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Minimum embedding of any Steiner triple system into a 3-sun system via matchings ☆

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Dedicated to the memory of our friend and colleague Lorenzo Milazzo

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ABSTRACT

Let G be a simple finite graph and G' be a subgraph of G. A G'-design $(X, \mathcal{B})$ of order n is said to be *embedded* into a G-design $(X \cup U, \mathcal{C})$ of order $n+u$, if there is an injective function $f : \mathcal{B} \to \mathcal{C}$ such that B is a subgraph of $f(B)$ for every $B \in \mathcal{B}$. The function f is called an *embedding* of $(X, \mathcal{B})$ into $(X \cup U, \mathcal{C})$. If u attains the minimum possible value, then f is a *minimum* embedding. Here, by means of König's Line Coloring Theorem and edge coloring properties, some results on the embedding of C_k-systems into k-sun systems are obtained and a complete solution to the problem of determining a minimum embedding of any Steiner Triple System into a 3-sun system is given.

1. Introduction

If G is a graph, then let $V(G)$ and $\mathcal{E}(G)$ denote the vertex-set and edge-set of G, respectively. Given a set Γ of pairwise non-isomorphic simple graphs, a Γ*-design or* Γ*-system of order* n is a pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a collection of graphs (called *blocks*) each isomorphic to some element of Γ, whose edges partition $\mathcal{E}(K_n)$, where K_n is the complete graph of order n on X; if the edges of the blocks of $\mathcal{B}$ partition a proper spanning subgraph of K_n, then we speak of *partial* Γ*-design of order* n. If $\Gamma = \{G\}$, then we simply write G-design. Let $\Sigma(G)$ denote the set of all integers n such that there exists a G-design of order n. A k*-cycle* $(x_1, x_2, \dots, x_k)$ consists of the k distinct vertices $x_1, x_2, \dots, x_k$ and the k edges $\{x_i, x_{i+1}\}$, $i = 1, 2, \dots k-1$, and $\{x_1, x_k\}$. By adding to a k-cycle $C_k = (x_1, x_2, \dots, x_k)$ an independent set of edges $\{\{x_i, x'_i\}, 1 \le i \le k\}$ we obtain the k*-sun* on $2k$ vertices denoted by $S(C_k) = (x_1, x_2, \dots, x_k; x'_1, x'_2, \dots, x'_k)$. Obviously, a necessary condition for the existence of a k-sun system of order n, shortly a kSS(n), is $n \ge 2k$ and $n(n-1) \equiv 0 \pmod{4k}$. The sufficiency of this condition was proved for $k \in \{3, 4, 5, 6, 8\}$ and only recently Buratti et al. ([2]) made remarkable progress towards solving the spectrum problem for kSS when k is odd and gave a complete solution whenever k is an odd prime. It is well-known that $\Sigma(C_k) = \{n \in N : n \equiv 1 \pmod 2, n(n-1) \equiv 0 \pmod{2k}\}$. A K_3-design (which is also a C_3-system) of order n is known as a *Steiner triple system* and denoted by STS(n); obviously, $\Sigma(K_3) = \{n \in N : n \equiv 1, 3 \pmod 6\}$.

Let G be a simple finite graph and G' be a subgraph of G. A G'-design $(X, \mathcal{B})$ of order n is said to be *embedded* into a G-design $(X \cup U, \mathcal{C})$ of order $n+u$, if there is an injective function $f : \mathcal{B} \to \mathcal{C}$ such that B is a subgraph of $f(B)$ for every

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$B \in \mathcal{B}$. The function f is called an *embedding* of $(X, \mathcal{B})$ into $(X \cup U, \mathcal{C})$. If u attains the minimum possible value, then f is a *minimum* embedding. Note that a special case occurs when $G = G'$ and the related embedding problem is better known as Doyen-Wilson problem (see [8,10,12–14]).

The embedding problems have interesting applications to networks ([7]), that is why they have been investigated in several papers (see [4–6,11,15–18]). In particular, the minimum embedding problem of STSs into G-designs has been studied in the case when $G = K_4$, $G = K_4 - e$ (the complete graph on four vertices with one deleted edge), or $G = K_3 + e$ (a *kite*, i.e., a triangle with one pendant edge) and solved in [5], [6], [11], [15].

In [9] the authors embed a cyclic STS of order $n \equiv 1 \pmod 6$ into a 3SS$(2n-1)$ and as an open problem they ask whether it is possible to embed any STS into a 3SS of some order. Here we give an answer to this open problem and for any STS, not only we determine the minimum order of a 3-sun system which embeds the STS, but also give a construction for this embedding. More precisely, for every integer $n \in \Sigma(K_3)$, denoted by $u_{min}(n)$ the minimum integer u such that any STS(n) can be embedded into a 3-sun system of order $n + u$, as main result we prove the following theorem.

Main Theorem.

(i) *If* $n \equiv 1, 3, 9, 19 \pmod{24}$, *then* $u_{min}(n) = \frac{n-1}{2}$ *for every* $n \neq 3, 9$, $u_{min}(3) = 6$, *and* $u_{min}(9) = 7$.
(ii) *If* $n \equiv 7, 13, 15, 21 \pmod{24}$, *then* $u_{min}(n) = \frac{n-1}{2} + 2$ *for every* $n \neq 7, 13$, $u_{min}(7) = 6$, *and* $u_{min}(13) = 11$.

Furthermore, embeddings of k-cycle systems of order $n \geq 2k$ into k-sun systems of order $\frac{3n-1}{2}$ are given when $\frac{n-1}{2} \in \Sigma(S(C_k))$.

To get our main goal we make use of some results on edge colorings and, in particular, of König's Line Coloring Theorem, which here, for convenience, is formulated in terms of matchings (for definitions and results on edge colorings or matchings, the reader is referred to [1]).

Theorem 1.1. (König's Line Coloring Theorem) *Let G be a bipartite multigraph with maximum degree Δ. Then $\mathcal{E}(G)$ can be partitioned into $M_1, M_2, \ldots, M_\Delta$ such that each M_i, $1 \leq i \leq \Delta$, is a matching in G.*

2. Embedding C_k-systems into kSSs, for $k \geq 3$

In this section we will give the necessary condition for embedding a C_k-system into a partial k-sun system. Moreover, if $n \in \Sigma(C_k)$ and $\frac{n-1}{2} \in \Sigma(S(C_k))$, we embed any C_k-system of order n into a kSS$(\frac{3n-1}{2})$ and prove some useful results to get our main result.

Lemma 2.1. *If there exists a kSS$(n+u)$ containing an embedded C_k-system of order n, then $u \geq \frac{n-1}{2}$.*

Proof. Since a C_k-system of order n has $\frac{n(n-1)}{2k}$ blocks, then in order to complete every k-cycle and obtain a k-sun, necessarily $n \cdot u \geq k\frac{n(n-1)}{2k}$ and so $u \geq \frac{n-1}{2}$. □

In general, to construct a kSS$(n+u)$ $(X \cup U, \mathcal{S})$ containing an embedded C_k-system of order n $(X, \mathcal{C})$, we need to complete each k-cycle of $\mathcal{C}$ to a k-sun by using some edges of the complete bipartite graph $K_{n,u}$ on $X \cup U$ and partition into k-suns the remaining edges of $K_{n,u}$ along with those of the complete graph K_u on U. In the following lemma a partial kSS containing an embedded C_k-system of order n is constructed by using all the edges of the above complete bipartite graph.

Lemma 2.2. *Any C_k-system of order $n \geq 2k+1$ can be embedded into a partial kSS$(\frac{3n-1}{2})$.*

Proof. Let $(X, \mathcal{C})$ be a C_k-system of order n and consider its incidence graph $\mathcal{I}$, i.e., the bipartite graph whose vertex set is $X \cup \mathcal{C}$ and whose edges are determined by joining $x \in X$ to $C \in \mathcal{C}$ if and only if $x \in C$. In the graph $\mathcal{I}$ every vertex of X has degree $\frac{n-1}{2}$ and every vertex of $\mathcal{C}$ has degree k. Since the maximum degree of $\mathcal{I}$ is $\Delta = \frac{n-1}{2}$, by König's Line Coloring Theorem the edges of $\mathcal{I}$ can be partitioned into Δ matchings $M_1, M_2, \ldots, M_\Delta$, each of which saturates the vertices of X, i.e., every vertex of X is incident to an edge of each matching. Let $\mathcal{S}$ be the set of k-suns on $X \cup \{M_1, M_2, \ldots, M_\Delta\}$ obtained by completing each k-cycle of $\mathcal{C}$ to a k-sun as follows: for every $C = (x_1, x_2, \ldots, x_k) \in \mathcal{C}$, consider the k-sun $(x_1, x_2, \ldots, x_k; M_{i_1}, M_{i_2}, \ldots, M_{i_k})$, where $\{x_j, C\} \in M_{i_j}$ for every $j = 1, 2, \ldots, k$. $(X \cup \{M_1, M_2, \ldots, M_\Delta\}, \mathcal{S})$ is a partial kSS$(\frac{3n-1}{2})$ which embeds $(X, \mathcal{C})$. □

The lower bound given by Lemma 2.1 is attained if there exists a kSS$(\frac{n-1}{2})$, as it is established by the following proposition.

Proposition 2.1. *Let $n \in \Sigma(C_k)$ such that $n \geq 2k+1$ and $\frac{n-1}{2} \in \Sigma(S(C_k))$. Then any C_k-system of order n can be embedded into a kSS$(\frac{3n-1}{2})$.*

Proof. Let $(X, \mathcal{C})$ be any C_k-system of order n. By Lemma 2.2, it can be embedded into a partial kSS$(\frac{3n-1}{2})$ $(X \cup \{M_1, M_2, \ldots, M_{\frac{n-1}{2}}\}, \mathcal{S})$. Since $\frac{n-1}{2} \in \Sigma(S(C_k))$, there exists a kSS$(\frac{n-1}{2})$ $(\{M_1, M_2, \ldots, M_{\frac{n-1}{2}}\}, \mathcal{S}')$. Then $(X \cup \{M_1, M_2, \ldots, M_{\frac{n-1}{2}}\}, \mathcal{S} \cup \mathcal{S}')$ is a kSS$(\frac{3n-1}{2})$ which embeds $(X, \mathcal{T})$. The result follows by Lemma 2.1. □

3. The case $k = 3$

In this section we completely solve the minimum embedding problem of any STS into a 3SS.

To start with, since a 3SS(n) exists if and only if $n \equiv 0, 1, 4, 9 \pmod{12}$, $n \geq 9$ (see [9]), when $k = 3$ Proposition 2.1 can be stated as follows.

Proposition 3.1. *For every $n \equiv 1, 3, 9, 19 \pmod{24}$, $n \geq 19$, $u_{min}(n) = \frac{n-1}{2}$.*

From now on, we will denote:

- the *kite* consisting of the triangle (a, b, c) and the pendant edge $\{c, d\}$ by $(a, b, c; d)$;
- the *bull* graph consisting of the triangle (a, b, c) and the pendant edges $\{b, d\}$ and $\{c, e\}$ by $(a, b, c; d, e)$;

If G is a kite, a bull, or a 3-sun, then its triangle will be denoted by $t(G)$. Finally, if f is an embedding of an STS $(X, \mathcal{T})$ into a 3SS $(X \cup U, \mathcal{S})$, then $f(\mathcal{T})$ will be denoted by $\mathcal{S}_{\mathcal{T}}$.

Lemma 3.1. *If $n = 3, 9$, then $u_{min}(n) = 6, 7$, respectively.*

Proof. Any STS(3) can be trivially embedded into a 3SS of any admissible order $v \geq 9$ and so $u_{min}(3) = 6$.

Let $(X \cup U, \mathcal{S})$ be a 3SS$(9 + u)$ containing an embedded STS(9) $(X, \mathcal{T})$. By Lemma 2.1 $u \geq 4$. If $u = 4$, then $\mathcal{S} \setminus \mathcal{S}_{\mathcal{T}}$ would contain only one 3-sun S such that $V(S) \subseteq U$, which is impossible and so $u_{min}(9) \geq 7$. The following set of graphs on Z_{16} is the block-set of a 3SS which embeds the unique STS(9) (whose triangles are in bold):

$$(\mathbf{0}, \mathbf{1}, \mathbf{2}; 9, 10, 11),\ ((\mathbf{0}, \mathbf{3}, \mathbf{6}; 10, 15, 9),\ (\mathbf{0}, \mathbf{4}, \mathbf{8}; 11, 9, 13),$$
$$(\mathbf{0}, \mathbf{5}, \mathbf{7}; 12, 9, 15),\ (\mathbf{1}, \mathbf{3}, \mathbf{8}; 9, 10, 11),\ (\mathbf{1}, \mathbf{4}, \mathbf{7}; 11, 10, 9),$$
$$(\mathbf{1}, \mathbf{5}, \mathbf{6}; 12, 10, 13),\ (\mathbf{2}, \mathbf{3}, \mathbf{7}; 9, 11, 12),\ (\mathbf{2}, \mathbf{4}, \mathbf{6}; 10, 11, 12),$$
$$(\mathbf{2}, \mathbf{5}, \mathbf{8}; 12, 13, 14),\ (\mathbf{3}, \mathbf{4}, \mathbf{5}; 9, 12, 14),\ (\mathbf{6}, \mathbf{7}, \mathbf{8}; 11, 10, 9),$$
$$(0, 13, 15; 14, 7, 8),\ (1, 14, 15; 13, 4, 9),\ (3, 12, 14; 13, 15, 11)$$
$$(2, 13, 14; 15, 4, 7),\ (5, 11, 12; 15, 7, 10),\ (6, 10, 14; 15, 8, 9),$$
$$(9, 12, 13; 10, 8, 11),\ (10, 11, 15; 13, 9, 4).\quad \square$$

Lemma 3.2. *Let $n \equiv 7, 13, 15, 21 \pmod{24}$. If there exists a $3SS(n + u)$ containing an embedded STS(n), then $u \geq \frac{n-1}{2} + 2$.*

Proof. Let $n = 24k + r$, $r \in \{7, 13, 15, 21\}$. If $(X, \mathcal{S})$ is a 3SS$(n + u)$ containing an embedded STS(n), then by Lemma 2.1 $n + u \geq \frac{3n-1}{2} = 36k + \frac{3r-1}{2}$, where $\frac{3r-1}{2} \in \{10, 19, 22, 31\}$. Since $n + u \equiv 0, 1, 4, 9 \pmod{12}$, this implies $u \geq \frac{n-1}{2} + 2$. □

Remark 3.1. For every $n \equiv 7, 13, 15, 21 \pmod{24}$, if $(X \cup U, \mathcal{S})$ is a 3SS$(n + \frac{n+3}{2})$ containing an embedded STS(n) $(X, \mathcal{T})$, then each vertex $x \in X$ appears in exactly two blocks of $\mathcal{S} \setminus \mathcal{S}_{\mathcal{T}}$ as a pendant vertex and so for every $S \in \mathcal{S} \setminus \mathcal{S}_{\mathcal{T}}$ the vertices of $t(S)$ are in U.

The lower bound established by Lemma 3.2 is not attained when $n = 7, 13$, as it is shown by the following lemma.

Lemma 3.3. *If $n = 7, 13$, then $u_{min}(n) = 6, 11$, respectively.*

Proof. Let $(X \cup U, \mathcal{S})$ be a 3SS$(n + u)$ containing an embedded STS(n) $(X, \mathcal{T})$, where $n = 7, 13$. By Lemma 3.2, $u \geq \frac{n+3}{2}$. For $n = 7$, by Remark 3.1 $|\mathcal{S} \setminus \mathcal{S}_{\mathcal{T}}| \geq 5$ and so $u_{min}(7) \geq 6$. To prove that $u_{min}(7) = 6$, on Z_{13} we give the blocks of a 3SS which embeds the unique STS(7):

$$(\mathbf{0}, \mathbf{1}, \mathbf{2}; 7, 8, 9),\ ((\mathbf{0}, \mathbf{3}, \mathbf{4}; 8, 7, 9),\ (\mathbf{0}, \mathbf{5}, \mathbf{6}; 9, 8, 10),\ (\mathbf{1}, \mathbf{3}, \mathbf{5}; 9, 8, 7),$$
$$(\mathbf{1}, \mathbf{4}, \mathbf{6}; 10, 7, 12),\ (\mathbf{2}, \mathbf{3}, \mathbf{6}; 7, 9, 8),\ (\mathbf{2}, \mathbf{4}, \mathbf{5}; 8, 11, 9),$$

$$(0, 10, 11; 12, 5, 3),\ (1, 7, 12; 11, 8, 5),\ (2, 10, 12; 11, 9, 8),\ (4, 8, 10; 12, 9, 3),$$
$$(9, 11, 12; 7, 5, 3),\ (6, 7, 11; 9, 10, 8).$$

For $n = 13$, if $u = 8$, then $|\mathcal{S} \setminus \mathcal{S}_{\mathcal{T}}| = 9$ and by Remark 3.1 a partial triple system on U with 9 triangles should exist, which is impossible because a maximum packing of K_8 with triangles (i.e., a partial K_3-design of order 8 with the maximum number of blocks) have 8 blocks. Therefore, $u_{min}(13) \geq 11$. Since there are two non-isomorphic STS(13)s, in order to prove that $u_{min}(13) = 11$ we need to embed each STS(13). Firstly, we embed the cyclic one into a 3SS on Z_{24} as follows:

$$(\mathbf{0}, \mathbf{1}, \mathbf{4}; 13, 18, 14),\ (\mathbf{1}, \mathbf{2}, \mathbf{5}; 13, 23, 14),\ (\mathbf{2}, \mathbf{3}, \mathbf{6}; 13, 18, 14),$$
$$(\mathbf{3}, \mathbf{4}, \mathbf{7}; 13, 15, 14),\ (\mathbf{4}, \mathbf{5}, \mathbf{8}; 13, 15, 14),\ (\mathbf{5}, \mathbf{6}, \mathbf{9}; 13, 18, 19),$$
$$(\mathbf{6}, \mathbf{7}, \mathbf{10}; 13, 15, 14),\ (\mathbf{7}, \mathbf{8}, \mathbf{11}; 13, 15, 14),\ (\mathbf{8}, \mathbf{9}, \mathbf{12}; 13, 20, 15),$$
$$(\mathbf{9}, \mathbf{10}, \mathbf{0}; 13, 15, 14),\ (\mathbf{10}, \mathbf{11}, \mathbf{1}; 13, 16, 14),\ (\mathbf{11}, \mathbf{12}, \mathbf{2}; 13, 16, 14),$$
$$(\mathbf{12}, \mathbf{0}, \mathbf{3}; 13, 15, 23),\ (\mathbf{0}, \mathbf{2}, \mathbf{7}; 16, 22, 21),\ (\mathbf{1}, \mathbf{3}, \mathbf{8}; 15, 19, 16),$$
$$(\mathbf{2}, \mathbf{4}, \mathbf{9}; 15, 16, 14),\ (\mathbf{3}, \mathbf{5}, \mathbf{10}; 14, 16, 17),\ (\mathbf{4}, \mathbf{6}, \mathbf{11}; 17, 15, 18),$$
$$(\mathbf{5}, \mathbf{7}, \mathbf{12}; 17, 16, 14),\ (\mathbf{6}, \mathbf{8}, \mathbf{0}; 16, 17, 18),\ (\mathbf{7}, \mathbf{9}, \mathbf{1}; 17, 15, 16),$$
$$(\mathbf{8}, \mathbf{10}, \mathbf{2}; 18, 16, 17),\ (\mathbf{9}, \mathbf{11}, \mathbf{3}; 16, 15, 17),\ (\mathbf{10}, \mathbf{12}, \mathbf{4}; 18, 17, 19),$$
$$(\mathbf{11}, \mathbf{0}, \mathbf{5}; 17, 19, 18),\ (\mathbf{12}, \mathbf{1}, \mathbf{6}; 18, 17, 19),$$
$$(0, 17, 20; 21, 6, 1),\ (1, 19, 21; 22, 2, 3),\ (2, 16, 18; 20, 3, 4),$$
$$(3, 15, 20; 22, 13, 4),\ (4, 21, 22; 23, 2, 0),\ (5, 19, 20; 21, 7, 6),$$
$$(6, 21, 23; 22, 8, 0),\ (7, 18, 20; 22, 9, 8),\ (8, 19, 22; 23, 10, 5),$$
$$(9, 17, 21; 22, 13, 10),\ (10, 20, 22; 23, 11, 12),\ (11, 19, 23; 21, 12, 1),$$
$$(12, 20, 21; 23, 13, 14),\ (13, 14, 16; 18, 15, 17),\ (13, 22, 23; 19, 11, 5),$$
$$(14, 17, 19; 18, 15, 16),\ (14, 20, 23; 22, 16, 7),\ (15, 16, 21; 19, 23, 13),$$
$$(15, 18, 22; 23, 19, 16),\ (17, 18, 23; 22, 21, 9).$$

A 3SS(24) which embeds the non cyclic STS(13) can be obtained from the above one by replacing the 3-suns

$$(\mathbf{0}, \mathbf{1}, \mathbf{4}; 13, 18, 14),\ (\mathbf{0}, \mathbf{2}, \mathbf{7}; 16, 22, 21),$$
$$(\mathbf{2}, \mathbf{4}, \mathbf{9}; 15, 16, 14),\ (\mathbf{7}, \mathbf{9}, \mathbf{1}; 17, 15, 16),$$

with

$$(\mathbf{9}, \mathbf{1}, \mathbf{4}; 14, 18, 16),\ (\mathbf{9}, \mathbf{2}, \mathbf{7}; 15, 22, 17),$$
$$(\mathbf{0}, \mathbf{2}, \mathbf{4}; 16, 15, 14),\ (\mathbf{0}, \mathbf{1}, \mathbf{7}; 13, 16, 21). \quad \square$$

In order to prove that for every $n \equiv 7, 13, 15, 21 \pmod{24}$, $n \neq 7, 13$, $u_{min}(n)$ equals the lower bound of Lemma 3.2, the following lemma is useful.

Lemma 3.4. ([1], Lemma 6.3) *Let M and N be disjoint matchings of a graph G with size $|M|$ and $|N|$ such that $|M| > |N|$. Then there are disjoint matchings M' and N' of G such that $|M'| = |M| - 1$, $|N'| = |N| + 1$ and $M' \cup N' = M \cup N$.*

Now, we determine $u_{min}(n)$ for every $n \equiv 7, 13, 15, 21 \pmod{24}$ with the exception of few small orders, which will be settled in Section 3.1.

In graph theory, the degree of a vertex of a graph G is the number of edges of G that are incident to the vertex; here, we define 2-*degree* of a vertex x of a Γ-design $\mathcal{D}$, and denote by $d_2(x)$, the number of blocks of $\mathcal{D}$ containing x as a vertex of degree 2. The 2-*degree sequence* of $\mathcal{D}$ is the non-decreasing sequence of its vertex 2-degrees.

In what follows, if G is a graph whose vertices belong to Z_u, then we call *orbit of B under Z_u* the set $(G) = \{G + i : i \in Z_u\}$, where $G + i$ is the graph with $V(G + i) = \{a + i : a \in V(G)\}$ and $\mathcal{E}(G + i) = \{\{a + i, b + i\} : \{a, b\} \in \mathcal{E}(G)\}$.

Lemma 3.5. *For any $u = 12k + h$, $h = 5, 8, 9, 12$ and $k \geq 3$, there exists a {bull, 3-sun}-design of order u whose 2-degree sequence is* $(2, 3, 3, 3, 3, 4, 4, \dots, 4)$.

Proof. Consider the following orbits under Z_u: for $i = 1, 2, 3$, $\mathcal{B}_i = (B_i)$, where $B_1 = (0, 6k-2, 4k+3; 3k, 6k-1)$, $B_2 = (6k, 0, 4k+1; 6k+2, 6k+1)$, and $B_3 = (0, 6k-1, 4k+2; 3k, 6k)$; for $j = 0, 1, \ldots, k-4$, $\mathcal{S}_j = (S_j)$, where $S_j = (5k+1+j, 5k-j, 0; 3k, k, u-2-2j)$. On Z_u define the set of graphs $\mathcal{A} = (\mathcal{B}_1 \cup \mathcal{B}_2^* \cup \mathcal{B}_3^* \cup \mathcal{B}) \cup [(\cup_{j=0}^{k-4}\mathcal{S}_j) \cup \mathcal{S}^* \cup \mathcal{S}]$, where $\mathcal{B}_2^* = \mathcal{B}_2 \setminus \{B_2\}$, $\mathcal{B}_3^* = \mathcal{B}_3 \setminus \{B_3 + i : i = 0, 4k+1, 6k, 6k+1, 6k+2\}$, $\mathcal{S}^* = \{(6k-1, 4k+2, 0; 3k, 6k, 4k+1), (10k, 8k+3, 4k+1; 7k+1, 10k+1, 6k), (12k-1, 10k+2, 6k; 9k, 12k, 0), (12k, 10k+3, 6k+1; 9k+1, 12k+1, 4k+1), (12k+1, 10k+4, 6k+2; 9k+2, 12k+2, 0)\}$; $\mathcal{B}$ and $\mathcal{S}$ depending on h and summarized as follows:
a) $h = 5$: $\mathcal{B}$ is the orbit of $(6k+1, 0, 3k; 3k+2, 6k+3)$ under Z_u; $\mathcal{S} = \emptyset$.
b) $h = 8$: $\mathcal{B} = \{(6k+3+i, i, 3k+i; 6k+4+i, 9k+1+i), (9k+5+i, 3k+2+i, 6k+2+i; 6k+4+i, 12k+3+i) : i = 0, 1, \ldots, 3k+1,\ i \in Z_u\} \cup \{(12k+7+i, 6k+4+i, 9k+4+i; 9k+5+i, 3k-3+i) : i = 0, 1, \ldots, 6k+3,\ i \in Z_u\}$; $\mathcal{S} = \{(i, 3k+2+i, 9k+6+i; 3k+1+i, 6k+3+i, 6k+4+i)) : i = 0, 1, \ldots, 3k+1,\ i \in Z_u\}$.
c) $h = 9$: $\mathcal{B}$ is the orbit of $(6k+1, 0, 3k; 3k+3, 9k+3)$ under Z_u; $\mathcal{S} = \{(3i, 3k+2+3i, 6k+4+3i; 6k+5+3i, 9k+7+3i, 9k+6+3i)) : i = 0, 1, \ldots, 4k+2,\ i \in Z_u\}$.
d) $h = 12$: $\mathcal{B} = \{(6k+1+i, i, 3k+i; 6k+6+i, 9k+5+i), (9k+4+i, 3k+3+i, 6k+3+i; 6k+6+i, 12k+8+i) : i = 0, 1, \ldots, 3k+2,\ i \in Z_u\} \cup \{(12k+7+i, 6k+6+i, 9k+6+i; 12k+9+i, 3k-1+i) : i = 0, 1, \ldots, 6k+5,\ i \in Z_u\}$; $\mathcal{S} = \{(i, 3k+3+i, 9k+9+i; 6k+3+i, 9k+6+i, 6k+6+i)) : i = 0, 1, \ldots, 3k+2,\ i \in Z_u\} \cup \{(3i, 3k+2+3i, 6k+4+3i; 6k+8+3i, 9k+10+3i, 9k+6+3i)) : i = 0, 1, \ldots, 4k+3,\ i \in Z_u\}$.
$(Z_u, \mathcal{A})$ is the required design, where $d_2(6k) = 2$, the vertices $d_2(0) = d_2(4k+1) = d_2(6k+1) = d_2(6k+2) = 3$, and the remaining vertices have 2-degree 4. □

Proposition 3.2. *For every $n \equiv 7, 13, 15, 21 \pmod{24}$, $n \geq 79$, $u_{min}(n) = \frac{n+3}{2}$.*

Proof. Let $(X, \mathcal{T})$ be an STS(n), $n \equiv 7, 13, 15, 21 \pmod{24}$, $n \geq 79$, and $\mathcal{I}$ be its incidence graph. $\mathcal{E}(\mathcal{I})$ can be partitioned into $\Delta = \frac{n-1}{2}$ matchings $M_1, M_2, \ldots, M_\Delta$ (see proof of Lemma 2.2). Therefore there exist $\Delta + 2$ mutually disjoint matchings $M_1, M_2, \ldots, M_\Delta, M_{\Delta+1}, M_{\Delta+2}$, with $M_{\Delta+1} = \emptyset = M_{\Delta+2}$, such that $\mathcal{E}(\mathcal{I}) = M_1 \cup M_2 \cup \cdots \cup M_{\Delta+2}$. By repeatedly applying Lemma 3.4 to pairs of those matchings that differ in size by more than one, we eventually obtain $\Delta + 2$ mutually disjoint matchings $M'_1, M'_2, \ldots, M'_{\Delta+2}$ of $\mathcal{E}(\mathcal{I})$ such that M'_i covers the vertices of $X \setminus X_i$, where $|X_1| = 2$, $|X_i| = 3$ for $i = 2, 3, 4, 5$, and $|X_i| = 4$ for $i = 6, 7, \ldots, \Delta + 2$ (note that each vertex of X is missing in exactly two matchings). If $\mathcal{S}$ denotes the set of 3-suns on $X \cup \{M'_1, M'_2, \ldots, M'_{\Delta+2}\}$ obtained by completing each triple of $\mathcal{T}$ as in the proof of Lemma 2.2, the pair $(X \cup \{M'_1, M'_2, \ldots, M'_{\Delta+2}\}, \mathcal{S})$ is a partial 3SS$(\frac{3(n+1)}{2})$ which embeds $(X, \mathcal{T})$. In order to complete the proof it will be sufficient to decompose the graph $K_{\Delta+2} \cup \mathcal{M}$ into 3-suns, where $K_{\Delta+2}$ is the complete graph induced by $\{M'_1, M'_2, \ldots, M'_{\Delta+2}\}$ and $\mathcal{M}$ is the bipartite graph on $X \cup \{M'_1, M'_2, \ldots, M'_{\Delta+2}\}$ such that $\{x, M'_i\} \in \mathcal{E}(\mathcal{M})$ if and only if $x \in X_i$ (i.e., x is missing in M'_i). By using Lemma 3.5, the complete graph $K_{\Delta+2}$ can be decomposed into bulls or 3-suns so that $d_2(M'_1) = 2$, $d_2(M'_i) = 3$ for $i = 2, 3, 4, 5$, and $d_2(M'_i) = 4$ for $i = 6, 7, \ldots, \Delta + 2$. To obtain the required decomposition it is sufficient to complete each bull to a 3-sun by using the edges of $\mathcal{M}$. □

3.1. Cases left

To determine $u_{min}(n)$ for the remaining orders $n \in \{15, 21, 31, 37, 39, 45, 55, 61, 63, 69\}$, we will start from an STS(n) $(X, \mathcal{T})$, with $X = \{x_1, x_2, \ldots, x_n\}$, and prove that $(X, \mathcal{T})$ can be embedded in a 3-sun system $(X \cup Z_{\frac{n+3}{2}}, \mathcal{S})$ by taking the following steps.
Step 1. Partition the edges of the complete graph on $Z_{\frac{n+3}{2}}$ into a set $\mathcal{A}$ of triangles, kites, bulls or 3-suns so that $|\mathcal{A}| = |\mathcal{S} \setminus \mathcal{T}| = (n^2 + 20n + 3)/48$ and $\sum_{i=0}^{(n+1)/2} d_2(i) = 2n$. For later convenience (see *Step* 4.), give $\mathcal{A}$ partitioned into suitable subsets $\mathcal{A}_j$, $j \in J$, such that for every $j \in J$ and for every vertex $i \in Z_{\frac{n+3}{2}}$, the number of blocks of $\mathcal{A}_j$ containing i as a vertex of degree 2 is at most 1.
Step 2. Partition the edge-set of the incidence graph $\mathcal{I}$ of $(X, \mathcal{T})$ into $\frac{n+3}{2}$ matchings $M_0, M_1, \ldots, M_{\frac{n+1}{2}}$ such that the set of vertices of X not saturated by M_i, denoted by X_i, has size $|X_i| = d_2(i)$ for each $i = 0, 1, \ldots, \frac{n+1}{2}$.
Step 3. Complete each triple of $\mathcal{T}$ as in the proof of Lemma 2.2 and obtain a partial 3-sun system $(X \cup \{M_0, M_1, \ldots, M_{\frac{n+1}{2}}\}, \mathcal{S})$ which embeds $(X, \mathcal{T})$.
Step 4. Call *missing graph* the bipartite graph $\mathcal{M}$ on $X \cup \{M_0, M_1, \ldots, M_{\frac{n+1}{2}}\}$ consisting of all the edges $\{x, M_i\}$ such that $x \in X_i$ and, for the sake of simplicity, for every $i = 0, 1, \ldots, \frac{n+1}{2}$ identify M_i with $i \in Z_{\frac{n+3}{2}}$.
Step 5. Partition the edges of the missing graph into suitable matchings M'_j, $j \in J$, such that for every $j \in J$ the edges of M'_j can be used to complete the blocks of $\mathcal{A}_j$ so to obtain a 3-sun system of order $\frac{3(n+1)}{2}$ which embeds $(X, \mathcal{T})$.

To begin with, we give an alternative solution for $n \equiv 15 \pmod{24}$ (which settles the orders $v = 15, 39, 63$ as well) by means of a technique used in [9] and involving the concepts of parallel classes and resolution of an STS.

A *parallel class* of an STS(n) is a set of $\frac{n}{3}$ triples such that no two triples in the set share an element; a partition of all triples of an STS(n) into parallel classes is a *resolution* and the STS is said to be *resolvable*. An STS(n) together with a resolution of its triples is a *Kirkman triple system*, shortly a KTS(n), and exists if and only if $n \equiv 3 \pmod{6}$ (see [3]).

Proposition 3.3. *For every $n \equiv 15 \pmod{24}$, $u_{min}(n) = \frac{n+3}{2}$.*

Proof. Let $(X, \mathcal{T})$ be an STS(n), $n = 24k+15$, $k \geq 0$. Consider a resolution P_i, $i = 1, 2, \dots, 6k+4$ of a KTS on $Z_{\frac{n+3}{2}}$. Without loss of generality, assume that P_1 contains the triangle $t = (0, 1, 2)$. Construct a set $\mathcal{K}$ of kites obtained by attaching the edges of t to the triangles t_1, t_2, t_3 of P_2 containing 0, 1, 2, respectively, and the set $\mathcal{A}_0$ of 3-suns obtained from the parallel classes P_i, $i = 5, 6, \dots, 6k+4$ by using the technique in Lemma 3.8 of [9]. The set $\mathcal{A} = \cup_{j=0}^{4} \mathcal{A}_j$, where $\mathcal{A}_1 = P_1 \setminus \{t\}$, $\mathcal{A}_2 = (P_2 \setminus \{t_1, t_2, t_3\}) \cup \mathcal{K}$ and $\mathcal{A}_j = P_j$ for $j = 3, 4$, is a partition of $\mathcal{E}(K_{\frac{n+3}{2}})$ such that $|\mathcal{A}| = (n^2+20n+3)/48$ and $\sum_{i=0}^{(n+1)/2} d_2(i) = 2n$. After applying *Step* 2., *Step* 3., and *Step* 4., proceed as follows. It is easy to see that the missing graph admits two matchings M'_1 and M'_2 both saturating the vertices $3, 4, \dots, \frac{n+1}{2}$; while, the edges of $\mathcal{M}$ not in M'_1 and M'_2 form a subgraph with maximum degree 2 and so can be partitioned into two matchings M'_3 and M'_4 both saturating all the vertices of $Z_{\frac{n+3}{2}}$. For every $j = 1, 2, 3, 4$, complete the blocks of $\mathcal{A}_j$ by using the edges of M'_j. □

Proposition 3.4. *For every $n \in \{21, 31, 37, 45, 55, 61, 69\}$, $u_{min}(n) = \frac{n+3}{2}$.*

Proof. Let $(X, \mathcal{T})$ be an STS(n).

For $n = 21$, partition the edges of the complete graph on Z_{12} into the following set $\mathcal{A}$:

$$\mathcal{A}_1 = \{(1, 2, 0; 11), (3, 7, 2; 5), (0, 4, 3; 9)\}\}$$

$$\mathcal{A}_2 = \{(0, 5, 6), (1, 8, 11), (7, 4, 10), (2, 9, 8; 10), (3, 1, 5; 10, 8)\}$$

$$\mathcal{A}_3 = \{(0, 9, 10), (3, 6, 8), (5, 7, 11), (2, 4, 11; 6), (1, 7, 9; 6, 11)\}$$

$$\mathcal{A}_4 = \{(0, 7, 8), (3, 10, 11), (5, 9, 4; 8), (1, 4, 6; 9), (2, 6, 10; 5)\}$$

where $d_2(i) = 3$ for $i \in \{5, 6, 8, 9, 10, 11\}$ and $d_2(i) = 4$ for $i \in \{0, 1, 2, 3, 4, 7\}$. After applying *Step* 2., *Step* 3., and *Step* 4., proceed as follows. Since $\mathcal{M}$ has maximum degree 4, it is easy to see that $\mathcal{M}$ admits a matching M'_1 saturating $\{0, 1, 2, 3, 4, 7\}$. Use M'_1 to complete the kites in $\mathcal{A}_1$. The graph obtained from $\mathcal{M}$ by deleting the edges of M'_1 is a bipartite graph such that all the vertices in Z_{12} has degree 3 and so its edges can be partitioned into three matchings M'_2, M'_3 and M'_4, each of which saturates the vertices of Z_{12}. For every $j = 2, 3, 4$, use the edges of M'_j to complete the blocks of $\mathcal{A}_j$.

For $n = 31$, partition the edges of the complete graph on Z_{17} into the following set $\mathcal{A}$:

$$\begin{aligned} \mathcal{A}_1 = &\{(0, 4, 1; 7) + i : i = 2, 3, 4, 5, 11, 12, 13, 14,\ i \in Z_{17}\} \cup \\ &\{(10, 12, 0; 3, 7)\} \\ \mathcal{A}_2 = &\{(0, 4, 1; 7) + i : i = 0, 1, 6, 7, 8, 9, 15, 16,\ i \in Z_{17}\} \cup \{(14, 7, 9; 2, 0)\} \\ \mathcal{A}_3 = &\{(0, 7, 2; 10) + i : i = 1, 4, 13, 15, 16,\ i \in Z_{17}\} \cup \{(10, 14, 11; 0), \\ &(9, 4, 2; 12, 0), (12, 2, 14; 10, 5)\} \\ \mathcal{A}_4 = &\{(0, 7, 2; 10) + i : i = 3, 5, 6, 8, 9, 11, 14,\ i \in Z_{17}\} \end{aligned}$$

where $d_2(i) = 2$ for $i \in \{0, 2, 7\}$ and $d_2(i) = 4$ for $i \in Z_{17} \setminus \{0, 2, 7\}$. After applying *Step* 2., *Step* 3., and *Step* 4., proceed as follows. Consider a subgraph $\mathcal{M}'$ of the missing graph such that each vertex in Z_{17} has degree 2. Partition the edges of $\mathcal{M}'$ into two matchings M'_1 and M'_2 and use them to complete the kites in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. After deleting the edges of M'_1 and M'_2 the remaining edges of $\mathcal{M}$ can be partitioned into two matchings M'_3 and M'_4, each of which saturates the vertices in $Z_{17} \setminus \{0, 2, 7\}$ and can be used to complete the kites in $\mathcal{A}_3$ and $\mathcal{A}_4$, respectively.

By similar arguments it is possible to settle the remaining cases $n = 37, 45, 55, 61, 69$, for which we refer to Appendix where we give the sets $\mathcal{A}_j$s, which automatically determine the matchings M'_js. □

4. Main result and conclusion

Combining Lemmas 2.1, 3.2, and Propositions 3.1, 3.2, 3.3, 3.4 gives our main result.

Main Theorem.

(i) *If $n \equiv 1, 3, 9, 19 \pmod{24}$, then $u_{min}(n) = \frac{n-1}{2}$ for every $n \neq 3, 9$, $u_{min}(3) = 6$, and $u_{min}(9) = 7$.*
(ii) *If $n \equiv 7, 13, 15, 21 \pmod{24}$, then $u_{min}(n) = \frac{n-1}{2} + 2$ for every $n \neq 7, 13$, $u_{min}(7) = 6$, and $u_{min}(13) = 11$.*

In [14] a complete solution to the Doyen-Wilson problem for 3-sun systems is given and it is proved that any 3SS(n) can be embedded in a 3SS(m) if and only if $m \geq \frac{7}{5}n + 1$ or $m = n$. For every integer $v \in \Sigma(K_3)$, combining Main Theorem with the above result gives an integer m_v such that any STS(v) can be embedded in a 3SS(m) for every admissible $m \geq m_v$. A question to be asked is the following.

Open Problem Can one embed any STS(v) in a 3SS(m) for every admissible m such that $v + u_{min}(v) < m < m_v$?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

$n = 37$:

$$\mathcal{A}_1 = \{(4, 11, 0; 8) + 2i : i = 0, 1, \ldots, 9,\ i \in Z_{20}\}$$
$$\mathcal{A}_2 = \{(5, 12, 1; 9) + 2i : i = 0, 1, \ldots, 9,\ i \in Z_{20}\}$$
$$\mathcal{A}_3 = \{(14, 16, 13; 2, 19), (4, 6, 3; 16, 9)\}$$
$$\mathcal{A}_4 = \{(1, 3, 0; 6) + i : i = 0, 12, 17,\ i \in Z_{20}\} \cup$$
$$\{(7, 12, 2; 17), (16, 17, 19; 14), (5, 7, 4; 17, 10), (6, 8, 5; 18, 11),$$
$$(8, 10, 7; 15, 13), (11, 9, 8; 4, 14), (10, 12, 9; 17, 15), (2, 4, 1; 14, 16)\}$$
$$\mathcal{A}_5 = \{(1, 3, 0; 6) + i : i = 14, 15,\ i \in Z_{20}\} \cup$$
$$\{(5, 10, 0; 15), (8, 13, 3; 18), (0, 2, 19; 4), (3, 5, 2; 15, 8), (7, 9, 6; 19, 12),$$
$$(11, 13, 10; 18, 16), (12, 14, 11; 19, 17), (1, 18, 19; 4, 5), (6, 11, 1; 16, 7)\}$$

$n = 45$:

$$\mathcal{A}_1 = \{(1, 13, 7; 19) + i : i = 0, 1, 2, 3, 4,\ i \in Z_{24}\} \cup$$
$$\{(8, 16, 0; 22, 12), (9, 17, 1; 23, 19), (10, 18, 2; 6, 20), (19, 11, 3; 0, 21),$$
$$(20, 12, 4; 6, 22), (21, 13, 5; 19, 23), (22, 14, 6; 20, 0), (7, 15, 23; 21, 12)\}$$
$$\mathcal{A}_2 = \{(0, 1, 5) + 3i : i = 0, 1, \ldots, 7,\ i \in Z_{24}\}$$
$$\mathcal{A}_3 = \{(1, 2, 6) + 3i : i = 0, 1, \ldots, 7,\ i \in Z_{24}\}$$
$$\mathcal{A}_4 = \{(2, 3, 7) + 3i : i = 0, 1, \ldots, 7,\ i \in Z_{24}\}$$
$$\mathcal{A}_5 = \{(1, 3, 10; 12, 6, 20) + i : i \in Z_{24} \setminus \{11, 23\}\} \cup \{(0, 2, 9; 18, 5, 19),$$
$$(12, 14, 21; 18, 17, 7)\}$$

$n = 55$:

$$\mathcal{A}_1 = \{(13, 27, 0; 25) + i : i = 3, 4, \ldots, 13,\ i \in Z_{29}\} \cup$$
$$\{(15, 0, 2; 25, 27), (12, 14, 27; 10, 13), (28, 13, 15; 0, 11), (14, 0, 16; 27, 12),$$
$$\mathcal{A}_2 = \{(13, 27, 0; 25) + i : i = 1, 17, 18, \ldots, 28,\ i \in Z_{29}\}$$
$$\mathcal{A}_3 = \{(0, 10, 11; 2, 6) + i : i \in Z_{29}\}$$
$$\mathcal{A}_4 = \{(0, 9, 12; 2, 6) + i : i \in Z_{29}\}$$

$n = 61$:

$$\begin{aligned}\mathcal{A}_1 =&\{(0,10,29;9)+i : i=0,1,2,3,6,17,18,19,20,\ i\in Z_{32}\}\cup\\ &\{(14,22,6;30)\}\cup\{(23,10,13;22,29)+i : i=0,1,2,\ i\in Z_{32}\}\cup\\ &\{(4,23,26;3,18)+i : i=0,1,3,4,5,\ i\in Z_{32}\}\cup\{(26,13,16;25,24),\\ &(21,18,31;30,7)\}\\ \mathcal{A}_2 =&\{(8,16,0;24)+i : i=0,1,\ldots,5,\ i\in Z_{32}\}\cup\{(4,14,1;13),\\ &(0,22,19;31),(2,24,21;1)\}\cup\\ &\{(1,20,23;0,15)+i : i=0,2,5,\ i\in Z_{32}\}\cup\{(5,2,15;14,7)\}\\ \mathcal{A}_3 =&\{(17,4,7;16,23)+i : i=0,1,2,3,4,5,\ i\in Z_{32}\}\\ \mathcal{A}_4 =&\{(9,0,2;11,17)+i : i\in Z_{32}\}\\ \mathcal{A}_5 =&\{(5,0,1;6,15)+i : i\in Z_{32}\}\end{aligned}$$

$n=69$:

$$\begin{aligned}\mathcal{A}_1 =&\{(4,2,0;6,34)+3i : i=5,6,7,8,9,10,\ i\in Z_{36}\}\\ \mathcal{A}_2 =&\{(4,2,0;6,34)+3i : i=0,1,2,3,4,11,\ i\in Z_{36}\}\cup\\ &\{(24,12,0;30,18)+i,(30,18,6;24)+i : i=0,1,2,3,4,5,\ i\in Z_{36}\}\\ \mathcal{A}_3 =&\{(0,7,15;1)+2i : i=0,1,\ldots,17,\ i\in Z_{36}\}\\ \mathcal{A}_4 =&\{(1,8,16;2)+2i : i=0,1,\ldots,17,\ i\in Z_{36}\}\\ \mathcal{A}_5 =&\{(9,20,0;3,13)+i : i\in Z_{36}\}\\ \mathcal{A}_6 =&\{(0,6,1;10,32,11)+9i,(1,7,2;5,11,12)+9i,(2,8,3;5,9,13)+9i,\\ &(3,9,4;6,14,8)+9i,(4,10,5;1,7,8)+9i,(5,11,6;35,8,9)+9i,\\ &(6,12,7;16,9,17)+9i,(7,13,8;4,23,18)+9i : i=0,1,2,3,\ i\in Z_{36}\}\end{aligned}$$

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