\rangle_1 \leq \langle\langle \tilde{p}_{t^*}, \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*} - \tilde{s}_{it^*})\rangle\rangle_1 \leq \langle\langle \tilde{p}_{t^*}, \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \tilde{z}_{i0}^{t^*j}\rangle\rangle_1 \qquad \forall p_{t^*} \in P_t$$

so that

$$\begin{aligned} &\langle\langle p_{t^*}, \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*} - \tilde{s}_{it^*})\rangle\rangle_1 \leq \langle\langle \tilde{p}_{t^*}, \sum_{i\in\mathcal{I}} \tilde{z}_{i0}^{t^*j^*}\rangle\rangle_1 \\ &\Downarrow \\ &\langle\langle p_{t^*}, \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*} - \tilde{s}_{it^*})\rangle\rangle_1 - \langle\langle \tilde{p}_{t^*}, \sum_{i\in\mathcal{I}} \tilde{z}_{i0}^{t^*j^*}\rangle\rangle_1 \leq 0 \qquad \forall p_{t^*} \in P_{t^*} \end{aligned} \tag{5.2.18}$$

where it results that $\langle\langle \tilde{p}_{t^*}, \sum_{i\in\mathcal{I}} \tilde{z}_{i0}^{t^*j^*}\rangle\rangle_1 < 0$. From Claim 3, one has that $\sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega)) \leq 0 \quad a.e.\ \omega \in \Omega$. So, if $\sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega)) = 0$, then the relation in (5.2.18) is contradicted for all $p_{t^*} \in P_{t^*}$. Instead, if it exists at least one $E_{t^*} \in \mathcal{F}_{t^*}$, with $\mathbb{P}(E_{t^*}) > 0$, for which $\sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega)) < 0$, we pose:

$$C := \int_\Omega \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*}(\omega) - \tilde{s}_{it^*}(\omega))\, d\,\mathbb{P}(\omega) \qquad \textit{and} \qquad D := \int_\Omega \tilde{p}_{t^*}(\omega) \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{t^*j}\, d\,\mathbb{P}(\omega)$$

with $C < 0$ and $D < 0$. Let $\hat{p}_{t^*} : \Omega \to \mathbb{R}$ be such that

$$\hat{p}_{t^*}(\omega) := \frac{\epsilon + D}{C} \quad a.e.\ \omega \in \Omega \qquad \text{with} \quad \epsilon \in [\max\{0, \overline{p}_{t^*}C - D\}, \underline{p}_{t^*}C - D]\,.$$

One has $\hat{p}_{t^*} \in P_{t^*}$ and:

$$\langle\langle \hat{p}_{t^*}, \sum_{i\in\mathcal{I}}(\tilde{x}_{it^*} - \tilde{s}_{it^*})\rangle\rangle_1 - \langle\langle \tilde{p}_{t^*}, \sum_{i\in\mathcal{I}} \tilde{z}_{i0}^{t^*j^*}\rangle\rangle_1 = \frac{\epsilon + D}{C}C - D = \epsilon > 0$$

which contradicts (5.2.18). Hence, $\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} \tilde{z}_{i0}^{tj} = 0$ for all $t \in \mathcal{T}$.

Then, thanks to Claim 1, 2, 3 and 4 and Remark 10 if $(\tilde{x}, \tilde{z}, \tilde{s}, \tilde{b}, \tilde{p}, \tilde{q}) \in B(\tilde{p}, \tilde{q}) \times P \times Q$ is a solution to $SGQVI$ (5.2.1), then it is an equilibrium vector of plans, prices, and price expectations for the Electricity Market $\mathcal{E}$. □

Theorem 26. *For all $i \in \mathcal{I}$, let Assumptions A be satisfied and $\widetilde{B} : P \times Q \rightrightarrows \mathcal{L}_I \times \mathcal{L}_{HTI}$ be lower semicontinuous, closed, and with compact values. Then, the $SGQVI$ (5.2.1) admits at least one solution, that is, there exists an equilibrium vector of plans, prices, and price expectations for the Electricity Market $\mathcal{E}$.*

Proof. Firstly, we pose

$$\widetilde{\mathcal{L}} := \mathcal{L}_I \times \mathcal{L}_{HTI} \times \mathcal{L}_I \times \mathcal{L}_{JTI} \times \mathcal{L}_1 \times \mathcal{L}_{HT}$$

so that, we can consider

$$\begin{aligned}
&\widetilde{\mathcal{K}} := \mathcal{L}_I \times \prod_{i\in\mathcal{I}} Z_i \times \prod_{i\in\mathcal{I}} S_i \times \prod_{i\in\mathcal{I}} C_i \times P \times Q \subset \widetilde{\mathcal{L}} \\
&\widetilde{\mathcal{S}} : \widetilde{\mathcal{K}} \rightrightarrows \widetilde{\mathcal{K}} \quad \text{s.t.} \quad \widetilde{\mathcal{S}}(x,z,s,b,p,q) := (\widetilde{B} \times P \times Q) \\
&\widetilde{\mathcal{F}} : \widetilde{\mathcal{L}} \rightrightarrows \widetilde{\mathcal{L}} \quad \text{s.t.} \quad \widetilde{\mathcal{F}}(x,z,s,b,p,q) := ((\nabla u_i(x_i,b_i))_{i\in\mathcal{I}}, \sum_{i\in\mathcal{I}}(x_i - s_i), \sum_{i\in\mathcal{I}} z_i) \ .
\end{aligned}$$

From Theorem 12, there exists a solution to the stochastic quasi-variational inequality (5.2.1). Then, from Theorem 25, there exists a equilibrium of plans, prices, and price expectations for the Electricity Market $\mathcal{E}$. □

As already anticipated, the strength point of this variational formulation is linked to the results obtained in Chapter 4. Indeed, most real-world phenomena vary with continuity and affect the agent's decision processes and all variables involved in the market. So, to capture these dynamics, in (5.2.1) we have introduced a quasi-variational problem on a continuous filtered probability space and we have linked it with an equilibrium solution for the electricity market. Now, we point out that the crucial aspect of this approach is that this formulation pays the way for several future developments. Indeed, recently in the literature [44, 14], they are been introduced discrete approximations of two-stage stochastic variational inequalities. Adapting opportunely these procedures to fit with the quasi-variational problem (5.2.1) under study, one could get an approximated stochastic variational formulation exhibiting an event-tree structure as in Chapter 4. In this way, it could be possible to introduce, as in (4.2.16), the extensive formulation of the approximated stochastic problem and, with similar arguments used in Proposition 12 and Corollary 2, to get the equivalence between the basic and extensive formulation of the approximated stochastic quasi-variational problem. In this way, the computational procedure studied in Section 4.3 can be applied to compute the approximated equilibrium solution to the electricity market. We stress out that this computational approach allows us to solve, efficiently and in parallel, large dimension problems which involve both time and uncertainty.

Chapter 6

Possible Future Research

The proposed research focuses on deterministic and stochastic variational inequalities problems and their applications to study equilibrium problems in which *time, uncertainty and information* play a central role.
As already seen in the previous Chapters, during the Ph.D. I have studied deterministic and stochastic variational problems, both by a quantitative and qualitative point of view, with particular attention to the Radner equilibrium model, due to its wide range of applications to real-life problems. In order to continue with these studies, it could be interesting to apply this stochastic variational approach to other Electricity Markets, as the framework of *retailers* that must provide the electricity demand of their clients in each period of the planning horizon. This case, in comparison with the configurations of consumers or producers, is more complicated due to the *elasticity behavior* of clients with respect to the selling price offered by the retailer. In addition, it could be interesting also incorporate in the formulation of these problems *measures of risk* (see, e.g. [15], Ch. 4) to express preferences between different manifestations of uncertain cost or loss arising from economics and finance problems.
To better deal with these real-world problems, it is necessary to develop also some theoretical aspects of the variational problems:

Qualitative Aspects
A first natural development could be to consider a suitable filtered probability space as in Chapter 5, but into a continuous-time framework, and study, relatively to it, optimality and equilibrium conditions by means of suitable stochastic variational problems under generalized assumptions, in terms of weak continuity and generalized monotonicity. In finance, for instance, it is quite natural to think that the system under analysis could change continuously in time from an instant to the next one.
Another interesting aspect to take into account comes from Theorem 11. Indeed,

in the recent work [9], authors, motivated by a real-life application on a bidding process for a deregulated electricity market, study the solution to a generalized quasi-variational inequality problem with non-self constraint map, in terms of projected solution as in Definition 8. This approach results to be relevant to find the solution of many real-world problems that, otherwise, could be hard to be mathematically treated. However, the authors established some existence results of the projected solution for the variational problems only in a *finite-dimensional setting.* This is due to the fact that the constraint set-valued map $S : K \rightrightarrows \mathbb{R}^G$ is assumed to be with $\text{int}S(x) \neq \emptyset$, for all $x \in K$, and $S(K)$ relatively compact. My proposal is to try to generalize these results in general infinite-dimensional spaces.
Other interesting aspects, could be the investigation of the relationship between these variational formulations and equilibrium models with bilevel optimization problems and vector variational inequalities problems.

Quantitative Aspects

Up to now, we have considered variational problems on continuous probability space in order to approach as closely as possible to the dynamics governing real-world problems. However, solving problems where uncertainty on input data is modeled by continuous stochastic processes is very difficult, or even impossible in many cases. So, from a computational viewpoint, a convenient way is to get a discrete approximation of the original problem. This could allow us to consider real or simulated data and use them to perform studies in terms of scenario reduction techniques, supported by opportune statistical methodologies (e.g. ARIMA model [50]) and the computational procedure introduced in Chapter 4.
Furthermore, in this context, investigate on *quantitative stability* (see, e.g. [69] [43]) of the system in question could be not only interesting but also crucial in order to retain most of the relevant information when discrete approximations are performed or scenario-reduction techniques are used to cut down the number of scenarios.

Appendix A

We devote this Appendix to recall some basic concepts of set-valued analysis, quasi-convex analysis, and probability theory which play an important role in the thesis. For the proofs and further details, we refer to [2, 45, 22] and the references therein.

A.1 Some concepts of set-valued maps

Let X be a Banach space and Y a topological vector space.

Definition 23. *An application $T : X \rightrightarrows Y$ that at any $x \in X$ associates a subset $T(x)$ of Y is called* set-valued map *or* multifunction*; the set $T(x)$ is called image of x under T.*

Of course, whenever the set-valued map is single-valued, then it is a function. The set-valued map T is called *proper* if there exists at least one element $x \in X$ such that $T(x) \neq \emptyset$. In this case, the set

$$Dom\ (T) = \{x \in X : T(x) \neq \emptyset\}$$

is called *domain* of T. It is called *graph* of T the subset of $X \times Y$ defined by

$$Graph\ (T) = \{(x, y) \in X \times Y : y \in T(x)\}\,.$$

Definition 24. *Let $T : X \rightrightarrows Y$ be such that $Dom\ (T) \neq \emptyset$ and $B(x_0, \delta)$ the ball on X centered at x_0 and with radius δ. We say that F is:*

(i) upper semicontinuous *at $x_0 \in Dom\ (T)$ if for any neighborhood U of $T(x_0)$, there exists $\delta > 0$ such that $T(x) \in U$ for all $x \in B(x_0, \delta)$; it is upper semicontinuous on X if it is upper semicontinuous at every point of $Dom\ (T)$;*

(ii) lower semicontinuous *at $x_0 \in Dom\ (T)$ if for any $y_0 \in T(x_0)$ and any neighborhood $V(y_0)$ of y_0 there exists $\delta > 0$ such that $T(x) \cap V(y_0) \neq \emptyset$ for any $x \in B(x_0, \delta)$; it is lower semicontinuous on X if it is upper semicontinuous at every point of $Dom\ (T)$;*

(iii) continuous *at $x_0 \in$ Dom (T) if is is both upper semicontinuous as well as lower semicontinuous at x_0; it is continuous on X if it is continuous at every point of Dom (T).*

Now, we state a useful characterization of upper / lower semicontinuity of a set-valued map:

- T *upper semicontinuous* at $x_0 \in$ *Dom* (T) if whenever $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $x_n \to x$ and $x \in X$, and $\{y_n\}_{n\in\mathbb{N}} \subset Y$ such that $y_n \in T(x_n)$ for all $n \in \mathbb{N}$ and $\lim_n y_n$ exists, then this limit point in as element of $T(x)$;

- T *lower semicontinuous* at $x_0 \in$ *Dom* (T) if for any sequence of elements $\{x_n\}_{n\in\mathbb{N}} \subset X$, $x_n \to x$, and for any $y \in T(x)$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset Y$, with $y_n \in T(x_n)$ for any $n \in \mathbb{N}$ and $y_n \to y$.

Another important concept in set-valued analysis is the following.

Definition 25. *A set-valued map $T : X \rightrightarrows Y$ is said to be* closed *at $x_0 \in$ Dom (T) if for any sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset$ Graph (T) converging to (x_0, y_0), one has $(x_0, y_0) \in$ Graph (T).*

If the images of a set-valued map T are closed, bounded, compact, and so on, we say that T is closed valued, bounded valued, compact valued, and so on.

Proposition 17. *Let $T : X \rightrightarrows Y$ be a set-valued map. The following holds:*

(i) T is upper semicontinuous with closed values. Then T is closed.

(ii) T is closed and $\overline{T(X)}$ is a compact subset. Then T is upper semicontinuous.

A.2 Some concepts of quasiconvexity and quasimonotonicity

Let X be a Banach space, denote by X^* its topological dual and $\langle \cdot, \cdot \rangle$ the duality pairing. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function.

Definition 26. *A function f is said to be:*

(i) quasiconvex *if, for any $x, y \in$ dom f and $\lambda \in [0, 1]$, one has*

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\};$$

(ii) semistrictly quasiconvex *if it is quasiconvex and for any $x, y \in dom\ f$ such that $f(x) \neq f(y)$, one has*

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\} \qquad \forall \lambda \in (0,1)\ ;$$

(iii) strictly quasiconvex *if for any $x, y \in dom\ f$ such that $x \neq y$, one has*

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\} \qquad \forall \lambda \in (0,1)\ .$$

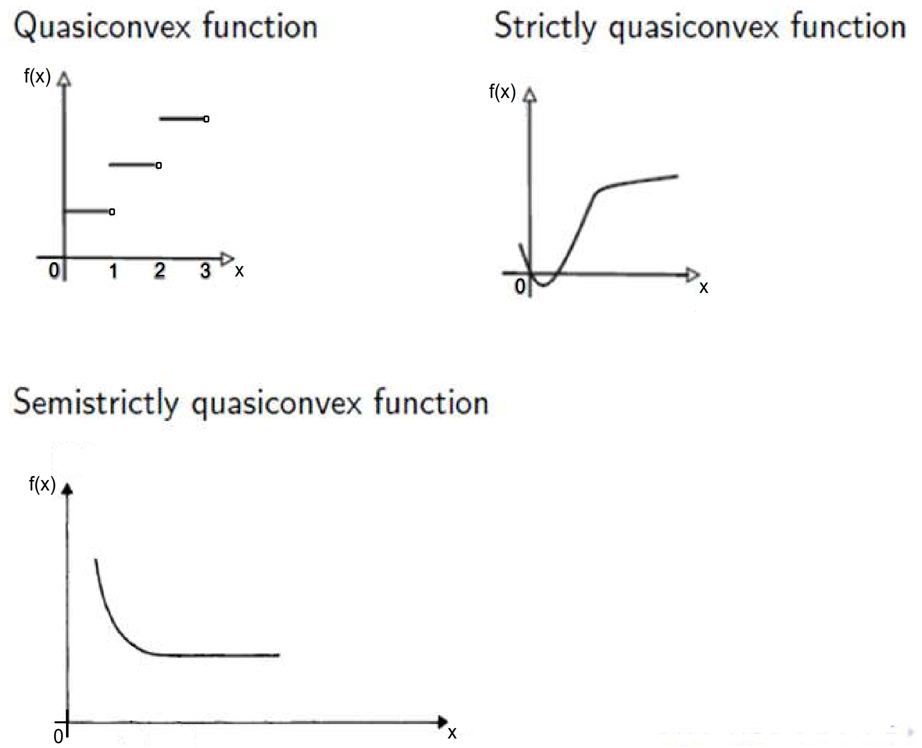


Figure A.1: Example

From Definition 26, a strictly quasiconvex function is semistrictly quasiconvex and quasiconvex. In general, a semistrictly quasiconvex function might not be quasiconvex. For instance, if we consider the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and $f(x) = 0$ for $x \neq 0$, it is semistrictly quasiconvex but not quasiconvex.

Proposition 18. *Let f be a semistrictly quasiconvex function. If f is lower semicontinuous on dom f, then f is quasiconvex.*

We point out that a function $f : X \to (-\infty, +\infty]$ is quasiconvex if and only if for every $x \in X$ and every unit vector $e \in X$, the function $g : \mathbb{R} \to (-\infty, +\infty]$ defined by $g(t) = f(x + te)$ is quasiconvex, that is, *its restriction on straight lines is quasiconvex.*

Proposition 19. *A function $f : \mathbb{R} \to (-\infty, +\infty]$ is quasiconvex if and only if there exists an interval I of the form $(-\infty, t)$ or $(-\infty, t]$, where $t \in [-\infty, +\infty]$, such that f is nonincreasing on I and nondecreasing on its complement*[1].

Every convex function is quasiconvex. The converse is not true. For instance, the function $\arctan(\cdot)$ is quasiconvex without being convex.

Remark 11. *The sum of two convex functions is a convex function. Instead, the sum of two quasiconvex functions need not be quasiconvex. Even if we add a linear function to a quasiconvex one, the result might not be quasiconvex.*

Proposition 20. *Let $f : X \to (-\infty, +\infty]$ be a proper function.*

(i) If f is quasiconvex, then the set of its global minima is convex.

(ii) If f is strictly quasiconvex, then it has at most one global minimum.

Semistrictly quasiconvex functions retain some localization properties from the class of convex functions.

Proposition 21. *Let f be a continuous and quasiconvex function defined on $X \subseteq \mathbb{R}^n$. Then f is semistrictly quasiconvex if and only if every local minimum $\bar{x} \in X$ is also a global minimum of f at X.*

It is possible to characterize the quasiconvex functions in terms of sublevel sets, defined as in (2.1.4).

Proposition 22. *A function is quasiconvex if and only if, for any $\alpha \in f(X)$, the sublevel set S_α is nonempty convex.*

For any lower semicontinuous function f, semistrictly quasiconvex on its domain *dom* f, one has:

$$\forall \alpha > \inf_X, \quad cl(S_\alpha^<) = S_\alpha \ .$$

This means that a lower semicontinuous semistrictly quasiconvex function f does not have any *flat part* with nonempty interior on *dom* $f \setminus \operatorname{argmin}_X f$.
Now, we recall the following definition for a set-valued operator.

[1]Note that, I or its complement might be empty, that is, the function may be nondecreasing or nonincreasing throughout $\mathbb{R}$.

Definition 27. *An operator* $T : X \rightrightarrows X^*$ *is said to be* quasimonotone *on a subset* K *iff, for all* $x, y \in K$,

$$\exists x^* \in T(x) : \langle x^*, y - x \rangle > 0 \quad \Rightarrow \quad \langle y^*, y - x \rangle \geq 0 \;\; \forall y^* \in T(y).$$

Let f be a function and, for any $x \in \textit{dom } f$, $N(x)$ and $N^<(x)$, respectively, the normal operator to $S_{f(x)}$ and $S^<_{f(x)}$. For general quasiconvex functions, however, one has that the operator N can fail to be closed and the operator $N^<$ can fail to be quasimonotone. To overcome this fact, Aussel and Hadjisavvas [8] introduced the concept of *adjusted sublevel set* of a function, as in Definition 7, with the relative normal operator.

A.3 Some concepts of probability theory

Let Ω be an abstract set, also known as *sample space.*

Definition 28. *A set* $\mathcal{F}$ *of subsets of* Ω *is called* σ-algebra *or* σ-field *if*

(i) $\Omega \in \mathcal{F}$;

(ii) $E \in \mathcal{F}$, *then* $\Omega \setminus E \in \mathcal{F}$;

(iii) $\{E_i\}_{i \in \mathbb{N}} \subseteq \mathcal{F}$, *then* $\cup_{i \in \mathbb{N}} E_i \in \mathcal{F}$.

The couple $(\Omega, \mathcal{F})$ is called *measurable space.* A set $E \subset \Omega$ is said to be $\mathcal{F}$-measurable if $E \in \mathcal{F}$. It is said that the σ-algebra $\mathcal{F}$ is *generated* by its subset $\mathcal{V}$ if $\mathcal{F}$ is the smallest σ-algebra containing $\mathcal{V}$. This means that any $\mathcal{F}$-measurable set can be obtained from sets belonging to $\mathcal{V}$ by set theoretic operations and by taking the union of a countable family of sets from $\mathcal{V}$.
The σ-algebra generated by the set of open (or closed) subsets of a finite-dimensional space $\mathbb{R}^G$ is called its *Borel σ-algebra*, $\mathcal{B}(\mathbb{R}^G)$. An element of this σ-algebra is called a Borel set.

Definition 29. *Let* $(\Omega, \mathcal{F})$ *be a measurable space. An increasing family* $\{\mathcal{F}\}_{t \geq 0}$ *of sub-σ-algebras of* $\mathcal{F}$ *is called* filtration.

Definition 30. *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A function* $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ *is called* probability measure *on* $(\Omega, \mathcal{F})$ *if*

(i) $\mathbb{P}(E) \geq 0$, *for all* $E \in \mathcal{F}$;

(ii) $\mathbb{P}(\Omega) = 1$;

(iii) *for every collection $E_i \in \mathcal{F}$, $i \in \mathbb{N}$, such that $E_i \cap E_j = \emptyset$ for all $i \neq j$, we have*

$$\mathbb{P}(\cup_{i \in \mathbb{N}} E_i) = \sum_{i \in \mathbb{N}} \mathbb{P}(E_i) \ .$$

A sample space $(\Omega, \mathcal{F})$ equipped with a probability measure $\mathbb{P}$ is called *probability space* and denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}\}_{t \geq 0}$ is called *filtered probability space* and denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, \mathbb{P})$. An event $E \in \mathcal{F}$ happens $\mathbb{P}$-*almost sure* (a.s.) or *almost everywhere* (a.e.) if $\mathbb{P}(E) = 1$.

Definition 31. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The measure $\mathbb{P}$ is called* non-atomic *if any set $E \in \mathcal{F}$, such that $\mathbb{P}(E) > 0$, contains a subset $D \in \mathcal{F}$ such that $\mathbb{P}(E) > \mathbb{P}(D) > 0$.*

A function $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}^G, \mathcal{B}(\mathbb{R}^G))$ is said to be *measurable* if for any Borel set $A \in \mathcal{B}(\mathbb{R}^G)$, its inverse imagine $Y^{-1}(A) := \{\omega \in \Omega : Y(\omega) \in A\} \in \mathcal{F}$.

Definition 32. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A measurable function $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^G, \mathcal{B}(\mathbb{R}^G))$ is called G-dimensional* random vector.

Definition 33. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A* stochastic process $Y := \left\{Y_i : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^G, \mathcal{B}(\mathbb{R}^G))\right\}_{i \in \mathcal{I}}$ *is a family of random vectors on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{I} \subset \mathbb{R}$ represents an arbitrary index set.*

If we identify the index set with a time set, that is $\mathcal{I} = \mathcal{T}$, it allows us to study the time evolution of the random vectors.

Definition 34. *A stochastic process $Y := \{Y_t\}_{t \in \mathcal{T}}$ is said to be* adapted to the filtration $\{\mathcal{F}\}_{t \in \mathcal{T}}$ *if, for each $t \in \mathcal{T}$, Y_t is $\mathcal{F}_t$-measurable.*

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