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A QUALITATIVE RESULT FOR HIGHER-ORDER DISCONTINUOUS IMPLICIT DIFFERENTIAL EQUATIONS

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ABSTRACT. Let $n,k \in \mathbf{N}$, and let $T > 0$, $Y \subseteq \mathbf{R}^n$ and $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$. Given a function $f : [0,T] \times (\mathbf{R}^n)^k \times Y \to \mathbf{R}$, we consider the Cauchy problem $f(t,u,u',\ldots,u^{(k)}) = 0$ in $[0,T]$, $u^{(i)}(0) = \xi_i$ for every $i = 0,1,\ldots,k-1$. We prove an existence and qualitative result for the generalized solutions of the above problem. In particular, we prove that, under suitable assumptions, the solution set $\mathscr{S}_T^f(\xi)$ of the above problem is nonempty, and the multifunction $\xi \in (\mathbf{R}^n)^k \to \mathscr{S}_T^f(\xi)$ admits an upper semicontinuous multivalued selection, with nonempty, compact and connected values. The assumptions of our result do not require any kind of continuity for the function $f(\cdot,\cdot,y)$. In particular, a function f satisfying our assumptions could be discontinuous, with respect to the second variable, even at all points $\xi \in (\mathbf{R}^n)^k$.

1. Introduction

Let $n,k \in \mathbf{N}$ be fixed, let $T > 0$, and let $p \in [1,+\infty]$. As usual, we denote by $W^{k,p}([0,T],\mathbf{R}^n)$ the space of all functions $u \in C^{k-1}([0,T],\mathbf{R}^n)$ such that $u^{(k-1)}$ is absolutely continuous in $[0,T]$ and $u^{(k)} \in L^p([0,T],\mathbf{R}^n)$. Now, let $Y \subseteq \mathbf{R}^n$ be a nonempty set, $f : [0,T] \times (\mathbf{R}^n)^k \times Y \to \mathbf{R}$ a given function, and let $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$. Let us consider the Cauchy problem

$$\begin{cases} f(t,u,u',\ldots,u^{(k-1)},u^{(k)}) = 0 & \text{in } [0,T], \\ u^{(i)}(0) = \xi_i & \text{for all } i = 0,\ldots,k-1\,. \end{cases} \tag{1.1}$$

We recall that a generalized solution of (**1.1**) is a function $u \in W^{k,1}([0,T],\mathbf{R}^n)$ such that

$$u^{(k)}(t) \in Y, \quad f(t,u(t),u'(t),\ldots,u^{(k-1)}(t),u^{(k)}(t)) = 0 \quad \text{for a.e.} \quad t \in [0,T],$$

and $u^{(i)}(0) = \xi_i$ for all $i = 0,\ldots,k-1$. For every $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, let us put

$$\mathscr{S}_T^f(\xi) := \{u \in W^{k,1}([0,T],\mathbf{R}^n) : \ u \text{ is a generalized solution of } (\mathbf{1.1})\ \}.$$

As remarked by Ricceri (1985) and Cubiotti (2018), the existence of generalized solutions of the implicit problem (**1.1**) associated with a discontinuous f has not been much studied in the literature, even for the case $k = 1$. Conversely, in the case $k = 1$, the existence of generalized solutions for the explicit case $f(t,u,u') = u' - g(t,u)$ (with g discontinuous in u) has been widely studied in the literature (see, for instance, Filippov 1964; Matrosov 1967; Cambini and Querci 1969; Pucci 1971; Pianigiani and Giuntini 1974; Sentis 1978; Binding 1979).

Very recently, the Cauchy problem (**1.1**) was studied in the particular case where $k = 1$ (Cubiotti 2018), and an existence and qualitative result for generalized solutions was proved by Cubiotti (2018, Theorem 3.1). In particular, it was proved that under suitable assumptions on f (which do not imply any kind of continuity with respect to the first two variables), the set $\mathscr{S}_T^f(\xi)$ is nonempty for every $\xi \in \mathbf{R}^n$, and the multifunction $\xi \in \mathbf{R}^n \to \mathscr{S}_T^f(\xi)$ admits an upper semicontinuous and compact valued multi-valued selection $\Phi : \mathbf{R}^n \to 2^{W^{1,\infty}([0,T],\mathbf{R}^n)}$.

Before stating explicitly the main result obtained by Cubiotti (2018), we need to introduce some notations. Firstly, we denote by $\mathscr{G}_n$ the family of all subsets $A \subseteq \mathbf{R}^n$ such that, for all $i = 1,\dots,n$, the supremum and the infimum of the projection of $\overline{\operatorname{conv}}(A)$ on the ith axis are both positive or both negative ("$\overline{\operatorname{conv}}$" standing for "closed convex hull"). Moreover, we denote by $\mathscr{D}$ the family of all sets $U \subseteq \mathbf{R} \times \mathbf{R}^n$ which can be represented as finite union of sets, each with at least one projection of null Lebesgue measure. Finally, the space $W^{1,\infty}([0,T].\mathbf{R}^n)$ is considered with the initial topology that makes the function $u \to (u,u')$ continuous from $W^{1,\infty}([0,T],\mathbf{R}^n)$ to $C^0([0,T],\mathbf{R}^n) \times L^\infty([0,T],\mathbf{R}^n)$, where the first space is considered with its strong topology, and the second with its weak-star topology. The following is Theorem 3.1 in (Cubiotti 2018).

Theorem 1.1. *Let $Y \in \mathscr{G}_n$ be a compact, connected and locally connected set. Let $f : [0,T] \times \mathbf{R}^n \times Y \to \mathbf{R}$ be a given function, D_1, D_2 two dense subsets of Y. Let $S := [0,T] \times \mathbf{R}^n$. Assume that there exists $U \subseteq S$, with $U \in \mathscr{D}$, such that:*

(i) *for every $y \in D_1$, the function $f(\cdot,\cdot,y)|_{S\setminus U}$ is lower semicontinuous;*
(ii) *for every $y \in D_2$, the function $f(\cdot,\cdot,y)|_{S\setminus U}$ is upper semicontinuous;*
(iii) *for every $(t,x) \in S \setminus U$, the function $f(t,x,\cdot)$ is continuous in Y, $0 \in \operatorname{int}_{\mathbf{R}}(f(t,x,Y))$ and*

$$\operatorname{int}_Y(\{y \in Y : \ f(t,x,y) = 0\}) = \emptyset.$$

Then, for every $\xi \in \mathbf{R}^n$, the solution set

$$\mathscr{S}(\xi) := \{u \in W^{1,1}([0,T],\mathbf{R}^n) : \ u(0) = \xi \text{ and } f(t,u(t),u'(t)) = 0 \text{ a.e. in } [0,T]\ \}$$

is nonempty. Moreover, there exists an upper semicontinuous multifunction

$$\Phi : \mathbf{R}^n \to 2^{W^{1,\infty}([0,T];\mathbf{R}^n)}$$

with nonempty compact acyclic values, such that:

(a) $\Phi(\xi) \subseteq \mathscr{S}_T(\xi)$ *for all* $\xi \in \mathbf{R}^n$*;*
(b) *the multifunction*

$$\xi \in \mathbf{R}^n \to \{u(T) : u \in \Phi(\xi)\}$$

is upper semicontinuous with nonempty connected and compact values;

(c) *the multifunction*

$$\xi \in \mathbf{R}^n \to \{u' \in L^\infty([0,T],\mathbf{R}^n) : \ u \in \Phi(\xi)\}$$

is upper semicontinuous (with compact values) from $\mathbf{R}^n$ to $L^\infty([0,T],\mathbf{R}^n)$, endowed with its weak-star topology;

(d) *for every* $\xi \in \mathbf{R}^n$ *and every* $u \in \Phi(\xi)$, *one has that* $(t, u(t)) \in S \setminus U$ *for a.e.* $t \in [0, T]$.

It is worth noticing that one of the peculiarities of Theorem **1.1** resides in the kind of discontinuity allowed for f. In particular, a function $f : [0,T] \times \mathbf{R}^n \times \mathrm{Y} \to \mathbf{R}$ satisfying the assumptions of Theorem **1.1** could be discontinuous, with respect to the second variable, even at all points $x \in \mathbf{R}^n$ (Cubiotti 2018, see Remark 3.3). As a matter of fact, the function f could be even defined only over the set $(S \setminus U) \times Y$, since its behaviour over the set $U \times Y$ plays no role.

At this point, it is natural to ask if Theorem **1.1** can be extended to the general k-order Cauchy problem (**1.1**). The aim of this paper is exactly to provide such an extension. We observe that such an extension is not immediate and requires a more articulate technical construction. In particular, we shall need to prove an existence and qualitative result for the generalized solutions of non-convex differential inclusions whose right-hand side may not have any property of lower semicontinuity. This will be done in Section 3, while in Section 2 we shall give some notations and preliminaries.

2. Preliminaries

As before, let $n, k \in \mathbf{N}$, and let $T > 0$. In what follows, we consider the space $W^{k,\infty}([0,T], \mathbf{R}^n)$ with the initial topology $\sigma^T_{n,k}$ that makes the function

$$u \in W^{k,\infty}([0,T], \mathbf{R}^n) \to (u, u^{(k)}) \in C^{k-1}([0,T], \mathbf{R}^n) \times L^\infty([0,T], \mathbf{R}^n)$$

continuous, where the space $C^{k-1}([0,T], \mathbf{R}^n)$ is considered with its strong topology, and the space $L^\infty([0,T], \mathbf{R}^n)$ with its weak-star topology.

Let $F : [0,T] \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ be a multifunction. Let $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$ be fixed, and let us consider the Cauchy problem

$$\begin{cases} u^{(k)} \in F(t, u, u', \ldots, u^{(k-1)}) & \text{in } [0,T], \\ u^{(i)}(0) = \xi_i & \text{for each } i = 0, \ldots, k-1\,. \end{cases} \qquad \textbf{(2.1)}$$

We recall that a generalized solution of problem (**2.1**) is a function $u \in W^{k,1}([0,T], \mathbf{R}^n)$ such that

$$u^{(k)}(t) \in F(t, u(t), u'(t), \ldots, u^{(k-1)}(t)) \quad \text{for a.e.} \quad t \in [0,T]$$

and $u^{(i)}(0) = \xi_i$ for all $i = 0, \ldots, k-1$. For each $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, we put

$$\mathscr{T}^F_T(\xi) := \left\{ u \in W^{k,1}([0,T], \mathbf{R}^n) : u \text{ is a generalized solution of (2.1)} \right\},$$

$$\mathscr{A}^F_T(\xi) := \left\{ u(T) : u \in \mathscr{T}^F_T(\xi) \right\}.$$

In the following, we shall often make the obvious identification $(\mathbf{R}^n)^k = \mathbf{R}^{nk}$. For all $i = 0, 1, \ldots, nk$, we denote by $P_j : \mathbf{R} \times \mathbf{R}^{nk} \to \mathbf{R}$ the projection over the j-th axis. That is, if $(t, x) = (t, x_1, x_2, \ldots, x_{nk}) \in \mathbf{R} \times \mathbf{R}^{nk}$, we put

$$P_j(t,x) = \begin{cases} t & \text{if } j = 0, \\ x_j & \text{if } j \in \{1, 2, \ldots, nk\}. \end{cases}$$

For every $j \in \mathbf{N}$, we shall denote by m_j the j-dimensional Lebesgue measure in $\mathbf{R}^j$. Moreover, we shall denote by $\mathscr{F}$ the family of all subsets $U \subseteq \mathbf{R} \times \mathbf{R}^{nk}$ such that there exist sets

$V_0, V_1, \ldots, V_{nk} \subseteq \mathbf{R} \times \mathbf{R}^{nk}$, with $m_1(P_j(V_j)) = 0$ for all $j = 0, 1 \ldots, nk$, such that $U = \bigcup_{j=0}^{nk} V_j$. Of course, any set $U \in \mathscr{F}$ satisfies $m_{nk+1}(U) = 0$.

Let $m \in \mathbf{N}$. We shall denote by $B_m(x,r)$ (resp., $\overline{B}_m(x,r)$) the open (resp., closed) ball in $\mathbf{R}^m$ with respect to the Euclidean norm $\|\cdot\|_m$ of $\mathbf{R}^m$. Finally, we shall denote by $\mathscr{B}(\mathbf{R})$ and $\mathscr{L}([a,b])$, respectively, the Borel family of $\mathbf{R}$ and the family of all Lebesgue measurable subsets of the interval $[a,b]$. For the reader's convenience, we now state some results that will be useful in the sequel.

Proposition 2.1. (Cubiotti and Yao 2015, Proposition 2.6). *Let $\psi : [a,b] \times \mathbf{R}^n \to \mathbf{R}^k$ be a given function, $E \subseteq \mathbf{R}^n$ a Lebesgue measurable set, with $m_n(E) = 0$, and let D be a countable dense subset of $\mathbf{R}^n$, with $D \cap E = \emptyset$. Assume that:*

(i) *for all $t \in [a,b]$, the function $\psi(t,\cdot)$ is bounded;*
(ii) *for all $x \in D$, the function $\psi(\cdot,x)$ is $\mathscr{L}([a,b])$-measurable.*

Let $G : [a,b] \times \mathbf{R}^n \to 2^{\mathbf{R}^k}$ be the multifunction defined by setting, for each $(t,x) \in [a,b] \times \mathbf{R}^n$,

$$G(t,x) := \bigcap_{m \in \mathbf{N}} \overline{\mathrm{conv}} \overline{\left(\bigcup_{\substack{y \in D \\ \|y-x\|_n \leq \frac{1}{m}}} \{\psi(t,y)\} \right)}.$$

Then, one has:

(a) *G has nonempty closed convex values;*
(b) *for all $x \in \mathbf{R}^n$, the multifunction $G(\cdot,x)$ is $\mathscr{L}([a,b])$-measurable;*
(c) *for all $t \in [a,b]$, the multifunction $G(t,\cdot)$ has closed graph;*
(d) *if $t \in [a,b]$, and $\psi(t,\cdot)|_{\mathbf{R}^n \setminus E}$ is continuous at $x \in \mathbf{R}^n \setminus E$, then one has*

$$G(t,x) = \{\psi(t,x)\}.$$

The following result summarizes several results proved in (Aubin and Cellina 1984, pp. 103–109).

Theorem 2.2. *Let $x^* \in \mathbf{R}^n$, and let $\Omega \subseteq \mathbf{R} \times \mathbf{R}^n$ be an open set, such that $(0,x^*) \in \Omega$. Let $G : \Omega \to 2^{\mathbf{R}^n}$ be an upper semicontinuous multifunction, with nonempty compact convex values. Assume that there exist $M > 0$, $b > 0$, $T > 0$ such that*

$$Q := [0,T] \times \overline{B}_n(x^*, b + MT) \subseteq \Omega \quad \text{and} \quad G(Q) \subseteq \overline{B}_n(0,M).$$

Then, one has:

(i) *For every $\xi \in B_n(x^*,b)$, the solution set*

$$\mathscr{T}_T^G(\xi) := \{u \in W^{1,1}([0,T],\mathbf{R}^n) : u(0) = \xi \text{ and } u'(t) \in G(t,u(t)) \text{ a.e. in } [0,T]\}$$

is nonempty. Moreover, the multifunction $\xi \to \mathscr{T}_T^G(\xi)$ is upper semicontinuous from $B_n(x^*,b)$ to $W^{1,\infty}([0,T],\mathbf{R}^n)$, with nonempty, compact, acyclic and connected values;

(ii) *The multifunction $\xi \to \mathscr{A}_T^G(\xi) := \{u(T) : u \in \mathscr{T}_T^G(\xi)\}$ is upper semicontinuous from $B_n(x^*,b)$ to $\mathbf{R}^n$, with nonempty compact connected values;*

For the basic facts and definitions about multifunctions, we refer the reader to Klein and Thompson (1984) and Aubin and Frankowska (1990).

3. Results

The following is our main result.

Theorem 3.1. *Let $n,k \in \mathbf{N}$, and let Y be a compact, connected and locally connected subset of $\mathbf{R}^n$, with $Y \in \mathscr{G}_n$. Let $T > 0$, and let $f : [0,T] \times (\mathbf{R}^n)^k \times Y \to \mathbf{R}$ be a given function. Let $S := [0,T] \times (\mathbf{R}^n)^k$. Assume that there exist a set $U \subseteq S$, with $U \in \mathscr{F}$, and two dense subsets D_1, D_2 of Y, such that:*

(i) *for every $y \in D_1$, the function $f(\cdot,\cdot,y)|_{S\setminus U}$ is lower semicontinuous;*
(ii) *for every $y \in D_2$, the function $f(\cdot,\cdot,y)|_{S\setminus U}$ is upper semicontinuous;*
(iii) *for every $(t,\xi) \in S \setminus U$, the function $f(t,\xi,\cdot)$ is continuous in Y, $0 \in \mathrm{int}_{\mathbf{R}}(f(t,\xi,Y))$ and*

$$\mathrm{int}_Y(\{y \in Y : \ f(t,\xi,y) = 0\}) = \emptyset.$$

Then, for every $\xi \in (\mathbf{R}^n)^k$, the solution set $\mathscr{S}_T^f(\xi)$ of problem (**1.1**) *is nonempty. Moreover, there exists a multifunction*

$$\Psi : (\mathbf{R}^n)^k \to 2^{W^{k,\infty}([0,T];\mathbf{R}^n)}$$

such that:

(a) *$\Psi(\xi) \subseteq \mathscr{S}_T^f(\xi)$ for all $\xi \in (\mathbf{R}^n)^k$;*
(b) *Ψ is upper semicontinuous (with respect to the topology $\sigma_{n,k}^T$), with nonempty connected and compact values;*
(c) *the multifunction*

$$\xi \in (\mathbf{R}^n)^k \to \{u(T) : u \in \Psi(\xi)\}$$

is upper semicontinuous with nonempty connected and compact values;
(d) *for every $\xi \in (\mathbf{R}^n)^k$ and every $u \in \Psi(\xi)$, one has that*

$$(t, u(t), u'(t), \dots, u^{(k-1)}(t)) \in S \setminus U \quad \text{for a.e.} \quad t \in [0,T].$$

(e) *the multifunction*

$$\xi \in (\mathbf{R}^n)^k \to \{u^{(k)} \in L^\infty([0,T],\mathbf{R}^n) : \ u \in \Psi(\xi)\}$$

is upper semicontinuous (with compact connected values), with respect to the weak-star topology of $L^\infty([0,T],\mathbf{R}^n)$.

We shall prove Theorem **3.1** as a consequence of the following qualitative result for higher-order differential inclusions.

Theorem 3.2. *Let $n,k \in \mathbf{N}$ and $T \in \mathbf{R}$, with $0 < T < 1$. Let $F : [0,T] \times (\mathbf{R}^n)^k \to 2^{\mathbf{R}^n}$ a given multifunction. Put $S := [0,T] \times (\mathbf{R}^n)^k$. Assume that there exist $M > 0$ and two sets $Q_0, Q \in \mathscr{B}(\mathbf{R})$, with $Q_0 \subseteq [0,T]$ and $m_1(Q_0) = m_1(Q) = 0$, such that, if one puts*

$$\Omega := ([0,T] \setminus Q_0) \times (\mathbf{R} \setminus Q)^{nk},$$

one has:

(i) *$F|_\Omega$ is lower semicontinuous with nonempty and closed values;*
(ii) *$F(\Omega) \in \mathscr{G}_n$ and $F(\Omega) \subseteq \overline{B}_n(0,M)$.*

Then, for every $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$*, the solution set* $\mathcal{T}_T^F(\xi)$ *of problem* **(2.1)** *is nonempty. Moreover, there exists a multifunction*

$$\Phi : (\mathbf{R}^n)^k \to 2^{W^{k,\infty}([0,T];\mathbf{R}^n)}$$

such that:

(a) $\Phi(\xi) \subseteq \mathcal{T}_T^F(\xi)$ *for all* $\xi \in (\mathbf{R}^n)^k$*;*

(b) Φ *is upper semicontinuous (with respect to the topology* $\sigma_{n,k}^T$*), with nonempty, compact and connected values;*

(c) *the multifunction*

$$\xi \in (\mathbf{R}^n)^k \to \{u(T) : u \in \Phi(\xi)\}$$

is upper semicontinuous with nonempty connected and compact values;

(d) *for every* $\xi \in (\mathbf{R}^n)^k$ *and every* $u \in \Phi(\xi)$*, one has that*

$$(t, u(t), u'(t), \dots, u^{(k-1)}(t)) \in \Omega \quad \textit{for a.e.} \quad t \in [0,T].$$

(e) *the multifunction*

$$\xi \in (\mathbf{R}^n)^k \to \{u^{(k)} \in L^\infty([0,T], \mathbf{R}^n) : \ u \in \Phi(\xi)\}$$

is upper semicontinuous (with compact connected values), whith respect to the weak-star topology of $L^\infty([0,T], \mathbf{R}^n)$*.*

Proof. By Lemma 2.4 of Cubiotti and Yao (2014), there exist two sets $Q_0', Q' \in \mathcal{B}(\mathbf{R})$, with $Q_0' \subseteq [0,T]$ and $m_1(Q_0') = m_1(Q') = 0$, and a function $\phi : \Omega \to \mathbf{R}^n$ such that:

(a)$'$ $\phi(t,\xi) \in F(t,\xi)$ for all $(t,\xi) \in \Omega$;

(b)$'$ ϕ is continuous at each point

$$(t,\xi) \in \left[([0,T] \setminus Q_0') \times (\mathbf{R} \setminus Q')^{nk}\right] \cap \Omega =$$
$$= ([0,T] \setminus (Q_0 \cup Q_0')) \times (\mathbf{R} \setminus (Q \cup Q'))^{nk}.$$

Fix any point $y^* \in \phi(\Omega)$, and let $\phi^* : \mathbf{R} \times \mathbf{R}^{nk} \to \mathbf{R}^n$ be defined by putting

$$\phi^*(t,\xi) = \begin{cases} \phi(t,\xi) & \text{if } (t,\xi) \in \Omega, \\ y^* & \text{if } (t,\xi) \in (\mathbf{R} \times \mathbf{R}^{nk}) \setminus \Omega. \end{cases}$$

Let

$$W := \left[(Q_0 \cup Q_0' \cup \{0,T\}) \times \mathbf{R}^{nk}\right] \cup \left[S \cap \bigcup_{i=1}^{nk} P_i^{-1}(Q \cup Q')\right].$$

Of course, $W \subseteq S$. We claim that $\phi^*|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus W}$ is continuous. To see this, fix $(t,\xi) \in (\mathbf{R} \times \mathbf{R}^{nk}) \setminus W$. Firstly, assume that $(t,\xi) \notin S$. Since $(\mathbf{R} \times \mathbf{R}^{nk}) \setminus S$ is open in $\mathbf{R} \times \mathbf{R}^{nk}$ and

$$(\mathbf{R} \times \mathbf{R}^{nk}) \setminus S \subseteq (\mathbf{R} \times \mathbf{R}^{nk}) \setminus W,$$

the set $(\mathbf{R} \times \mathbf{R}^{nk}) \setminus S$ is open in $(\mathbf{R} \times \mathbf{R}^{nk}) \setminus W$. Since ϕ^* is constant over $(\mathbf{R} \times \mathbf{R}^{nk}) \setminus S$, we get that $\phi^*|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus W}$ is continuous at (t,ξ), as desired. Conversely, assume that $(t,\xi) \in S$. Hence,

$$(t,\xi) \in S \setminus W = (]0,T[\times \mathbf{R}^{nk}) \setminus W.$$

Since $S \setminus W \subseteq \Omega$, we get

$$\phi^*|_{(]0,T[\times\mathbf{R}^{nk})\setminus W} = \phi|_{(]0,T[\times\mathbf{R}^{nk})\setminus W}. \tag{3.1}$$

By (b)$'$, the function $\phi|_{(]0,T[\times\mathbf{R}^{nk})\setminus W}$ is continuous. Consequently, by (**3.1**), the function $\phi^*|_{(]0,T[\times\mathbf{R}^{nk})\setminus W}$ is continuous. Since the set $(]0,T[\times\mathbf{R}^{nk})\setminus W$ is an open neighborhood of (t,ξ) in $(\mathbf{R}\times\mathbf{R}^{nk})\setminus W$, the function $\phi^*|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus W}$ is continuous at (t,ξ), as desired.

Now, observe that, by construction and by assumptions (i) and (ii), we have

$$\phi^*(\mathbf{R}\times\mathbf{R}^{nk}) \subseteq F(\Omega) \subseteq \overline{B}_n(0,M)$$

and $\phi^*(\mathbf{R}\times\mathbf{R}^{nk}) \in \mathscr{G}_n$.

Let $D \subseteq \mathbf{R}\times\mathbf{R}^{nk}$ be a countable set, dense in $\mathbf{R}\times\mathbf{R}^{nk}$, such that $D\cap W = \emptyset$. Of course, such a set D exists since $m_{1+nk}(W) = 0$.

Let $G : \mathbf{R}\times\mathbf{R}^{nk} \to 2^{\mathbf{R}^n}$ be the multifunction defined by putting, for each $(t,\xi) \in \mathbf{R}\times\mathbf{R}^{nk}$,

$$G(t,\xi) := \bigcap_{m\in\mathbf{N}} \overline{\operatorname{conv}\left(\bigcup_{\substack{(\lambda,\eta)\in D\\ \|(\lambda,\eta)-(t,\xi)\|_{1+nk}\leq\frac{1}{m}}} \{\phi^*(\lambda,\eta)\}\right)}.$$

By Proposition **2.1**, taking into account that

$$\phi^*|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus W} = \phi|_{(\mathbf{R}\times\mathbf{R}^{nk})\setminus W}$$

and that the latter function is continuous by (b)$'$, we get that:

(a)$''$ G has nonempty closed convex values;
(b)$''$ the multifunction G has closed graph;
(c)$''$ for every $(t,\xi) \in (\mathbf{R}\times\mathbf{R}^{nk})\setminus W$, one has

$$G(t,\xi) = \{\phi^*(t,\xi)\} = \{\phi(t,\xi)\}.$$

Moreover, $G(\mathbf{R}\times\mathbf{R}^{nk}) \subseteq \overline{\operatorname{conv}}(F(\Omega))$, hence

$$G(\mathbf{R}\times\mathbf{R}^{nk}) \in \mathscr{G}_n$$

and

$$G(\mathbf{R}\times\mathbf{R}^{nk}) \subseteq \overline{B}_n(0,M). \tag{3.2}$$

By (**3.2**) and (b)$''$, we have that G is upper semicontinuous (Klein and Thompson 1984, see Theorem 7.1.16). From now on, for the reader's convenience, we shall divide the proof into steps.

STEP 1. Of course, we can regard the multifunction G as defined on $\mathbf{R}\times(\mathbf{R}^n)^k$, by means of the obvious identification $(\mathbf{R}^n)^k = \mathbf{R}^{nk}$. For every $\xi = (\xi_0,\xi_1,\ldots,\xi_{k-1}) \in (\mathbf{R}^n)^k$, let us consider the Cauchy problem

$$\begin{cases} u^{(k)} \in G(t,u,u',\ldots,u^{(k-1)}) & \text{in } [0,T],\\ u^{(i)}(0) = \xi_i & \text{for each } i = 0,\ldots,k-1\,, \end{cases} \tag{3.3}$$

and its solution set

$$\mathscr{T}_T^G(\xi) := \left\{u \in W^{k,1}([0,T],\mathbf{R}^n) : u \text{ is a generalized solution of } (\mathbf{3.3})\right\}.$$

We claim that

$$\mathscr{T}_T^G(\xi) \subseteq \mathscr{T}_T^F(\xi) \quad \text{for all} \quad \xi \in (\mathbf{R}^n)^k. \tag{3.4}$$

To see this, fix $\xi \in (\mathbf{R}^n)^k$, and let $u \in \mathscr{T}_T^G(\xi)$. Hence, there exists $\Sigma_0 \in \mathscr{L}([0,T])$, with $m_1(\Sigma_0) = 0$, such that

$$u^{(k)}(t) \in G(t, u(t), u'(t), \ldots, u^{(k-1)}(t)) \quad \text{for all} \quad t \in [0,T] \setminus \Sigma_0 . \tag{3.5}$$

In particular, we have

$$u^{(k)}(t) \in \overline{\operatorname{conv}}\,(F(\Omega)) \quad \text{for all} \quad t \in [0,T] \setminus \Sigma_0 . \tag{3.6}$$

Fix $i \in \{1, \ldots, n\}$, and let us denote by u_i the i-th component of the function u. Since $F(\Omega) \in \mathscr{G}_n$, by (**3.6**) we get that $u_i^{(k)}(t)$ has constant sign for all $t \in [0,T] \setminus \Sigma_0$. Assume that

$$u_i^{(k)}(t) > 0 \quad \text{for all} \quad t \in [0,T] \setminus \Sigma_0$$

(if $u_i^{(k)}(t) < 0$ for all $t \in [0,T] \setminus \Sigma_0$, then the argument is analogous). This implies that the absolutely continuous function $u_i^{(k-1)}$ is strictly increasing in $[0,T]$ (with a.e. positive derivative). By Theorem 2 of Villani (1984), the function $(u_i^{(k-1)})^{-1}$ is absolutely continuous, hence by Theorem 18.25 of Hewitt and Stromberg (1965) the set

$$\Sigma_{k-1,i} := (u_i^{(k-1)})^{-1}(Q \cup Q') = \{t \in [0,T] : u_i^{(k-1)}(t) \in Q \cup Q'\}$$

has null Lebesgue measure. Since $u_i^{(k-1)}$ is strictly increasing in $[0,T]$, there exists a partition

$$0 = t_{k-1,0} < \ldots t_{k-1,j_{k-1}} = T$$

(with $j_{k-1} \leq 2$) such that $u_i^{(k-1)}$ has constant sign over each interval $]t_{k-1,l-1}, t_{k-1,l}[$ (in particular, $u_i^{(k-1)}(t) \neq 0$ on each interval $]t_{k-1,l-1}, t_{k-1,l}[$). This implies that, for every $l = 1, \ldots, j_{k-1}$, the function $u_i^{(k-2)}|_{[t_{k-1,l-1}, t_{k-1,l}]}$ is strictly monotone. Consequently, for each $l = 1, \ldots, j_{k-1}$, by Theorem 2 of Villani (1984) the function

$$(u_i^{(k-2)}|_{[t_{k-1,l-1}, t_{k-1,l}]})^{-1}$$

is absolutely continuous, hence (Theorem 18.25 of Hewitt and Stromberg (1965)) it maps null sets into null sets. Consequently, for every $l = 1, \ldots, j_{k-1}$ the set

$$(u_i^{(k-2)}|_{[t_{k-1,l-1}, t_{k-1,l}]})^{-1}(Q \cup Q') = \left\{t \in [t_{k-1,l-1}, t_{k-1,l}] : \, u_i^{(k-2)}(t) \in Q \cup Q'\right\}$$

has null Lebesgue measure. Thus, we easily get that the whole set

$$\Sigma_{k-2,i} := (u_i^{(k-2)})^{-1}(Q \cup Q') = \{t \in [0,T] : u_i^{(k-2)}(t) \in Q \cup Q'\}$$

has null Lebesgue measure. Since the function $u_i^{(k-2)}$ is strictly monotone on each interval $[t_{k-1,l-1}, t_{k-1,l}]$, with $l = 1, \ldots, j_{k-1}$, there exists a partition

$$0 = t_{k-2,0} < \ldots t_{k-2,j_{k-2}} = T$$

(with $j_{k-2} \leq 4$) such that $u_i^{(k-2)}$ has constant sign over each interval $]t_{k-2,l-1}, t_{k-2,l}[$ (in particular, $u_i^{(k-2)}(t) \neq 0$ on each interval $]t_{k-2,l-1}, t_{k-2,l}[$). This implies that, for every

$l=1,\dots,j_{k-2}$, the function $u_i^{(k-3)}|_{[t_{k-2,l-1},t_{k-2,l}]}$ is strictly monotone. Consequently, for each $l=1,\dots,j_{k-2}$, by Theorem 2 of Villani (1984) the function

$$(u_i^{(k-3)}|_{[t_{k-2,l-1},t_{k-2,l}]})^{-1}$$

is absolutely continuous, hence (Theorem 18.25 of Hewitt and Stromberg (1965)) it maps null sets into null sets. Consequently, the set

$$\Sigma_{k-3,i}:=[u_i^{(k-3)}]^{-1}(Q\cup Q')=\{t\in[0,T]:u_i^{(k-3)}(t)\in Q\cup Q'\}$$

has null Lebesgue measure. If we now apply recursively the same argument, we have that, if for each $j=0,\dots,k-1$ we put

$$\Sigma_{j,i}:=[u_i^{(j)}]^{-1}(Q\cup Q')=\{t\in[0,T]:u_i^{(j)}(t)\in Q\cup Q'\},$$

then we get $m_1(\Sigma_{j,i})=0$. Now, put

$$\Sigma:=\{0,T\}\cup Q_0\cup Q_0'\cup\Sigma_0\cup\Big[\bigcup_{\substack{i=1,\dots,n\\ j=0,\dots,k-1}}\Sigma_{j,i}\Big].$$

By the above argument, we get $m_1(\Sigma)=0$. Fix $\hat{t}\in[0,T]\setminus\Sigma=]0,T[\setminus\Sigma$. Since $\hat{t}\notin\bigcup_{\substack{i=1,\dots,n\\ j=0,\dots,k-1}}\Sigma_{j,i}$, we have that

$$u_i^{(j)}(\hat{t})\notin Q\cup Q'$$

for every $i=1,\dots,n$ and every $j=0,\dots,k-1$. Consequently, we have

$$(\hat{t},u(\hat{t}),u'(\hat{t}),\dots,u^{(k-1)}(\hat{t}))\notin W, \tag{3.7}$$

hence by (c)$''$ we get

$$\begin{aligned}G(\hat{t},u(\hat{t}),u'(\hat{t}),\dots,u^{(k-1)}(\hat{t}))&=\{\phi^*(\hat{t},u(\hat{t}),u'(\hat{t}),\dots,u^{(k-1)}(\hat{t}))\}=\\&=\{\phi(\hat{t},u(\hat{t}),u'(\hat{t}),\dots,u^{(k-1)}(\hat{t}))\}.\end{aligned}$$

By (**3.5**) and (a)$'$ we get

$$\begin{aligned}u^{(k)}(\hat{t})=\phi(\hat{t},u(\hat{t}),u'(\hat{t}),\dots,u^{(k-1)}(\hat{t}))\in\\ \in F(\hat{t},u(\hat{t}),u'(\hat{t}),\dots,u^{(k-1)}(\hat{t})).\end{aligned}$$

Therefore, $u\in\mathscr{T}_T^F(\xi)$. Consequently, (**3.4**) is proved. Moreover, by (**3.7**), the above argument shows that, for every $\xi\in(\mathbf{R}^n)^k$ and every $u\in\mathscr{T}_T^G(\xi)$, one has

$$(t,u(t),u'(t),\dots,u^{(k-1)}(t))\in S\setminus W\subseteq\Omega\quad\text{for a.e.}\quad t\in[0,T]. \tag{3.8}$$

STEP 2. Let $\Psi:\mathbf{R}\times(\mathbf{R}^n)^k\to 2^{(\mathbf{R}^n)^k}$ be the multifunction defined by setting, for each $\xi=(\xi_0,\xi_1,\dots,\xi_{k-1})\in(\mathbf{R}^n)^k$,

$$\Psi(t,\xi)=\Psi(t,\xi_0,\xi_1,\dots,\xi_{k-1})=\{\xi_1\}\times\{\xi_2\}\times\cdots\times\{\xi_{k-1}\}\times G(t,\xi).$$

By the upper semicontinuity of G and by Theorem 7.3.14 of Klein and Thompson (1984), we have that Ψ is upper semicontinuous. For every fixed $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, let us consider the first-order Cauchy problem

$$\begin{cases} \lambda' \in \Psi(t,\lambda) & \text{in } [0,T] \\ \lambda(0) = \xi\,, \end{cases} \tag{3.9}$$

with solution set

$$\mathscr{T}_T^{\Psi}(\xi) := \Big\{ \lambda(t) = (\lambda_0(t), \lambda_1(t), \dots, \lambda_{k-1}(t)) \in (W^{1,1}([0,T], \mathbf{R}^n))^k : \\ : \lambda(t) \text{ is a generalized solution of } (\mathbf{3.9}) \Big\}$$

and reachable set

$$\mathscr{A}_T^{\Psi}(\xi) = \Big\{ \lambda(T) = (\lambda_0(T), \lambda_1(T), \dots, \lambda_{k-1}(T)) : \lambda \in \mathscr{T}_T^{\Psi}(\xi) \Big\}.$$

Fix $\xi^* \in (\mathbf{R}^n)^k$ and choose any $b > 0$. Since $T \in]0,1[$, we have

$$\lim_{h \to +\infty} \big[h^2 - \big((b + hT + \|\xi^*\|_{nk})^2 + M^2 \big) \big] = +\infty,$$

hence there exists $L > 0$ such that

$$(b + LT + \|\xi^*\|_{nk})^2 + M^2 < L^2. \tag{3.10}$$

We claim that

$$\Psi\big([0,T] \times \overline{B}_{nk}(\xi^*, b + LT)\big) \subseteq \overline{B}_{nk}(0, L). \tag{3.11}$$

In order to prove (**3.11**), fix $t \in [0,T]$ and $\xi = (\xi_0, \xi_1, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, with $\xi \in \overline{B}_{nk}(\xi^*, b + LT)$. Let $\eta = (\eta_0, \eta_1, \dots, \eta_{k-1}) \in \Psi(t, \xi)$, and let $z \in G(t, \xi)$ such that $\eta = (\xi_1, \xi_2, \dots, \xi_{k-1}, z)$. Of course, we have

$$\|\xi\|_{nk}^2 \le (b + LT + \|\xi^*\|_{nk})^2.$$

By (**3.2**) and (**3.10**) we get

$$\begin{aligned} \|\eta\|_{nk}^2 &\le \|\xi_1\|_n^2 + \|\xi_2\|_n^2 + \dots + \|\xi_{k-1}\|_n^2 + \|z\|_n^2 \le \\ &\le \|\xi\|_{nk}^2 + M^2 \le M^2 + (b + LT + \|\xi^*\|_{nk})^2 < L^2, \end{aligned}$$

hence (**3.11**) is proved. By Theorem **2.2**, we then get:

(i)′ For every $\xi = (\xi_0, \dots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, with $\xi \in B_{nk}(\xi^*, b)$, the solution set $\mathscr{T}_T^{\Psi}(\xi)$ is nonempty. Moreover, the multifunction $\xi \to \mathscr{T}_T^{\Psi}(\xi)$ is upper semicontinuous from $B_{nk}(\xi^*, b)$ to

$$W^{1,\infty}([0,T], \mathbf{R}^{nk}) = \big(W^{1,\infty}([0,T], \mathbf{R}^n)\big)^k,$$

with nonempty compact connected values (it is routine matter to check that the topology $\sigma_{nk,1}^T$ coincides with the product topology $(\sigma_{n,1}^T)^k$);

(ii)′ The multifunction $\xi \to \mathscr{A}_T^{\Psi}(\xi)$ is upper semicontinuous in $B_{nk}(\xi^*, b)$, with nonempty compact connected values.

By the arbitrariness of $\xi^* \in (\mathbf{R}^n)^k$ it follows at once that:

(i)″ For every $\xi \in (\mathbf{R}^n)^k$, the solution set $\mathscr{T}_T^{\Psi}(\xi)$ is nonempty. Moreover, the multifunction $\xi \to \mathscr{T}_T^{\Psi}(\xi)$ is upper semicontinuous from $(\mathbf{R}^n)^k$ to $[W^{1,\infty}([0,T], \mathbf{R}^n)]^k$, with nonempty compact connected values;

(ii)$''$ The multifunction $\xi \to \mathscr{A}_T^{\Psi}(\xi)$ is upper semicontinuous from $(\mathbf{R}^n)^k$ to $(\mathbf{R}^n)^k$, with nonempty compact connected values.

STEP 3. For each $j = 1,\dots,k$, we denote by

$$P_j^* : \left[W^{1,\infty}([0,T],\mathbf{R}^n)\right]^k \to W^{1,\infty}([0,T],\mathbf{R}^n)$$

the j-th projection. For each $v \in W^{k,\infty}([0,T],\mathbf{R}^n)$, let

$$\lambda_v : [0,T] \to (\mathbf{R}^n)^k$$

be defined by putting, for every $t \in [0,T]$,

$$\lambda_v(t) = (v(t), v'(t), \dots, v^{(k-1)}(t)).$$

Clearly, one has $\lambda_v \in [W^{1,\infty}([0,T],\mathbf{R}^n)]^k$ for every $v \in W^{k,\infty}([0,T],\mathbf{R}^n)$. Now, let

$$H := \left\{ \lambda_v :\ v \in W^{k,\infty}([0,T],\mathbf{R}^n) \right\}.$$

It is easy to check that H is a closed subset of $[W^{1,\infty}([0,T],\mathbf{R}^n)]^k$. Of course, $P_1^*(H) = W^{k,\infty}([0,T],\mathbf{R}^n)$. We claim that the function

$$P_1^*|_H : (H, (\sigma_{n,1}^T)^k) \to (W^{k,\infty}([0,T],\mathbf{R}^n), \sigma_{n,k}^T)$$

is continuous. To see this, consider a net $(w_\alpha)_{\alpha\in\Lambda}$ in H, converging to a point $w \in H$ with respect to the product topology $(\sigma_{n,1}^T)^k$. By the definition of H, for each $\alpha \in \Lambda$ there exists $v_\alpha \in W^{k,\infty}([0,T],\mathbf{R}^n)$ such that $w_\alpha = \lambda_{v_\alpha}$. Moreover, there exists $v \in W^{k,\infty}([0,T],\mathbf{R}^n)$ such that $w = \lambda_v$. Hence, $(\lambda_{v_\alpha})_{\alpha\in\Lambda} = ((v_\alpha, v'_\alpha, \dots, v_\alpha^{(k-1)}))_{\alpha\in\Lambda}$ converges to $\lambda_v = (v, v', \dots, v^{(k-1)})$ with respect to the product topology $(\sigma_{n,1}^T)^k$. This means that

$$(v_\alpha^{(j)})_{\alpha\in\Lambda} \to v^{(j)} \quad \text{for every} \quad j = 0,\dots,k-1,$$

with respect to the topology $\sigma_{n,1}^T$. By the definition of $\sigma_{n,1}^T$, this means that

$$((v_\alpha^{(j)}, v_\alpha^{(j+1)}))_{\alpha\in\Lambda} \to (v^{(j)}, v^{(j+1)}) \quad \text{for every} \quad j = 0,\dots,k-1,$$

in $C([0,T],\mathbf{R}^n) \times L^\infty([0,T],\mathbf{R}^n$ (where the first space is taken with its strong topology, the second one with its weak-star topology). This implies that

$$(v_\alpha^{(j)})_{\alpha\in\Lambda} \to v^{(j)} \quad \text{for every} \quad j = 0,\dots,k-1$$

in $C([0,T],\mathbf{R}^n)$ (with respect to the strong topology), and

$$(v_\alpha^{(k)})_{\alpha\in\Lambda} \to v^{(k)}$$

in $L^\infty([0,T],\mathbf{R}^n)$ (with respect to the weak-star topology). Hence, $(v_\alpha)_{\alpha\in\Lambda} \to v$ in $C^{k-1}([0,T],\mathbf{R}^n)$ (with respect to the strong topology), and $(v_\alpha^{(k)})_{\alpha\in\Lambda} \to v^{(k)}$ weakly-star in $L^\infty([0,T],\mathbf{R}^n)$. Consequently, we have that

$$((v_\alpha, v_\alpha^{(k)}))_{\alpha\in\Lambda} \to (v, v^{(k)})$$

in $C^{k-1}([0,T],\mathbf{R}^n) \times L^\infty([0,T],\mathbf{R}^n)$. Therefore, $(v_\alpha)_{\alpha\in\Lambda} \to v$ with respect to the topology $\sigma_{n,k}^T$, hence $(P_1^*(w_\alpha))_{\alpha\in\Lambda} \to P_1^*(w)$ with respect to the topology $\sigma_{n,k}^T$, as desired.

STEP 4. Fix $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, and let

$$w(t) = (w_0(t), w_1(t), \ldots, w_{k-1}(t)) \in \mathscr{T}_T^{\Psi}(\xi) \subseteq \left[W^{1,\infty}([0,T], \mathbf{R}^n)\right]^k.$$

Hence, we have $w'(t) \in \Psi(t, w(t))$ for a.e. $t \in [0,T]$. By the definition of Ψ we have, for a.e. $t \in [0,T]$,

$$w_1(t) = w_0'(t), \quad w_2(t) = w_1'(t), \quad \ldots, \quad w_{k-1}(t) = w_{k-2}'(t),$$

$$w_{k-1}'(t) \in G(t, w_0(t), \ldots, w_{k-1}(t)).$$

Since the functions $w_0, w_1, \ldots, w_{k-1}$ are absolutely continuous, by a standard argument it follows that $w_0 \in C^{k-1}([0,T], \mathbf{R}^n)$, and one has, for every $t \in [0,T]$,

$$w_1(t) = w_0'(t), \quad w_2(t) = w_0''(t), \quad \ldots \quad w_{k-1}(t) = w_0^{(k-1)}(t).$$

In particular, $w_0^{(k-1)}(t)$ is absolutely continuous in $[0,T]$ (hence, $w_0 \in W^{k,\infty}([0,T], \mathbf{R}^n)$) and one has, for a.e. $t \in [0,T]$,

$$w_0^{(k)}(t) = w_{k-1}'(t) \in G(t, w_0(t), \ldots, w_0^{(k-1)}(t)).$$

Moreover, for every $j = 0, \ldots, k-1$ one has

$$w_0^{(j)}(0) = w_j(0) = \xi_j,$$

hence $w_0 \in \mathscr{T}_T^G(\xi)$ and $w = \lambda_{w_0} \in H$. Consequently, for every $\xi = (\xi_0, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, we have

$$\mathscr{T}_T^{\Psi}(\xi) \subseteq H, \qquad \text{and} \quad P_1^*(\mathscr{T}_T^{\Psi}(\xi)) \subseteq \mathscr{T}_T^G(\xi). \tag{3.12}$$

STEP 5. Let $\Phi : (\mathbf{R}^n)^k \to 2^{W^{k,\infty}([0,T],\mathbf{R}^n)}$ be defined by putting, for each $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$,

$$\Phi(\xi) = P_1^*(\mathscr{T}_T^{\Psi}(\xi)).$$

By (**3.4**) and (**3.12**) we get that

$$\Phi(\xi) \subseteq \mathscr{T}_T^G(\xi) \subseteq \mathscr{T}_T^F(\xi). \tag{3.13}$$

Moreover, by (i)$'$ and by the continuity of the function

$$P_1^*|_H : (H, (\sigma_{n,1}^T)^k) \to (W^{k,\infty}([0,T], \mathbf{R}^n), \sigma_{n,k}^T),$$

we have that Φ is upper semicontinuous (with respect to the topology $\sigma_{n,k}^T$), with nonempty compact connected values. Now, let

$$P_1^{**} : (\mathbf{R}^n)^k \to \mathbf{R}^n$$

by the first projection. By the above construction, it follows easily that

$$\{u(T) : u \in \Phi(\xi))\} = P_1^{**}(\mathscr{A}_T^{\Psi}(\xi))$$

for every $\xi = (\xi_0, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$. Consequently, by (ii)$''$ and by the continuity of P_1^{**}, conclusion (c) follows at once. Moreover, conclusion (d) follows immediately by (**3.8**) and (**3.13**). Finally, conclusion (e) follows at once by conclusion (b) and by the continuity of the function

$$u \in W^{k,\infty}([0,T], \mathbf{R}^n) \to u^{(k)} \in L^{\infty}([0,T], \mathbf{R}^n),$$

where the last space is considered with its weak-star topology. This completes the proof. □

Proof of Theorem 3.1. By the definition of the family $\mathscr{F}$, there exists sets $V_0, V_1, \dots, V_{nk} \subseteq \mathbf{R} \times \mathbf{R}^{nk}$, with $m_1(P_j(V_j)) = 0$ for all $j = 0, 1 \dots, nk$, such that $U = \bigcup_{i=0}^{nk} V_j$. Consequently, there exists two sets $Q_0, Q \in \mathscr{B}(\mathbf{R})$, with $Q_0 \subseteq [0,T]$ and $m_1(Q_0) = m_1(Q) = 0$, such that

$$P_0(V_0) \subseteq Q_0 \quad \text{and} \quad \bigcup_{j=1}^{nk} P_j(V_j) \subseteq Q.$$

Of course, we have that

$$\Omega := ([0,T] \setminus Q_0) \times (\mathbf{R} \setminus Q)^{nk} \subseteq S \setminus U. \tag{3.14}$$

Fix $a \in]0,1[$. Put

$$Y^* := \frac{T^k}{a^k} Y, \qquad D_1^* := \frac{T^k}{a^k} D_1, \qquad D_2^* := \frac{T^k}{a^k} D_2 .$$

Of course, Y^* is compact, connected and locally connected. Moreover, $Y^* \in \mathscr{G}_n$. Let $f^* : [0,a] \times (\mathbf{R}^n)^k \times Y^* \to \mathbf{R}$ be defined by putting, for each $(s, \xi_0, \xi_1, \dots, \xi_{k-1}, z) \in [0,a] \times (\mathbf{R}^n)^k \times Y^*$,

$$f^*(s,\, \xi_0\,, \xi_1, \dots, \xi_{k-1}, z) = f\Big(\, \frac{T}{a} s,\; \xi_0,\; \frac{a}{T} \xi_1,\; \frac{a^2}{T^2} \xi_2,\; \dots,\; \frac{a^{k-1}}{T^{k-1}} \xi_{k-1},\; \frac{a^k}{T^k} z\, \Big).$$

Consider the function $\phi : [0,a] \times (\mathbf{R}^n)^k \to [0,T] \times (\mathbf{R}^n)^k$ defined by putting, for each $(s, \xi_0, \xi_1, \dots, \xi_{k-1}) \in [0,a] \times (\mathbf{R}^n)^k$,

$$\phi(s,\, \xi_0\,, \xi_1, \dots, \xi_{k-1}) = \Big(\, \frac{T}{a} s,\; \xi_0,\; \frac{a}{T} \xi_1,\; \frac{a^2}{T^2} \xi_2,\; \dots,\; \frac{a^{k-1}}{T^{k-1}} \xi_{k-1}\, \Big).$$

Of course, ϕ is continuous, and

$$f^*(s,\, \xi_0\,, \xi_1, \dots, \xi_{k-1}, z) = f\Big(\phi(s,\, \xi_0\,, \xi_1, \dots, \xi_{k-1}),\; \frac{a^k}{T^k} z\Big) \tag{3.15}$$

for every $(s, \xi_0, \xi_1, \dots, \xi_{k-1}, z) \in [0,a] \times (\mathbf{R}^n)^k \times Y^*$. Now, put

$$Q_0^* := \frac{a}{T} Q_0, \qquad Q^* := \bigcup_{i=0}^{k-1} \Big(\frac{T^i}{a^i} Q\Big).$$

Of course, we have that $Q_0^*, Q^* \in \mathscr{B}(\mathbf{R})$, $Q_0^* \subseteq [0,a]$ and $m_1(Q_0^*) = m_1(Q^*) = 0$. Moreover, put

$$S^* := [0,a] \times (\mathbf{R}^n)^k, \qquad \Omega^* := ([0,a] \setminus Q_0^*) \times (\mathbf{R} \setminus Q^*)^{nk}$$

We observe the following facts.

- (i)′ $\phi(\Omega^*) \subseteq \Omega$.
- (ii)′ For every $z \in D_1^*$, the function $f^*(\cdot,\cdot,z)|_{\Omega^*}$ is lower semicontinuous, and for every $z \in D_2^*$, the function $f^*(\cdot,\cdot,z)|_{\Omega^*}$ is upper semicontinuous. This follows at once from assumptions (i) and (ii) and the continuity of ϕ, taking into account (**3.15**) and (i)′.
- (iii)′ for every $(s,\xi) \in \Omega^*$, $f^*(s,\xi,\cdot)$ is continuous in Y^* (this follows at once from assumption (iii), taking into account (**3.15**) and (i)′.
- (iv)′ for every $(s,\xi) \in \Omega^*$, we have $0 \in \operatorname{int}_{\mathbf{R}}(f^*(s,\xi,Y^*))$ (this follows at once from assumption (iii), taking into account (**3.15**), (i)′, (iii)′ and the connectedness of Y^*.

(v)′ for every $(s,\xi)\in\Omega^*$, we have

$$\mathrm{int}_{Y^*}(\{z\in Y^*:\ f^*(s,\xi,z)=0\})=\emptyset$$

(this follows at once from assumption (iii), taking into account (**3.15**) and (i)′).

For each $(s,\xi)\in\Omega^*$, let

$$H(s,\xi):=\{z\in Y^*:f^*(s,\xi,z)=0\},$$

$$E(s,\xi):=\{z\in Y^*:\ z\text{ is a local extremum for }f^*(s,\xi,\cdot)\},$$

$$J(s,x):=H(s,\xi)\setminus E(s,\xi).$$

By Theorem 2.2 of Ricceri (1982), the multifunction

$$J:\Omega^*\to 2^{Y^*}$$

is lower semicontinuous in Ω^* with nonempty closed (in Y^*, hence in $\mathbf{R}^n$) values. Moreover, $J(\Omega^*)$ is bounded and belongs to $\mathscr{G}_n$. Let y^* be any point in $J(\Omega^*)$, and let $F:[0,a]\times(\mathbf{R}^n)^k\to 2^{\mathbf{R}^n}$ be defined by

$$F(t,\xi)=\begin{cases}J(t,\xi) & \text{if }(t,\xi)\in\Omega^*,\\ \{y^*\} & \text{otherwise.}\end{cases}$$

For any fixed $\xi=(\xi_0,\xi_1,\dots,\xi_{k-1})\in(\mathbf{R}^n)^k$, let us consider the Cauchy problem

$$\begin{cases}u^{(k)}\in F(t,u,u',\dots,u^{(k-1)}) & \text{in }[0,a],\\ u^{(i)}(0)=\xi_i & \text{for each }i=0,\dots,k-1,\end{cases}\tag{3.16}$$

and its solution set

$$\mathscr{T}_a^F(\xi):=\left\{u\in W^{k,1}([0,a],\mathbf{R}^n):u\text{ is a generalized solution of }(\mathbf{3.16})\right\}.$$

By Theorem **3.2**, for every $\xi=(\xi_0,\xi_1,\dots,\xi_{k-1})\in(\mathbf{R}^n)^k$, the solution set $\mathscr{T}_a^F(\xi)$ of problem (**3.16**) is nonempty. Moreover, there exists a multifunction

$$\Phi:(\mathbf{R}^n)^k\to 2^{W^{k,\infty}([0,a];\mathbf{R}^n)}$$

such that:

(i)″ $\Phi(\xi)\subseteq\mathscr{T}_a^F(\xi)$ for all $\xi\in(\mathbf{R}^n)^k$;

(ii)″ Φ is upper semicontinuous (with respect to the topology $\sigma^a_{n,k}$ of $W^{k,\infty}([0,a];\mathbf{R}^n)$), with nonempty compact and connected values;

(iii)″ the multifunction

$$\xi\in(\mathbf{R}^n)^k\to\{u(a):u\in\Phi(\xi)\}$$

is upper semicontinuous with nonempty connected and compact values;

(iv)″ for every $\xi\in(\mathbf{R}^n)^k$ and every $u\in\Phi(\xi)$, one has that

$$(s,u(s),u'(s),\dots,u^{(k-1)}(s))\in\Omega^*\quad\text{for a.e.}\quad s\in[0,a].$$

Fix $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$ and $u \in \Phi(\xi)$. Taking into account (iv)$''$, for a.e. $s \in [0,a]$ we get

$$\begin{aligned} u^{(k)}(s) &\in F(s, u(s), u'(s), \ldots, u^{(k-1)}(s)) = \\ &= J(s, u(s), u'(s), \ldots, u^{(k-1)}(s)) \subseteq H(s, u(s), u'(s), \ldots, u^{(k-1)}(s)), \end{aligned}$$

hence u is a generalized solution of the Cauchy problem

$$\begin{cases} f^*(s, u, u', \ldots, u^{(k-1)}, u^{(k)}) = 0 & \text{in } [0,a], \\ u^{(i)}(0) = \xi_i & \text{for every } i = 0, \ldots, k-1. \end{cases} \tag{3.17}$$

Consequently, if for every $\xi = (\xi_0, \xi_1, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$, we put

$$\mathscr{S}_a^{f^*}(\xi) := \{u \in W^{k,1}([0,a], \mathbf{R}^n) : \ u \text{ is a generalized solution of } (\mathbf{3.17}) \text{ in } [0,a]\},$$

we get that

$$\Phi(\xi) \subseteq \mathscr{S}_a^{f^*}(\xi) \quad \text{for all} \quad \xi \in (\mathbf{R}^n)^k. \tag{3.18}$$

Let

$$\psi : W^{k,\infty}([0,a], \mathbf{R}^n) \to W^{k,\infty}([0,T], \mathbf{R}^n)$$

be defined by putting, for each $u \in W^{k,\infty}([0,a], \mathbf{R}^n)$,

$$\psi(u)(t) = u\Big(\frac{a}{T} t\Big) \quad \text{for every} \quad t \in [0,T].$$

It is routine matter to check that ψ is continuous (with respect to the topologies $\sigma_{n,k}^a$ and $\sigma_{n,k}^T$, respectively). Let $g : (\mathbf{R}^n)^k \to (\mathbf{R}^n)^k$ be defined by putting, for every $\xi = (\xi_0, \ldots, \xi_{k-1}) \in (\mathbf{R}^n)^k$,

$$g(\xi) = (\xi_0, \frac{T}{a} \xi_1, \frac{T^2}{a^2} \xi_2, \ldots, \frac{T^{k-1}}{a^{k-1}} \xi_{k-1}).$$

Of course, g is continuous. Moreover, it is immediate to check that for every $\xi \in (\mathbf{R}^n)^k$ one has

$$\psi(\mathscr{S}_a^{f^*}(g(\xi))) \subseteq \mathscr{S}_T^{f}(\xi). \tag{3.19}$$

At this point, it suffices to put, for every $\xi \in (\mathbf{R}^n)^k$,

$$\Psi(\xi) = \psi(\Phi(g(\xi))).$$

By the continuity of ψ and g and by (ii)$''$, it follows that Ψ is upper semicontinuous (with respect to the topology $\sigma_{n,k}^T$ of $W^{k,\infty}([0,T], \mathbf{R}^n)$) with non empty compact connected values. Moreover, conclusion (b) follows at once from (**3.18**) and (**3.19**). In order to prove conclusion (c), observe that for every $\xi \in (\mathbf{R}^n)^k$ one has

$$\{v(T) : v \in \Psi(\xi)\} = \{u(a) : u \in \Phi(g(\xi))\}.$$

Hence, conclusion (c) follows by (iii)$''$ and by the continuity of g.

In order to prove conclusion (d), fix $\xi \in (\mathbf{R}^n)^k$ and $v \in \Psi(\xi)$. Hence, there exists $u \in \Phi(g(\xi))$ such that

$$v(t) = \psi(u)(t) = u(\frac{a}{T} t)$$

for all $t \in [0,T]$. By (iv)$''$, there exists $K \subseteq [0,a]$, with $m_1(K) = 0$, such that

$$(s, u(s), u'(s), \ldots, u^{(k-1)}(s)) \in \Omega^* \quad \text{for every} \quad s \in [0,a] \setminus K. \tag{3.20}$$

Let $t \in [0,T] \setminus \frac{T}{a}K$. Putting $s = (a/T)t \in [0,a] \setminus K$, and taking into account (**3.14**), (**3.20**) and (i)$'$, we get

$$\begin{aligned}(t, v(t), v'(t); \ldots, v^{(k-1)}(t)) &= (t, \psi(u)(t), \psi(u)'(t); \ldots, \psi(u)^{(k-1)}(t)) = \\ &= (t, u(\frac{a}{T}t), \frac{a}{T}u'(\frac{a}{T}t), \ldots, \frac{a^{k-1}}{T^{k-1}}u^{(k-1)}(\frac{a}{T}t)) = \\ &= (\frac{T}{a}s, u(s), \frac{a}{T}u'(s), \ldots, \frac{a^{k-1}}{T^{k-1}}u^{(k-1)}(s)) = \\ &= \phi(s, u(s), u'(s), \ldots, u^{(k-1)}(s)) \in \Omega \subseteq S \setminus U.\end{aligned}$$

Therefore, conclusion (d) is proved. Conclusion (e) follows easily by (b) and by the continuity of the function

$$u \in W^{k,\infty}([0,T], \mathbf{R}^n) \to u^{(k)} \in L^\infty([0,T], \mathbf{R}^n),$$

where $L^\infty([0,T], \mathbf{R}^n)$ is considered with its weak-star topology. The proof is now complete. $\square$

Remark 3.3. The example given in Remark 3.2 of Cubiotti (2018) (which is, in substance, Example 1 of Ricceri (1985)) shows that Theorem **3.1** does not hold without the assumption $Y \in \mathscr{G}_n$. Moreover, the example in Remark 3.3 of Cubiotti (2018) shows that a function $f : [0,T] \times (\mathbf{R}^n)^k \times Y \to \mathbf{R}$ can satisfy the assumption of Theorem **3.1** even if it is discontinuous, with respect to the second variable, even at all points $\xi \in (\mathbf{R}^n)^k$. As a matter of fact, as it happens for Theorem **1.1**, the function f in the statement of Theorem **3.1** could be defined only on the set $(S \setminus U) \times Y$, since its behaviour over the set $U \times Y$ plays no role. This fact represents the main peculiarity of Theorem **3.1**, since in the literature (as far as we know) the function $f(t, \cdot, y)$ is usually required to be defined either on the whole space $(\mathbf{R}^n)^k$, or on a closed set with empty interior, or on a ball (see, for instance, Webb and Welsh 1989; Ricceri 1991; Heikkilä *et al.* 1996; Heikkilä and Lakshmikantham 1996; Carl and Heikkilä 1998; Pouso 2001; Cid 2003; Cid *et al.* 2006, and references therein).

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