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SEQUENCES OF WEAK SOLUTIONS TO A FOURTH-ORDER ELLIPTIC PROBLEM

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ABSTRACT. This paper contains some results of existence of infinitely many solutions to an elliptic equation involving the $p(x)$-biharmonic operator coupled with Navier boundary conditions where the nonlinearities depend on two real parameters and do not possess any symmetric property. The approach is variational and the main tool is an abstract result of Ricceri.

1. Introduction

The aim of this paper is to establish the existence of infinitely many solutions to the following $p(x)$-biharmonic elliptic equation with Navier boundary conditions,

$$\begin{cases} \Delta^2_{p(x)} u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$, $p \in C^0(\overline{\Omega})$ satisfies $\max\{1, n/2\} < \inf_{\overline{\Omega}} p \leq \sup_{\overline{\Omega}} p$, $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions with suitable behaviors, $\lambda \in \mathbb{R}$ and $\mu > 0$. The operator $\Delta^2_{p(x)}$ is the natural generalization to the variable exponent framework of the standard p-biharmonic ($p > 1$ constant).

Equations with variable exponent growth conditions model various phenomena, for instance, the image restoration or the motion of the so called electrorheological fluids, characterized by their ability to drastically change their mechanical properties under the influence of an exterior electromagnetic field (see for instance Růžička (2000) and Chen *et al.* (2004) and the survey paper of Rădulescu (2015)). The most suitable contexts in which this kind of problems can be studied is represented by the Lebesgue and Sobolev spaces with variable exponents; for more informations about this topic, the reader is invited to consult Kováčik and Rákosník (1991), Fan *et al.* (2001), Fan and Zhao (2001), and Diening *et al.* (2011) together with the further sources of Cammaroto and Vilasi (2013a,b, 2014) and Vilasi (2016) in which several classes of variable exponent problems (and related multiplicity results) are investigated, still from a variational perspective.

In the last years several authors have showed their interest in fourth-order differential problems involving p-biharmonic and $p(x)$-biharmonic operators, motivated by the fact that this type of equations arise in many domains like micro electromechanical systems, surface diffusion on solids, thin film theory, flow in Hele-Shaw cells and phase fieldmodels of multiphasic systems and, more generically, in fields such as the elasticity theory, or more in general, in continuous mechanics. Some recent existence results for Navier problems driven by the p-biharmonic and $p(x)$-biharmonic operators are provided by Furusho and Takaŝi (1998), Ayoujil and El Amrouss (2009), Wang and Shen (2009), Candito and Livrea (2010), Ayoujil and El Amrouss (2011), Candito *et al.* (2012), Massar *et al.* (2012), Yin and Liu (2013), Kong (2014), Kefi and Rădulescu (2017), and Bueno *et al.* (2018).
In this paper, the existence of infinitely many solutions to $(P_{\lambda,\mu})$ is obtained by finding suitable conditions on the nonlinearities; unlike much of the existing literature on the subject this approach does not require any symmetry condition on the nonlinearities, but rather an opportune behaviour of the term μg at infinity, expressed in terms of its primitive.
The approach used in this paper is variational; more precisely we will apply the following critical point theorem that Ricceri established in 2000 (Theorem 2.5 of Ricceri (2000)), recalled below for the reader's convenience.

Theorem 1.1. *Let E be a reflexive real Banach space and let $\Phi, \Psi : E \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that Ψ is strongly continuous and coercive.*

For each $r > \inf_E \Psi$, define φ to be

$$\varphi(r) := \inf_{x\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(x) - \inf_{\overline{\Psi^{-1}(]-\infty,r[)}_w} \Phi}{r - \Psi(x)},$$

where $\overline{\Psi^{-1}(]-\infty,r[)}_w$ is the closure of $\Psi^{-1}(]-\infty,r[)$ in the weak topology.

Fixed $L \in \mathbb{R}$, then

a) if $\{r_k\}$ is a real sequence such that $\lim_{k\to\infty} r_k = +\infty$ and $\varphi(r_k) < L$ for all $k \in \mathbb{N}$, the following alternative holds: either $\Phi + L\Psi$ has a global minimum or there exists a sequence $\{x_k\} \subset E$ of critical points of $\Phi + L\Psi$ such that $\lim_{k\to\infty} \Psi(x_k) = +\infty$;

b) if $\{s_k\}$ is a real sequence such that $\lim_{k\to\infty} s_k = (\inf_E \Psi)^+$ and $\varphi(s_k) < L$ for all $k \in \mathbb{N}$, the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi + L\Psi$ or there exists a sequence $\{x_k\} \subset E$ of pairwise distinct critical points of $\Phi + L\Psi$ with $\lim_{k\to\infty} \Psi(x_k) = \inf_E \Psi$ which weakly converges to a global minimum of Ψ.

Since its appearance in 2000 until our days, Theorem **1.1** has been a powerful tool to get multiplicity results for different kinds of problems. In particular, it has been applied to obtain the existence of infinitely many solutions for a vast range of differential problems. In each of these applications, in order to guarantee that $\varphi(r_k) < L$ (or $\varphi(s_k) < L$), for each $k \in \mathbb{N}$, and that the functional $\Phi + L\Psi$ has no global minimum, it is necessary to construct suitable sequences of functions. Generally, in these constructions, one uses the norm of the variable raised to a suitable power which depends on the nature of the problem and that gives to the functions the requested regularity properties: in some application the norm is used without power (see, for instance, Cammaroto *et al.* (2005), Kristály (2006), Bonanno *et al.* (2010), and Dai and Wei (2010)), in some others it is raised to the second (Candito

and Livrea (2010), Bonanno and Di Bella (2011), Candito *et al.* (2012), and Massar *et al.* (2012)) or to the third (Heidarkhani (2014) and Makvand Chaharlanga and Razani (2018)) or to the forth power (Afrouzi and Shokooh (2015)); Hadjian and Ramezani (2017) and Reza Heidari Tavani and Nazari (2019) combined the norm with trigonometric functions.
The choice of a particular sequence of functions inside the proof reflects heavily on the assumptions and while there are some cases in which probably the choice is optimal, in some other cases it could happen that a different choice of the sequence would make the result applicable in a greater number of cases. This is the reason why in this paper an abstract class of test functions is introduced; some examples presented in Section **4** will clarify this fact.
A similar procedure is used by Cammaroto and Genoese (2018) and Cammaroto and Vilasi (2019) and above all by Song (2014) where he doesn't choose the test functions arbitrarily during the proof but uses two generic functions whose properties are described in the initial assumptions.
The paper is then structured as follows. Section **2** includes all the basic results about Lebesgue and Sobolev variable exponent spaces necessary for the variational set-up of Problem $(P_{\lambda,\mu})$. In Section **3** the multiplicity results will be presented and, finally, in Section **4** some concrete examples of nonlinearities will be exhibited.

2. Variational framework

We collect here some preliminary results about Lebesgue and Sobolev variable exponent spaces, which are used in our investigations.
To begin with, we fix some notation for the sequel. Given a measurable function $u : \Omega \to \mathbb{R}$, we set

$$u^- := \operatorname*{essinf}_{\Omega} u, \quad u^+ := \operatorname*{esssup}_{\Omega} u.$$

We denote by $\omega := \pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}+1\right)$ the measure of the unit ball in $\mathbb{R}^n$. If X is a Banach space, the symbol $B(x,r)$ denotes the open ball centered at $x \in X$ and of radius $r > 0$ and $\overline{B}(x,r)$ its clousure.
Let Ω be a bounded smooth domain of $\mathbb{R}^n$, $n \geq 1$, and let $p \in C^0(\overline{\Omega})$ satisfy $1 < p^- \leq p^+$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable: } \int_\Omega |u|^{p(x)} dx < +\infty \right\}.$$

It is a reflexive Banach space when endowed with the Luxemburg norm

$$|u|_{p(x)} = \inf\left\{ \sigma > 0 : \int_\Omega \left|\frac{u}{\sigma}\right|^{p(x)} dx \leq 1 \right\}.$$

If we denote by $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ the functional defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx$$

for all $u \in L^{p(x)}(\Omega)$, it is the so-called *modular* of the space $L^{p(x)}(\Omega)$ and the following proposition clarifies its relations with the Luxemburg norm.

Proposition 2.1. *Let $u \in L^{p(x)}(\Omega)$ and let $\{u_k\}$ be a sequence in $L^{p(x)}(\Omega)$; then*

(1) $|u|_{p(x)} < 1\ (=1;>1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1\ (=1;>1)$;
(2) $|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$;
(3) $|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$;
(4) $|u_k - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_k - u) \to 0$.

For any $k \in \mathbb{N}$, define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega) \text{ for any } |\alpha| \leq k \right\},$$

where $D^\alpha u$ is the partial derivative of u with respect to the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. The space $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space under the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)}.$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ and $X := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ will be the space naturally associated with $(P_{\lambda,\mu})$. On X the functional

$$\|u\| := \inf \left\{ \sigma > 0 : \int_\Omega \left| \frac{\Delta u}{\sigma} \right|^{p(x)} dx \leq 1 \right\}$$

for any $u \in X$, defines a norm equivalent to $\|\cdot\|_{2,p(x)}$ (see for instance Zang and Fu (2008)).

Clearly the embedding $X \hookrightarrow W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega)$ is continuous and, by Rellich- Kondrachov's theorem, $W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ compactly when Ω is bounded and $p^- > \dfrac{n}{2}$. This leads immediately to the following important result.

Proposition 2.2. *The embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact provided that $p^- > \dfrac{n}{2}$.*

The previous result implies that there exists a constant $c_\infty > 0$ such that

$$\|u\|_\infty \leq c_\infty \|u\| \tag{1}$$

for every $u \in X$. In addition, defining the modular $\Upsilon : X \to \mathbb{R}$ by

$$\Upsilon(u) = \rho_{p(x)}(\Delta u) = \int_\Omega |\Delta u|^{p(x)} dx$$

for all $u \in X$, it is well known that a similar result as Proposition **2.1** holds:

Proposition 2.3. *Let $u \in X$ and let $\{u_k\}$ be a sequence in X. Then*
(1) $\|u\| < 1\ (=1;>1) \quad \Leftrightarrow \quad \Upsilon(u) < 1\ (=1;>1)$;
(2) $\|u\| \geq 1 \quad \Rightarrow \quad \|u\|^{p^-} \leq \Upsilon(u) \leq \|u\|^{p^+}$;
(3) $\|u\| \leq 1 \quad \Rightarrow \quad \|u\|^{p^+} \leq \Upsilon(u) \leq \|u\|^{p^-}$;
(4) $\|u_k - u\| \to 0 \quad \Leftrightarrow \quad \Upsilon(u_k - u) \to 0$.

Now, for the motivations illustrated in the Introduction, let us introduce the following class of functions.

If $\{a_k\}, \{b_k\}, \{\sigma_k\}$ are three real sequences with $0 < a_k < b_k$ and $\sigma_k > 0$ for all $k \in \mathbb{N}$, let us denote by $\mathscr{H}(\{a_k\}, \{b_k\}, \{\sigma_k\})$ the space of all sequences $\{\chi_k\}$ satisfying

(i) $\chi_k \in W^{2,p^+}(a_k,b_k)$, for each $k \in \mathbb{N}$;
(ii) $0 \leq \chi_k(x) \leq \sigma_k$ for a.e. $x \in (a_k,b_k)$ and for each $k \in \mathbb{N}$;
(iii) $\lim\limits_{x \to a_k^+} \chi_k(x) = \sigma_k, \quad \lim\limits_{x \to b_k^-} \chi_k(x) = 0$;
(iv) $\lim\limits_{x \to a_k^+} \chi_k'(x) = \lim\limits_{x \to b_k^-} \chi_k'(x) = 0$;
(v) for all $j \in \{1,2\}$ there exists $c_j > 0$, independent of k, such that

$$|\chi_k^{(j)}(x)| \leq c_j \frac{\sigma_k}{(b_k - a_k)^j} \tag{2}$$

for a.e. $x \in (a_k,b_k)$ and for each $k \in \mathbb{N}$.

If $x_0 \in \Omega$, $\{b_k\} \subset]0,+\infty[$ such that $B(x_0,b_k) \subset \Omega$, for each $k \in \mathbb{N}$, and $\{\chi_k\} \in \mathscr{H}(\{a_k\},\{b_k\},\{\sigma_k\})$, consider the function $u_k : \Omega \to \mathbb{R}$ defined as follows:

$$u_k(x) := \begin{cases} 0 & \text{in } \Omega \setminus B(x_0,b_k) \\ \sigma_k & \text{in } B(x_0,a_k) \\ \chi_k(|x-x_0|) & \text{in } B(x_0,b_k) \setminus B(x_0,a_k). \end{cases} \tag{3}$$

Owing to the embedding $W^{2,p^+}(\Omega) \hookrightarrow W^{2,p(x)}(\Omega)$, it is clear that $\{u_k\} \subset X$. Moreover, a simple computation shows that, fixed $k \in \mathbb{N}$, for any $i = 1,2,\dots,n$ one has

$$\frac{\partial^2 u_k}{\partial x_i^2}(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(x_0,b_k) \\ 0 & \text{in } B(x_0,a_k) \\ \chi_k''(|x-x_0|)\dfrac{(x_i - x_i^0)^2}{|x-x_0|^2} + & \\ +\chi_k'(|x-x_0|)\dfrac{|x-x_0|^2 - (x_i - x_i^0)^2}{|x-x_0|^3} & \text{in } B(x_0,b_k) \setminus B(x_0,a_k) \end{cases} \tag{4}$$

From (2) (4) we obtain the following inequality

$$\begin{aligned} |\Delta u_k(x)| &= \chi_k''(|x-x_0|) + \chi_k'(|x-x_0|)\frac{n-1}{|x-x_0|} \\ &\leq \frac{\sigma_k}{(b_k-a_k)^2}c_2 + \frac{\sigma_k}{b_k-a_k}\frac{n-1}{a_k}c_1, \end{aligned} \tag{5}$$

thanks to which we are able to get the following estimation of the modular Υ at u_k

$$\begin{aligned} \Upsilon(u_k) &\leq \left(\frac{\sigma_k}{(b_k-a_k)^2}c_2 + \frac{\sigma_k}{b_k-a_k}\frac{n-1}{a_k}c_1\right)^{p^+} \omega(b_k^n - a_k^n) \\ &\leq \frac{\omega \sigma_k^{p^+}(b_k^n - a_k^n)}{a_k^{p^+}(b_k-a_k)^{2p^+}}(a_k c_2 + (n-1)(b_k-a_k)c_1)^{p^+} \end{aligned} \tag{6}$$

that is valid for those $k \in \mathbb{N}$ such that $\dfrac{\sigma_k}{(b_k-a_k)^2}c_2 + \dfrac{\sigma_k}{b_k-a_k}\dfrac{n-1}{a_k}c_1 \geq 1$.

In the sequel, without further mentioning, we always assume that $p \in C^0(\overline{\Omega})$ satisfies

$$\max\left\{1, \frac{n}{2}\right\} < p^- \leq p^+.$$

Let us denote by $\mathscr{C}$ the class of all Carathéodory functions $\zeta : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying $\sup_{|t| \leq \xi} |\zeta(\cdot,t)| \in L^1(\Omega)$ for all $\xi > 0$ and let $f, g \in \mathscr{C}$.
A weak solution to $(P_{\lambda,\mu})$ is any function $u \in X$ such that

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx = \lambda \int_\Omega f(x,u) v dx + \mu \int_\Omega g(x,u) v dx$$

for all $v \in X$. Obviously the weak solutions of problem $(P_{\lambda,\mu})$ are nothing but the critical points of the functional $\mathscr{E} : X \to \mathbb{R}$ defined by

$$\mathscr{E}(u) := \Psi(u) + \lambda J_F(u) + \mu J_G(u),$$

for any $u \in X$, where

$$\begin{aligned} \Psi(u) &:= \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx, \\ J_F(u) &:= -\int_\Omega F(x,u) dx, \\ J_G(u) &:= -\int_\Omega G(x,u) dx, \end{aligned} \tag{7}$$

for all $u \in X$, where

$$F(x,t) := \int_0^t f(x,s) ds, \quad G(x,t) := \int_0^t g(x,s) ds$$

for all $(x,t) \in \Omega \times \mathbb{R}$.

3. Multiplicity results

The main result of this paper reads as follows:

Theorem 3.1. *Let $f, g \in \mathscr{C}$ satisfy:*

(f_1) *there exist a measurable function $m : \Omega \to \mathbb{R}$, with $1 \leq m \leq p$ in Ω and $m^+ < p^-$ and a function $h \in L^1(\Omega)$, such that*

$$|f(x,t)| \leq h(x) \left(1 + |t|^{m(x)-1}\right)$$

for a.e. $x \in \Omega$, for all $t \in \mathbb{R}$;

(g_1) $G(x,t) \geq 0$ *for a.e. $x \in \Omega$, for all $t \geq 0$;*

(g_2) *there exist $x_0 \in \Omega$, $\rho, s_1, s_2 > 0$, such that $\overline{B}(x_0,\rho) \subset \Omega$ and*

$$a := \liminf_{t \to +\infty} \frac{\int_\Omega \max_{|\xi| \leq t} G(x,\xi) dx}{t^{s_1}} < +\infty, \quad b := \limsup_{t \to +\infty} \frac{\int_{B(x_0,\rho)} G(x,t) dx}{t^{s_2}} > 0.$$

Then the following facts hold:

(i) *if $s_1 < p^-$ and $s_2 > p^+$, for all $\lambda \in \mathbb{R}$ and for all $\mu > 0$, $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

(ii) *if $s_1 < p^-$ and $s_2 = p^+$, there exists $\mu_1 > 0$ such that, for all $\lambda \in \mathbb{R}$ and for all $\mu > \mu_1$, $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

(iii) *if $s_1 = p^-$ and $s_2 > p^+$, there exists $\mu_2 > 0$ such that, for all $\lambda \in \mathbb{R}$ and for all $\mu \in (0, \mu_2)$, $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

(iv) *if $s_1 = p^-$ and $s_2 = p^+$, there exist $\gamma > 1$ and $C_{\gamma,\rho} > 0$ such that, if*

$$C_{\gamma,\rho} < \frac{bp^-}{ap^+\omega c_\infty^{p^-}}, \tag{8}$$

(the previous inequality always being satisfied whether $a = 0$ or $b = +\infty$) then $\mu_1 < \mu_2$ and for all $\lambda \in \mathbb{R}$ and every $\mu \in (\mu_1, \mu_2)$, $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions.

Proof. To prove (i), fixed $\lambda \in \mathbb{R}$ and $\mu > 0$, we use Theorem **1.1**, item $a)$, with $E = X$, Ψ as in (7), $\Phi = \lambda J_F + \mu J_G$ and $L = 1$. It is clear that Ψ is C^1, sequentially weakly lower semicontinuous and, being

$$\Psi(u) \geq \frac{1}{p^+} \|u\|^{p^-},$$

coercive as well. The sequential weak lower semicontinuity of Φ follows, by standard arguments, from (f_1), (g_2) and Proposition **2.2**. Now, we define

$$\varphi(r) = \inf_{\Psi(u)<r} \frac{\sup_{\Psi(w)\leq r} \Phi - \Phi(u)}{r - \Psi(u)}$$

for all $r > 0$, and we will find a sequence $\{r_k\} \subset \mathbb{R}$, diverging to $+\infty$, such that $\varphi(r_k) < 1$ for all $k \in \mathbb{N}$. To this aim, thanks to the definition of φ, it sufficies to build a sequence $\{u_k\} \subset X$, with $\Psi(u_k) < r_k$, for all $k \in \mathbb{N}$, and satisfying

$$\begin{aligned} &\sup_{\Psi(w)\leq r_k} \left(-\lambda \int_\Omega F(x,w)dx - \mu \int_\Omega G(x,w)dx \right) + \lambda \int_\Omega F(x,u_k)dx + \\ &+ \mu \int_\Omega G(x,u_k)dx < r_k - \Psi(u_k). \end{aligned} \tag{9}$$

We choose $u_k = 0$, for any $k \in \mathbb{N}$. Thanks to (g_2), fixed $\tilde{a} > a$, for any $k \in \mathbb{N}$ there exists $\alpha_k \geq k$ such that

$$\int_\Omega \max_{|\xi|\leq\alpha_k} G(x,\xi)dx \leq \tilde{a}\alpha_k^{s_1}.$$

For any $k \in \mathbb{N}$ define

$$r_k := \frac{1}{p^+ c_\infty^{p^-}} \alpha_k^{p^-}.$$

It is clear that $r_k \to +\infty$ as $k \to \infty$ and $\Psi(u_k) < r_k$. To verify (9), observe that one has

$$\|w\|_\infty \leq c_\infty \|w\| \leq c_\infty \max\left\{ (p^+\Psi(w))^{\frac{1}{p^+}}, (p^+\Psi(w))^{\frac{1}{p^-}} \right\}$$

for any $w \in X$. Taking this fact into account, if $w \in X$ and $\Psi(w) \leq r_k$, one has, for k large enough

$$\|w\|_\infty \leq c_\infty (p^+ r_k)^{1/p^-} = \alpha_k$$

and by (f_1), (g_1), (g_2) we obtain, for k large enough,

$$\begin{aligned} &-\lambda \int_\Omega F(x,w)dx - \mu \int_\Omega G(x,w)dx \le |\lambda| \int_\Omega |h(x)| \left(w + \frac{|w|^{m(x)}}{m(x)} \right) dx + \\ &+ \mu \int_\Omega \max_{|\xi| \le \alpha_k} G(x,\xi)dx \le |\lambda| \, \|h\|_{L^1(\Omega)} \left(\alpha_k + \frac{\alpha_k^{m^+}}{m^-} \right) + \mu \tilde{a} \alpha_k^{s_1} \le \\ &\le |\lambda| \, \|h\|_{L^1(\Omega)} c_\infty (p^+)^{\frac{1}{p^-}} r_k^{\frac{1}{p^-}} + \frac{|\lambda| \, \|h\|_{L^1(\Omega)}}{m^-} c_\infty^{m^+} (p^+)^{\frac{m^+}{p^-}} r_k^{\frac{m^+}{p^-}} + \\ &+ \mu \tilde{a} c_\infty^{s_1} (p^+)^{\frac{s_1}{p^-}} r_k^{\frac{s_1}{p^-}} < r_k. \end{aligned} \tag{10}$$

According to part $a)$ of Theorem **1.1**, either the functional $\Phi + \Psi$ has a global minimum or there exists a sequence of weak solutions $\{v_k\} \subset X$ such that $\|v_k\| \to +\infty$ as $k \to \infty$. Let us show that $\Phi + \Psi$ is unbounded from below. Thanks to (g_2), fixed $0 < \tilde{b} < b$, for any $k \in \mathbb{N}$ there exists $\beta_k \ge k$ such that

$$\int_{B(x_0,\rho)} G(x,\beta_k)dx \ge \tilde{b}\beta_k^{s_2}.$$

Let $\gamma > 1$ such that $B(x_0,\gamma\rho) \subset \Omega$ and $\{\chi_k\} \in \mathscr{H}(\rho,\gamma\rho,\beta_k)$. Similarly to (3), consider the function u_k defined by

$$u_k(x) := \begin{cases} 0 & \text{in } \Omega \setminus B(x_0,\gamma\rho) \\ \beta_k & \text{in } B(x_0,\rho) \\ \chi_k(|x-x_0|) & \text{in } B(x_0,\gamma\rho) \setminus B(x_0,\rho). \end{cases} \tag{11}$$

Since $\beta_k \to +\infty$ as $k \to \infty$, we can use (6) and we get

$$\Upsilon(u_k) \le \frac{\omega \beta_k^{p^+}(\gamma^n - 1)}{\rho^{2p^+-n}(\gamma-1)^{2p^+}} (c_2 + (n-1)(\gamma-1)c_1)^{p^+}$$

and hence

$$\Psi(u_k) \le \frac{1}{p^-}\Upsilon(u_k) \le \frac{1}{p^-}\omega C_{\gamma,\rho}\beta_k^{p^+}, \tag{12}$$

where

$$C_{\gamma,\rho} = C_{\gamma,\rho}(\{\eta_k\}) := \frac{\gamma^n - 1}{\rho^{2p^+-n}(\gamma-1)^{2p^+}} (c_2 + (n-1)(\gamma-1)c_1)^{p^+}. \tag{13}$$

So, one has

$$\begin{aligned} &\Phi(u_k) + \Psi(u_k) \le |\lambda| \int_\Omega |h(x)| \left(u_k + \frac{|u_k|^{m(x)}}{m(x)} \right) dx - \mu \tilde{b} \beta_k^{s_2} + \frac{1}{p^-}\omega C_{\gamma,\rho}\beta_k^{p^+} \le \\ &\le |\lambda| \, \|h\|_{L^1(B(x_0,\rho))} \beta_k + \frac{|\lambda| \, \|h\|_{L^1(B(x_0,\rho))}}{m^-} \beta_k^{m^+} - \mu \tilde{b} \beta_k^{s_2} + \frac{1}{p^-}\omega C_{\gamma,\rho}\beta_k^{p^+} \end{aligned} \tag{14}$$

and, since $m^+ < p^+ < s_2$ and $\beta_k \to +\infty$ as $k \to \infty$, $\Phi(u_k) + \Psi(u_k) \to -\infty$, namely the functional $\Psi + \Phi$ does not possess any global minimum, as desired. This concludes the proof of (i).

The proof of (ii) and (iii) follows from slight modifications. When $s_1 < p^-$ and $s_2 = p^+$, set

$\mu_1 := \dfrac{\omega C_{\gamma,\rho}}{bp^-}$ (if $b = +\infty$, we agree to read, as usual, $\mu_1 = 0$). Then, if $\lambda \in \mathbb{R}$ and $\mu > \mu_1$, choosing $\dfrac{\omega C_{\gamma,\rho}}{\mu p^-} < \tilde{b} < b$, in the wake of the proof of (i), we obtain

$$\Phi(u_k) + \Psi(u_k) \le |\lambda| \, \|h\|_{L^1(B(x_0,\rho))} \beta_k + \frac{|\lambda| \, \|h\|_{L^1(B(x_0,\rho))}}{m^-} \beta_k^{m^+} - \mu \tilde{b} \beta_k^{p^+} + \frac{1}{p^-} \omega C_{\gamma,\rho} \beta_k^{p^+}$$

and also in this case $\Phi(u_k) + \Psi(u_k) \to -\infty$ thanks to the choice of $\tilde{b}$. On the other hand, in the case $s_1 = p^-$ and $s_2 > p^+$, it suffices to pick $\mu_2 := \dfrac{1}{ap^+ c_\infty^{p^-}}$ (as before, if $a = 0$, we set $\mu_2 = +\infty$). Then, fixing $\lambda \in \mathbb{R}$, $\mu < \mu_2$ and choosing $a < \tilde{a} < \dfrac{1}{\mu p^+ c_\infty^{p^-}}$, similarly to (i), using (g_2) with such an $\tilde{a}$ we get

$$\begin{aligned} &-\lambda \int_\Omega F(x,w)dx - \mu \int_\Omega G(x,w)dx \le |\lambda| \, \|h\|_{L^1(\Omega)} c_\infty (p^+)^{\frac{1}{p^-}} r_k^{\frac{1}{p^-}} + \\ &+ \frac{|\lambda| \, \|h\|_{L^1(\Omega)}}{m^-} c_\infty^{m^+} (p^+)^{\frac{m^+}{p^-}} r_k^{\frac{m^+}{p^-}} + \mu \tilde{a} c_\infty^{p^-} p^+ r_k < r_k \end{aligned}$$

for k large enough, due to the choice of $\tilde{a}$.
Finally, in the last case (iv), assumption (8) ensures that $\mu_1 < \mu_2$. So, in the light of (ii) and (iii), the conclusion is achieved for any $\lambda \in \mathbb{R}$ and $\mu \in (\mu_1, \mu_2)$. □

The next result is a direct consequence of Theorem **3.1** and deals with the case that the nonlinearities f and g have a particular form.

Theorem 3.2. *Let $h \in L^1(\Omega)$, $h_1 \in L^1(\Omega) \setminus \{0\}$ with $h_1 \ge 0$ in Ω, $m : \Omega \to \mathbb{R}$ measurable with $1 \le m \le p$ in Ω and $m^+ < p^-$. Let $\tilde{g} : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\int_0^t \tilde{g}(\xi) d\xi \ge 0$ for all $t \ge 0$. Finally assume that there exist two real sequences $\{\alpha_k\}$, $\{\beta_k\}$, with $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = +\infty$, and $s_1, s_2, \alpha, \beta > 0$, such that*

$$\max_{|\xi| \le \alpha_k} \int_0^\xi \tilde{g}(t)dt \le \alpha \alpha_k^{s_1}, \qquad \int_0^{\beta_k} \tilde{g}(t)dt \ge \beta \beta_k^{s_2}.$$

Then, considering the problem

$$\begin{cases} \Delta^2_{p(x)} u = \lambda h(x)|u|^{m(x)-2}u + \mu h_1(x)\tilde{g}(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \qquad (\tilde{P}_{\lambda,\mu})$$

the following facts hold:

- (i) *if $s_1 < p^-$ and $s_2 > p^+$, for all $\lambda \in \mathbb{R}$ and for all $\mu > 0$, $(\tilde{P}_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*
- (ii) *if $s_1 < p^-$ and $s_2 = p^+$, there exists $\mu_1 > 0$ such that, for all $\lambda \in \mathbb{R}$ and for all $\mu > \mu_1$, $(\tilde{P}_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*
- (iii) *if $s_1 = p^-$ and $s_2 > p^+$, there exists $\mu_2 > 0$ such that, for all $\lambda \in \mathbb{R}$ and for all $\mu \in (0, \mu_2)$, $(\tilde{P}_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

(*iv*) *if* $s_1 = p^-$ *and* $s_2 = p^+$, *there exist* $x_0 \in \Omega$, $\rho > 0$, $\gamma > 1$ *and* $C_{\gamma,\rho} > 0$ *such that, if*

$$C_{\gamma,\rho} < \frac{\beta p^- \|h_1\|_{L^1(B(x_0,\rho))}}{\alpha p^+ \|h_1\|_{L^1(\Omega)} \omega c_\infty^{p^-}}, \tag{15}$$

then $\mu_1 < \mu_2$ *and for all* $\lambda \in \mathbb{R}$ *and every* $\mu \in (\mu_1, \mu_2)$, $(\tilde{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions.*

Proof. The proof follows immediately by applying Theorem **3.1** to the nonlinearities

$$f(x,t) = h(x)|t|^{m(x)-2}t, \quad g(x,t) = h_1(x)\tilde{g}(t)$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Assumptions (f_1) and (g_1) are immediate to verify. Since $h_1 \not\equiv 0$, let us choose $x_0 \in \Omega$ and $\rho > 0$ such that $\overline{B}(x_0, \rho) \subset \Omega$ and $h_1 > 0$ in $B(x_0, \rho)$. One has

$$\int_\Omega \max_{|\xi|\le\alpha_k} G(x,\xi)dx = \int_\Omega \max_{|\xi|\le\alpha_k} \left(\int_0^\xi h_1(x)\tilde{g}(t)dt\right) dx =$$

$$= \|h_1\|_{L^1(\Omega)} \max_{|\xi|\le\alpha_k} \int_0^\xi \tilde{g}(t)dt \le \alpha \|h_1\|_{L^1(\Omega)} \alpha_k^{s_1}$$

and thus

$$\liminf_{t\to+\infty} \frac{\int_\Omega \max_{|\xi|\le t} G(x,\xi)dx}{t^{s_1}} \le \alpha \|h_1\|_{L^1(\Omega)} < +\infty. \tag{16}$$

In a similar way,

$$\int_{B(x_0,\rho)} G(x,\beta_k)dx = \|h_1\|_{L^1(B(x_0,\rho))} \int_0^{\beta_k} \tilde{g}(t)dt \ge \beta \|h_1\|_{L^1(B(x_0,\rho))} \beta_k^{s_2}$$

and therefore

$$\limsup_{t\to+\infty} \frac{\int_{B(x_0,\rho)} G(x,t)dx}{t^{s_2}} \ge \beta \|h_1\|_{L^1(B(x_0,\rho))} > 0. \tag{17}$$

So, the conclusions $(i) - (iii)$ follow directly from Theorem **3.1** with $a = \alpha \|h_1\|_{L^1(\Omega)}$ and $b = \beta \|h_1\|_{L^1(B(x_0,\rho))}$. As for (iv), inequality (8) is verified by the joint use of (15), (16) and (17). □

Obviously, Theorem **3.1** can be applied even in the case of a constant exponent p. Anyway, in this case, the assumption $m^+ < p^-$ doesn't allow to cover the case $m = p$. For this reason, the last theorem of this section concerns the case $m = p$. In this situation the existence of infinite weak solutions will be obtained not for each $\lambda \in \mathbb{R}$ but for λ running in an appropriate interval.

Theorem 3.3. *Let* $p > \max\{1, n/2\}$, $f, g \in \mathscr{C}$ *such that* (g_1) *and* (g_2) *are verified. Moreover, suppose that:*

$(\tilde{f}_1)$ *there exist* $h \in L^1(\Omega)$ *such that* $|f(x,t)| = h(x)\left(1 + |t|^{p-1}\right)$ *for a.e. in* Ω *and for all* $t \in \mathbb{R}$.

Then, considering the problem

$$\begin{cases} \Delta_p^2 u = \lambda h(x)|u|^{p-2}u + \mu g(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{$P^*_{\lambda,\mu}$}$$

the following facts hold:

(i) *if $s_1 < p < s_2$, for all λ such that $|\lambda| < \frac{1}{\|h\|_{L^1(\Omega)} c_\infty^p}$ (for all λ if $h = 0$) and for all $\mu > 0$, the problem ($P^*_{\lambda,\mu}$) admits a sequence of non-zero weak solutions;*

(ii) *if $s_1 < p = s_2$, there exists $\mu_1 > 0$ such that, for all $\mu > \mu_1$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$, the problem ($P^*_{\lambda,\mu}$) admits a sequence of non-zero weak solutions;*

(iii) *if $s_1 = p < s_2$, there exists $\mu_2 > 0$ such that, for all $\mu \in (0, \mu_2)$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$, the problem ($P^*_{\lambda,\mu}$) admits a sequence of non-zero weak solutions;*

(iv) *if $s_1 = s_2 = p$, there exists $\gamma > 1$ and $C_{\gamma,\rho} > 0$ such that, if*

$$C_{\gamma,\rho} < \frac{b}{a\omega c_\infty^p} \tag{18}$$

then $\mu_1 < \mu_2$ and for all $\mu \in (\mu_1, \mu_2)$, there exists $\lambda_\mu > 0$ such that, for all $|\lambda| < \lambda_\mu$ the problem ($P^*_{\lambda,\mu}$) admits a sequence of non-zero weak solutions.

Proof. The proof is similar to that of Theorem **3.1**; the two main evaluations (10) and (14), when $m = p$, read as follows

$$\begin{aligned} -\lambda \int_\Omega F(x,w)dx - \mu \int_\Omega G(x,w)dx \leq |\lambda|\, \|h\|_{L^1(\Omega)}\, c_\infty p^{\frac{1}{p}} r_k^{\frac{1}{p}} + \\ + |\lambda|\, \|h\|_{L^1(\Omega)}\, c_\infty^p r_k + \mu \tilde{a} c_\infty^{s_1} p^{\frac{s_1}{p}} r_k^{\frac{s_1}{p}} \end{aligned} \tag{19}$$

and

$$\begin{aligned} \Phi(u_k) + \Psi(u_k) \leq |\lambda|\, \|h\|_{L^1(B(x_0,\rho))}\, \beta_k + \\ + \frac{|\lambda|\, \|h\|_{L^1(B(x_0,\rho))}}{p} \beta_k^p - \mu \tilde{b} \beta_k^{s_2} + \frac{1}{p} \omega C_{\gamma,\rho} \beta_k^p \end{aligned} \tag{20}$$

with the same meaning of symbols as before.
To prove (i), fix λ such that $|\lambda| \leq \frac{1}{\|h\|_{L^1(\Omega)} c_\infty^p}$ and $\mu > 0$. Thanks to the choice of λ and to the fact that $s_1 < p$ then, from (19) we get

$$\lambda \int_\Omega F(x,w)dx + \mu \int_\Omega G(x,w)dx < r_k \tag{21}$$

for k large enough (remember that $\lim_{k\to\infty} r_k = +\infty$); moreover, from (20) we obtain

$$\lim_{k\to\infty} \Phi(u_k) + \Psi(u_k) = -\infty \tag{22}$$

because $p < s_2$.
To prove (ii), it is sufficient to choose $\mu_1 = \dfrac{\omega C_{\gamma,\rho}}{bp}$. Fixed $\mu > \mu_1$ and $\tilde{b}$ in a similar way as done in Theorem **3.1**, we define $\lambda_\mu = \min\left\{\dfrac{1}{\|h\|_{L^1(\Omega)} c_\infty^p}, \dfrac{\mu\tilde{b}p - \omega C_{\gamma,\rho}}{\|h\|_{L^1(B(x_0,\rho))}}\right\}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (19), we get (21) (for k large enough) because $s_1 < p$ and thanks to the choice of λ. Moreover, using (20), the choice of λ and μ guarantees that (22) holds.
To prove (iii), it is sufficient to choose $\mu_2 = \dfrac{1}{ac_\infty^p p}$. Fixed $\mu \in (0, \mu_2)$ and $\tilde{a}$ in a similar way as done in Theorem **3.1**, we choose $\lambda_\mu = \dfrac{1 - \mu\tilde{a}c_\infty^p p}{\|h\|_{L^1(\Omega)} c_\infty^p}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (20), we get (22) because $p < s_2$. Moreover, using (19), the choice of λ and μ guarantees that (21) holds.
In the last case, to prove (iv), we observe that, thanks to (18), we have $\mu_1 < \mu_2$. So, fixed $\mu \in (\mu_1, \mu_2)$, and choosing $\tilde{a}$ and $\tilde{b}$ in a similar way as done in Theorem **3.1**, we define $\lambda_\mu = \min\left\{\dfrac{1 - \mu\tilde{a}c_\infty^p p}{\|h\|_{L^1(\Omega)} c_\infty^p}, \dfrac{\mu\tilde{b}p - \omega C_{\gamma,\rho}}{\|h\|_{L^1(B(x_0,\rho))}}\right\}$. Fixed λ such that $|\lambda| < \lambda_\mu$, obviously, from (19), we get (21) (for k large enough) because of the choice of λ and μ. Moreover, using (20), the choice of λ and μ guarantees that (22) holds.

□

4. Examples

In this section we supply some examples related to the previous results. The first one concerns Theorem **3.2** (case (i)) and works as a prototype for this kind of nonlinearities.

Example 4.1. Let $h_1 \in L^1(\Omega) \setminus \{0\}$, $h_1 \geq 0$ in Ω. Choose $\alpha, \beta, s_1, s_2 > 0$ with $s_1 < p^-$ and $s_2 > p^+$, and let $\{\beta_r\}$ be a non-decreasing real sequence such that $\lim_{r\to\infty} \beta_r = +\infty$. Define a subsequence $\{\beta_{r_k}\}$ of $\{\beta_r\}$ and a new sequence $\{\alpha_k\}$ recursively as follows:

$$\beta_{r_1} > \left(\frac{\alpha}{\beta}\right)^{\frac{1}{s_2 - s_1}}, \quad \beta_{r_k} > \left(\frac{\beta}{\alpha}\right)^{\frac{1}{s_1}} \beta_{r_{k-1}}^{\frac{s_2}{s_1}} := \alpha_{k-1}, \quad \text{for all } k \geq 2. \tag{23}$$

Now, define $\hat{g} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{g}(t) := \begin{cases} 0 & \text{for } t \in (-\infty, 0] \\ At^3 + Bt^2 & \text{for } t \in (0, \beta_{r_1}] \\ \beta\beta_{r_k}^{s_2} & \text{for } t \in (\beta_{r_k}, \alpha_k], \quad k \geq 1 \\ C_k t^3 + D_k t^2 + E_k t + F_k & \text{for } t \in (\alpha_k, \beta_{r_{k+1}}], \quad k \geq 1 \end{cases} \tag{24}$$

where

$$
\begin{aligned}
A &:= -2\eta\beta_{r_1}^{s_2-3},\\
B &:= 3\beta\beta_{r_1}^{s_2-2},\\
C_k &:= -\frac{2(\beta\beta_{r_k}^{s_2}-\alpha\alpha_{k-1})}{(\beta_{r_k}-\alpha_{k-1})^3},\\
D_k &:= \frac{3(\beta_{r_k}+\alpha_{k-1})(\beta\beta_{r_k}^{2}-\alpha\alpha_{k-1}^{s_1})}{(\beta_{r_k}-\alpha_{k-1})^3},\\
E_k &:= -\frac{6\alpha_{k-1}\beta_{r_k}(\beta\beta_{r_k}^{s_2}-\alpha\alpha_{k-1}^{s_1})}{(\beta_{r_k}-\alpha_{k-1})^3},\\
F_k &:= \frac{\alpha\beta_{r_k}^{2}\alpha_{k-1}^{s_1}(\beta_{r_k}-3\alpha_{k-1})+\beta\beta_{r_k}^{s_2}\alpha_{k-1}^{2}(3\beta_{r_k}-\alpha_{k-1})}{(\beta_{r_k}-\alpha_{k-1})^3}.
\end{aligned}
$$

It is straightforward to verify that the sequences $\{\alpha_k\}$, $\{\beta_{r_k}\}$ defined by (23) and the function $\int_0^t \tilde{g}(\xi)d\xi := \hat{g}(t)$ satisfy all the requirements of Theorem **3.2**. Indeed, by construction, one has

$$
\max_{|\xi|\le\alpha_k}\hat{g}(\xi)=\beta\beta_{r_k}^{s_2}=\alpha\alpha_k^{s_1},
$$
$$
\hat{g}(\beta_{r_k})=\beta\beta_{r_k}^{s_2},
$$

for all $k\in\mathbb{N}$.

The second example is related to Theorem **3.2** (case (iv)). In this circumstance, for the sake of concreteness we limit ourselves to the one-dimensional setting, providing an explicit estimate of the constant c_∞ in (15).

Example 4.2. Let $n=1$, $\Omega=(-1,1)$, $p(x)=-2x^2+4$ for all $x\in(-1,1)$, $s_1=p^-=2$, $s_2=p^+=4$, $h_1\in L^1((-1,1))\setminus\{0\}$, $h_1\ge 0$ in $(-1,1)$ and $\int_{-1/2}^{1/2}h_1(x)dx>0$.

Assume $\{\alpha_k\}$, $\{\beta_{r_k}\}$, $\tilde{g}$ as in Example **4.1**. It is well-known that, for all $u\in W^{2,2}((-1,1))\cap W_0^{1,2}((-1,1))$, one has

$$
\max_{x\in(-1,1)}|u(x)|\le\frac{\sqrt{2}}{2}\left\|u'\right\|_{L^2((-1,1))}
$$

and

$$
\left\|u'\right\|_{L^2((-1,1))}\le\frac{2}{\pi}\left\|u''\right\|_{L^2((-1,1))},
$$

so

$$
\max_{x\in(-1,1)}|u(x)|\le\frac{\sqrt{2}}{\pi}\left\|u''\right\|_{L^2((-1,1))}.
$$

Now, since $L^{p(x)}((-1,1))\hookrightarrow L^2((-1,1))$ and

$$
\|u\|_{L^2((-1,1))}\le 2\max\left\{2^{\left(\frac{x^2-1}{2(x^2-2)}\right)^+},2^{\left(\frac{x^2-1}{2(x^2-2)}\right)^-}\right\}|u|_{p(x)}=2\sqrt[4]{2}|u|_{p(x)}
$$

(cf. Corollary 3.3.4 in Diening *et al.* 2011), collecting the previous estimates we finally get

$$\max_{x\in(-1,1)} |u(x)| \leq \frac{2\sqrt[4]{8}}{\pi} \|u\| .$$

Now choose $\rho = \frac{1}{2}$, $\gamma = \frac{3}{2}$ and for any $k \in \mathbb{N}$ let $\chi_k \in \mathscr{H}\left(\frac{1}{2}, \frac{3}{4}, \beta_{r_k}\right)$ be the function

$$\chi_k(x) := 64\beta_{r_k}\left(2x^3 - \frac{15}{4}x^2 + \frac{9}{4}x - \frac{27}{64}\right),$$

for any $x \in \left(\frac{1}{2}, \frac{3}{4}\right)$. The computations of the first two derivatives of η_k yield

$$|\chi'(x)| \leq 6\beta_{r_k}, \quad |\chi''(x)| \leq 96\beta_{r_k},$$

so (2) is satisfied by $c_1 = \dfrac{3}{2}$ and $c_2 = 6$, respectively.

As a next step, consider the sequence $\{u_k\} \subset W^{2,p(x)}((-1,1)) \cap W_0^{1,p(x)}((-1,1))$ defined by

$$u_k(x) := \begin{cases} 0 & \text{in } (-1,1) \setminus (-\frac{3}{4}, \frac{3}{4}) \\ \beta_{r_k} & \text{in } (-\frac{1}{2}, \frac{1}{2}) \\ \chi_k(|x|) & \text{in } (-\frac{3}{4}, \frac{3}{4}) \setminus (-\frac{1}{2}, \frac{1}{2}). \end{cases} \tag{25}$$

It turns out that $C_{\gamma,\rho} = 2^{18}3^4$. Hence, inequality (8) is fulfilled provided that

$$\frac{\beta}{\alpha} > \frac{2^{22}3^4\sqrt{2}}{\pi^2} \frac{\|h_1\|_{L^1((-1,1))}}{\|h_1\|_{L^1((-1/2,1/2))}}.$$

The third example is again related to Theorem **3.2** and it concerns a similar type of nonlinearity as in Example **4.1**.

Example 4.3. Let $p > 1$, $\delta > 1$ and let $\tilde{g} : \mathbb{R} \to \mathbb{R}$ be the function such that

$$\int_0^t \tilde{g}(\xi)d\xi = \begin{cases} 0, & \text{in } (-\infty, 0], \\ -2\delta t^3 + 3\delta t^2, & \text{in } (0,1], \\ 2^{p(k-1)}\delta^k & \text{in } \left(2^{k-1}\delta^{\frac{k-1}{p}}, 2^{k-1}\delta^{\frac{k}{p}}\right] \quad k \geq 1, \\ A_k t^3 + B_k t^2 + C_k t + D_k & \text{in } \left(2^{k-1}\delta^{\frac{k}{p}}, 2^k\delta^{\frac{k}{p}}\right] \quad k \geq 1 \end{cases}$$

where

$$A_k := -2^{(p-3)k+4}\delta^{\frac{(p-3)k}{p}}\left(\delta - 2^{-p}\right),$$

$$B_k := 9 \cdot 2^{(p-2)k+2}\delta^{\frac{(p-2)k}{p}}\left(\delta - 2^{-p}\right),$$

$$C_k := -3 \cdot 2^{(p-1)k+3}\delta^{\frac{(p-1)k}{p}}\left(\delta - 2^{-p}\right),$$

$$D_k := 2^{pk}\delta^k\left(5\delta - 2^{2-p}\right).$$

The function s satisfies all the assumption of Theorem **3.2** with $\alpha = 1$, $\beta = \delta$, $\alpha_k = 2^{k-1}\delta^{\frac{k}{p}}$ and $\beta_k = 2^k\delta^{\frac{k}{p}}$, for each $k \in \mathbb{N}$. In particular

$$\max_{|\xi|\leq\alpha_k}\int_0^{\xi}\tilde{g}(t)dt = \int_0^{2^{k-1}\delta^{\frac{k}{p}}}\tilde{g}(t)dt = 2^{p(k-1)}\delta^k = \alpha_k^p$$

and

$$\int_0^{\beta_k}\tilde{g}(t)dt = 2^{pk}\delta^{k+1} = \delta\beta_k^p$$

for all $k \in \mathbb{N}$.

The last example shows a case in which the choice of a sequence of functions with the norm raised to a second power makes Theorem **3.2** inapplicable, while the use of a different sequence solves the problem.

Example 4.4. Let $p > \max\{1, n/2\}$, $\Omega = B(0,1)$ in $\mathbb{R}^n$, $x_0 = 0$, $h_1 \in L^1(\Omega)\setminus 0$, with $h_1 \geq 0$, $\rho = \dfrac{1}{2}$, $\gamma = 2$ and $\{\sigma_k\} \subset]0,+\infty[$ with $\lim_{k\to\infty}\sigma_k = +\infty$. Let $\{\chi_k^1\}, \{\chi_k^2\} \in \mathscr{H}(\frac{1}{2}, 1, \{\sigma_k\})$ the sequences defined by

$$\chi_k^1(x) = 4\sigma_k(4x^3 - 9x^2 + 6x - 1)$$

and

$$\chi_k^2(x) = \frac{\sigma_k}{2}\cos(\pi(2x-1)+1)$$

for all $x \in \left(\dfrac{1}{2}, 1\right)$ and for each $k \in \mathbb{N}$. We observe that, for each $x \in \left(\dfrac{1}{2}, 1\right)$,

$$|\chi_k^{1'}(x)| \leq 3\sigma_k, \qquad |\chi_k^{1''}(x)| \leq 24\sigma_k$$

and then the constants $c_j(\{\chi_k^1\})$, defined in (2), are respectively $c_1(\{\chi_k^1\}) = \dfrac{3}{2}$ and $c_2(\{\chi_k^1\}) = 6$. In a similar way, for each $x \in \left(\dfrac{1}{2}, 1\right)$, we have

$$|\chi_k^{2'}(x)| \leq \pi\sigma_k, \qquad |\chi_k^{2''}(x)| \leq 2\pi^2\sigma_k$$

and, in this case, the constants $c_j(\{\chi_k^2\})$ are respectively $c_1(\{\chi_k^2\}) = \dfrac{\pi}{2}$ and $c_2(\{\chi_k^1\}) = \dfrac{\pi^2}{2}$.

The last sequence of test function that we take in consideration has been used by Candito *et al.* 2012 for the same kind of problem; in this case the norm is raised to the second power; namely

$$\chi_k^3(x) = \begin{cases} \sigma_k\left(-8x^2 + 8x - 1\right) & \text{in } (\frac{1}{2}, \frac{3}{4}) \\ \sigma_k(8x^2 - 16x + 8) & \text{in } (\frac{3}{4}, 1) \end{cases} \tag{26}$$

for each $k \in \mathbb{N}$. In this case

$$|\chi_k^{3'}(x)| \leq 4\sigma_k, \qquad |\chi_k^{3''}(x)| \leq 16\sigma_k$$

and then $c_1(\{\chi_k^3\}) = 2$ and $c_2(\{\chi_k^1\}) = 4$. With respect to these three sequences of test functions the best $C_{\gamma,\rho}$ depends on the values of n and p. For instance, for $n = 3$ and $p = 2$ the best $C_{\gamma,\rho}$ is the one in correspondance with the sequence $\{\chi_k^3\}$; in fact

$$C_{\gamma,\rho}(\{\chi_k^1\}) = 1134, \quad C_{\gamma,\rho}(\{\chi_k^2\}) \approx 913, \quad C_{\gamma,\rho}(\{\chi_k^3\}) = 896.$$

But, for instance, for $n = 4$ and $p = 3$, the best $C_{\gamma,\rho}$ is the one in correspondance with the sequence $\{\chi_k^2\}$ being

$$C_{\gamma,\rho}(\{\chi_k^1\}) = 69457,5 \quad C_{\gamma,\rho}(\{\chi_k^2\}) \approx 53871, \quad C_{\gamma,\rho}(\{\chi_k^3\}) = 60000.$$

Now, in any case, if we consider the function $\tilde{g}$ of Example **4.3**, taking $\delta > \dfrac{\omega c_\infty^p \|h_1\|_{L^1(\Omega)} C_{\gamma,\rho}}{\|h_1\|_{L^1(B(0,\frac{1}{2})}}$ the corresponding problem admits a sequence of non-zero weak solutions.

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