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What symmetries can do for you

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Several applications of Lie symmetries and its generalisation are presented: from turning butterflies into tornados, to its applications in epidemics, population dynamics, and ultimately converting classical problems into the quantum realm. Applications of nonclassical symmetries are also illustrated.

Keywords: Lie symmetry; population dynamics; Lagrangian; Jacobi last multiplier; Noether symmetry; classical quantization; nonclassical symmetries.

1. Introduction

In his inaugural address at King's College London in 1860, James Clerk Maxwell said:[1] *We, while following out the discoveries of the teachers of science, must experience in some degree the same desire to know and the same joy in arriving at knowledge which encouraged and animated them.*

This paper is a review of some of the author's work in the area of symmetries, those symmetries initiated by a great teacher of science, Sophus Lie, who based his idea on the work of another sublime teacher of science, Karl Gustav Jacob Jacobi[a]. Which symmetries are we dealing with? In the Introduction of his book,[3] Olver tersely stated: *The applications of Lie's continuous symmetry groups include such diverse fields as algebraic topology, differential geometry, invariant theory, bifurcation theory, special functions, numerical analysis, control theory, classical mechanics quantum mechanics, relativity, continuum mechanics and so on. It is impossible to*

[a]Hawkins has established in Ref. 2 *the nature and extent of Jacobi's influence upon Lie.* As Hawkins clearly stated,[2] *given the fact that the Jacobi Identity is fundamental to the theory of Lie groups, Jacobis influence upon Lie will come as no surprise. But the bald fact that he inherited the Identity from Jacobi fails to convey fully or accurately the historical dimension of the impact of Jacobi's work on partial differential equations.*

overestimate the importance of Lie's contribution to modern science and mathematics. Nevertheless, anyone who is already familiar with one of these modern manifestations of Lie group theory is perhaps surprise to learn that its original inspirational source was the field of differential equations.

Nearly thirty years later, and after hundreds of papers and many books, e.g. Refs. 4-13 just to cite a few, have been published on this subject, a young theoretical mathematician still states: *The only useful symmetry is dilation (they can be used to determine embeddings into Sobelev or Lebesgue spaces) and all other ones are useless,* while a senior theoretical physicist asks: *What are these symmetries?* We shall show by means of some examples what symmetries can do[b]. Of course, we do not claim to cover every aspect of the applications of Lie symmetries in the field of differential equations. We hope to inspire some curiosity in both young and senior mathematical physicists, in such a way that they wish to know what Lie symmetries can do for them.

2. From a Butterfly to a Tornado with Symmetries

The motion of a heavy rigid body about a fixed point is one of the most famous problems of classical mechanics.[15] In 1857 Maxwell himself wrote[16] *To those who study the progress of exact science, the common spinning-top is a symbol of the labours and the perplexities of men who had successfully threaded the mazes of the planetary motions. The mathematicians of the last age, searching through nature for problems worthy of their analysis, found in this toy of their youth, ample occupation for their highest mathematical powers.* In 1750 it was Euler[17] who derived the equations of motion, which now bear his name, and described what is nowadays known as the Euler-Poinsot case because of the geometrical description given by Poinsot[c] about a hundred years later.[18] It was Jacobi[19] who integrated this case by using the elliptic functions which he had developed (along with Legendre, Abel and Gauss[20]) and mastered[21] – we have translated this fundamental text into Italian and commented extensively.[22]

More than 200 years later, in 1963, a paper was published[23] in which was presented a system of three ordinary differential equations. The author considered a hydrodynamical system developed by Rayleigh[24] and reduced it by applying a double Fourier series as in Ref. 25. Thus he obtained what nowadays is the famous Lorenz system.[26] Three parameters are part of the Lorenz system. For particular values of those parameters the Lorenz system can be integrated in closed form by means of Jacobi elliptic functions.[27] We call this system the Lorenz integrable system.

[b]Since the task of finding symmetries can be quite cumbersome, we have used our interactive REDUCE programs[14] throughout.

[c]Again in Ref. 16 Maxwell stated *M. Poinsôt has brought the subject under the power of a more searching analysis than that of the calculus, in which ideas take the place of symbols, and intelligible propositions supersede equations.*

In Ref. 28 we applied Lie group analysis to a third-order differential equation, which is equivalent to the Lorenz integrable system, and obtained a two-dimensional Lie symmetry algebra, which we then used to integrate the Lorenz integrable system in terms of Jacobi elliptic functions. In Ref. 29 we showed that the same Lie symmetry algebra is admitted by a third-order differential equation which is equivalent to the Euler equations of a torque-free rigid body moving about a fixed point. Then a transformation was easily derived by which the Lorenz integrable system becomes the Euler equations of a torque-free rigid body moving about a fixed point. Thus, it can be stated that "the Lorenz integrable system moves à la Poinsot".

In Ref. 29 the same transformation was applied to the Lorenz system with any value of parameters, and consequently the Euler equations of a rigid body moving about a fixed point and subjected to a torsion depending on time and angular velocity was obtained. The numerical solution of this system yields a three-dimensional picture which resembles a tornado. Thus Lorenz's butterfly was transformed into a tornado[d].

Consider the Lorenz system:[23]

$$x' = \tilde{\sigma}(y - x), \tag{1}$$

$$y' = -xz + \tilde{r}x - y, \tag{2}$$

$$z' = xy - \tilde{b}z, \tag{3}$$

where $\tilde{\sigma}$, $\tilde{b}$ and $\tilde{r}$ are parameters (a prime denotes differentiation with respect to τ). This system can be reduced to a single third-order ordinary differential equation[30] for x, which admits a two-dimensional Lie symmetry algebra if $\tilde{\sigma} = 1/2$, $\tilde{b} = 1$ and $\tilde{r} = 0$. System (1-3) becomes

$$x' = \frac{(y - x)}{2}, \tag{4}$$

$$y' = -xz - y, \tag{5}$$

$$z' = xy - z. \tag{6}$$

The corresponding third-order equation is:

$$2xx''' - 2x'x'' + 5xx'' - 3x'^2 + 2x^3x' + 3xx' + x^4 + x^2 = 0, \tag{7}$$

and admits a two-dimensional Lie symmetry algebra L_2 with basis:

$$X_1 = \partial_\tau, \quad X_2 = e^{\tau/2}\left(\partial_\tau - \frac{1}{2}x\partial_x\right). \tag{8}$$

A basis of its differential invariants of order ≤ 2 is given by:

$$\phi = \left(x' + \frac{x}{2}\right)x^{-2}, \quad \psi = \left(x'' + \frac{3}{2}x' + \frac{x}{2}\right)x^{-3}. \tag{9}$$

[d] In 1972, Lorenz gave a talk at the December meeting of the American Association for the Advancement of Science in Washington, entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?*

Equation (7) is reduced to the following first-order equation:

$$\left(\psi - 2\phi^2\right)\frac{\mathrm{d}\psi}{\mathrm{d}\phi} = -2\psi\phi - \phi, \tag{10}$$

which can be easily integrated[31] to give:

$$\frac{1 + 4\psi - 4\phi^2}{(1+2\psi)^2} = c_1, \tag{11}$$

where c_1 is an arbitrary constant. Substitution of x and its derivatives into (11) yields a second-order ordinary differential equation

$$\frac{1 + 4\left(x'' + \frac{3}{2}x' + \frac{1}{2}x\right)x^{-3} - 4\left(x' + \frac{1}{2}x\right)^2 x^{-4}}{\left(x^3 + 2x'' + 3x' + x\right)^2 x^{-6}} = c_1, \tag{12}$$

which admits the Lie symmetry algebra L_2. Lie's classification of two-dimensional algebras into four canonical types[32] allows us to integrate (12) by quadrature if we introduce the canonical variables:

$$v = -2e^{-\tau/2}, \quad u = \frac{e^{-\tau/2}}{x}, \tag{13}$$

which transform equation (12) into

$$\frac{1 + 4\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 - 4u\frac{\mathrm{d}^2u}{\mathrm{d}v^2}}{\left[2u\frac{\mathrm{d}^2u}{\mathrm{d}v^2} - 4\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 - 1\right]^2} = c_1, \tag{14}$$

and operators (8) into

$$\bar{X}_1 = \partial_v, \quad \bar{X}_2 = v\partial_v + u\partial_u. \tag{15}$$

Then the general solution of (14) can be easily derived[32] to be:

$$\int \left(-c_1 \mp 2c_2u^2 - c_2^2u^4\right)^{-1/2}\mathrm{d}u = \pm\frac{v}{2\sqrt{c_1}} + c_3, \tag{16}$$

with c_2 and c_3 arbitrary constants. This solution which involves an elliptic integral has already been obtained by Sen and Tabor[30] by means of a lengthier analysis.

The Euler equations describing the motion of a heavy rigid body about a fixed point with no torsion are

$$\dot{p} = \frac{(B-C)}{A}qr, \tag{17}$$

$$\dot{q} = \frac{(C-A)}{B}pr, \tag{18}$$

$$\dot{r} = \frac{(A-B)}{C}pq, \tag{19}$$

with A, B and C being the principal moments of inertia, and $p(t), q(t)$ and $r(t)$ the components of the angular velocity (a dot denotes differentiation with respect to

t). This system can be reduced to a single third-order ordinary differential equation for, say, p, viz

$$p\frac{\mathrm{d}^3p}{\mathrm{d}t^3} - \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\mathrm{d}^2p}{\mathrm{d}t^2} - \frac{4(C-A)(A-B)}{BC}p^3\frac{\mathrm{d}p}{\mathrm{d}t} = 0, \tag{20}$$

which admits a two-dimensional Lie symmetry algebra $\mathcal{L}_2$ with basis:

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = t\partial_t - p\partial_p. \tag{21}$$

The two Lie symmetry algebras L_2 and $\mathcal{L}_2$ that we have found are actually the same, i.e. Type IV in Lie's classification.[32] Therefore they are linked by a transformation which takes (τ, x) into (t, p). Prolongation to the second-order of the two equivalent Lie symmetry algebras yields a transformation which takes the system (17)-(19) into the system (4)-(6) as

$$\begin{aligned} \tau &= \log\left(\frac{4}{t^2}\right), \\ x &= \frac{p\,t}{2}, \\ y &= \frac{C-B}{2\,A}qrt^2, \\ z &= \frac{C-B}{2A}\left[\frac{(C-A)}{B}r^2 + \frac{(A-B)}{C}q^2\right]t^2, \end{aligned} \tag{22}$$

with the following condition on the moments of inertia

$$\frac{(A-B)(A-C)}{BC} = \frac{1}{4}. \tag{23}$$

A slightly more general condition could have been considered if one replaces $1/4$ with $k/4$ (k an arbitrary parameter). If one derives B from (23) by assuming $A - C > 0$ and $4A - 3C > 0$, i.e.

$$B = \frac{4A(A-C)}{4A-3C}, \tag{24}$$

then the transformation (22) turns into the following

$$\begin{aligned} \tau &= \log\left(\frac{4}{t^2}\right), \\ x &= \frac{p\,t}{2}, \\ y &= -\frac{(2A-C)(2A-3C)}{2A(4A-3C)}qrt^2, \\ z &= -\frac{(2A-C)(2A-3C)\left[4A^2q^2 - (4A-3C)^2r^2\right]}{8A^2(4A-3C)^2}t^2, \end{aligned} \tag{25}$$

and the system (17)-(19) assumes the form

$$\dot{p} = \frac{(2A-C)(2A-3C)}{A(4A-3C)}qr, \tag{26}$$

$$\dot{q} = \frac{3C-4A}{4A}pr, \tag{27}$$

$$\dot{r} = \frac{A}{4A-3C}pq. \tag{28}$$

We also derived the inverse transformation, i.e.

$$\begin{aligned} t &= 2e^{-\tau/2},\\ p &= xe^{\tau/2},\\ q &= -\frac{(4A-3C)ye^{\tau/2}}{2\sqrt{(2A-C)(2A-3C)(\sqrt{y^2+z^2}+z)}},\\ r &= \frac{Ae^{\tau/2}\sqrt{\sqrt{y^2+z^2}+z}}{\sqrt{(2A-C)(2A-3C)}}, \end{aligned} \tag{29}$$

which takes the system (4)-(6) into the system (17)-(19) after substituting B as in (24).

If one applies the transformation (29) to the general Lorenz system (1)-(3), then the following equations are obtained

$$\dot{p} = \frac{2(2A-C)(2A-3C)\tilde{\sigma}}{A(4A-3C)}qr + (2\tilde{\sigma}-1)\,\frac{p}{t}, \tag{30}$$

$$\begin{aligned} \dot{q} = {} & \frac{3C-4A}{4A}pr + (\tilde{b}-1)\frac{4A^2q^2-(4A-3C)^2r^2}{4A^2q^2+(4A-3C)^2r^2}\,\frac{q}{t}\\ & +\tilde{r}\frac{2(4A-3C)^3A}{(2A-C)(2A-3C)\left[4A^2q^2+(4A-3C)^2r^2\right]}\,\frac{pr}{t^2}, \end{aligned} \tag{31}$$

$$\begin{aligned} \dot{r} = {} & \frac{A}{4A-3C}pq - (\tilde{b}-1)\frac{4A^2q^2-(4A-3C)^2r^2}{4A^2q^2+(4A-3C)^2r^2}\,\frac{r}{t}\\ & +\tilde{r}\frac{8(4A-3C)A^3}{(2A-C)(2A-3C)\left[4A^2q^2+(4A-3C)^2r^2\right]}\,\frac{pq}{t^2}. \end{aligned} \tag{32}$$

They can be interpreted as the Euler equations of a rigid body moving about a fixed point and subjected to a torsion which depends on time t and angular velocity (p, q, r) in the body-frame reference. Also the moments of inertia are related by means of (24).

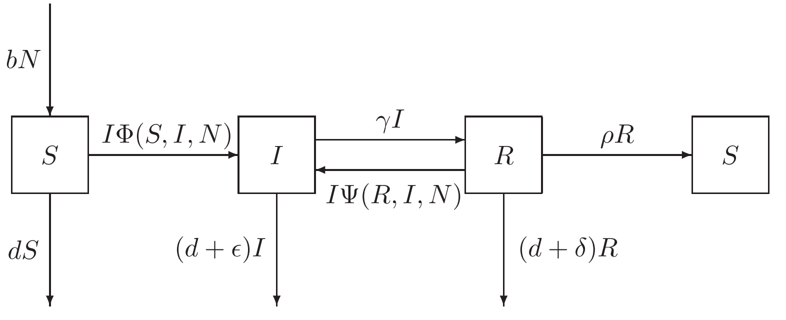
We used MAPLE in order to draw a three-dimensional plot of system (30)-(32), and a tornado was indeed obtained.[29]

3. Epidemics: Periodic Solutions with Symmetries

In Ref. 33 we showed that for a certain relationship among the involved parameters Lie group analysis, when applied to a SIRI disease transmission model formulated by Derrick and van den Driessche,[34] leads to a periodic general solution in apparent contrast to the qualitative analysis performed in Ref. 34.

Derrick and van den Driessche formulated a model of disease transmission in a nonconstant population of size N divided into three classes: susceptibles (S), infectives (I) and recovereds (R).

Individuals move from one compartment to the next according to the following flow diagram:



The parameter b is per capita birth-rate, d per capita disease free death rate, ϵ excess per capita death rate of infectives, δ excess per capita death rate of recovereds, γ per capita recovery rate of infectives, and ρ per capita loss of immunity rate of recovereds. The incidence of disease in the susceptible class is given by the function $I\Phi(S, I, N)$, while $I\Psi(R, I, N)$ is the transfer rate of the recovered class into the infective class. The above hypotheses lead to the following differential equations[e]:

$$S' = bN - dS + \varrho R - I\Phi(S, I, N), \tag{33}$$

$$I' = I[\Phi(S, I, N) + \Psi(R, I, N) - (d + \epsilon + \gamma)], \tag{34}$$

$$R' = \gamma I - (d + \delta + \varrho)R - I\Psi(R, I, N). \tag{35}$$

The analysis in Ref. 34 was mainly dedicated to show existence (or nonexistence) of periodic solutions for the SIRS model (33)-(35) when proportions of individuals in the three epidemiological classes are considered, i.e.

$$s = S/N, \quad i = I/N, \quad r = R/N. \tag{36}$$

With these variables system (33)-(35) becomes

$$s' = b(1 - s) + \varrho r + \epsilon si + \delta sr - i\Phi(s, i), \tag{37}$$

$$i' = -(b + \epsilon + \gamma)i + \epsilon i^2 + \delta ir + i\Phi(s, i) + i\Psi(r, i), \tag{38}$$

$$r' = \gamma i - (b + \varrho + \delta)r + \epsilon ri + \delta r^2 - i\Psi(r, i), \tag{39}$$

where $\Phi(s, i) = \Phi(s, i, 1) = \Phi(S/N, I/N, N/N) = \Phi(S, I, N)$ and $\Psi(r, i) = \Psi(r, i, 1)$ $= \Psi(R/N, I/N, N/N) = \Psi(R, I, N)$.

[e]Here $'$ denotes a derivative by t.

In Ref. 34 a theorem was presented and proved in order to establish under what conditions system (37)-(39) does not possess periodic solutions in the feasibility region

$$\mathcal{D} = \{s \geq 0, i \geq 0, r \geq 0 : s + i + r = 1\}. \tag{40}$$

An example of the nonexistence of periodic solutions was then introduced, namely a special SIRI case of the general model (37)-(39) with $\varrho = \delta = 0$, $\Phi(s,i) = \phi s$, and $\Psi(r,i) = \psi r$. Since $s + i + r = 1$ it is possible to eliminate r and finally obtain the following system:

$$s' = b(1-s) - (\phi - \epsilon)si, \tag{41}$$

$$i' = i[(\phi - \psi)s + (\epsilon - \psi)i - (\epsilon + b + \gamma - \psi)]. \tag{42}$$

In Ref. 35 Lie group analysis was applied to system (41)-(42), namely to either the second-order equation in the unknown s that one obtains by deriving i from (41) or the second-order equation in the unknown i that one obtains by deriving s from (42). Several cases were found, even instances of hidden linearity. In particular, it was found[33] that when

$$b = 0, \quad \phi = 2\epsilon - \psi, \quad \gamma = \psi - \epsilon$$

then a two-dimensional Lie symmetry algebra is admitted by equation

$$\begin{aligned} i'' = & -((\psi i^2 - i')i' + \gamma\phi i^3 + b^2 i^2 + (i-1)\epsilon^2 i^3 + ((i-1)\psi i + i')\phi i^2 \\ & -(\gamma i + 2i' + (\phi + \psi)(i-1)i)\epsilon i^2 + (\psi i^2 + i' + \gamma i \\ & +(i-1)\phi i - (2i-1)\epsilon i)bi)/i, \end{aligned} \tag{43}$$

which is obtained from system (41)-(42) by deriving s from equation (42), i.e.

$$s = \frac{[b + \gamma - (\epsilon - \psi)(i-1)]i + i'}{(\phi - \psi)i}, \tag{44}$$

and substituting it into equation (41). The Lie symmetry algebra is generated by the operators

$$\Gamma_1 = t\partial_t - i\partial_i, \qquad \Gamma_2 = \partial_t\,. \tag{45}$$

This means that equation (43) can be easily integrated by quadrature. Its general solution is

$$i = \frac{a_1}{\sin\left(\dfrac{a_1\ a_2 - a_1\ t}{\epsilon - \psi}\right)\,(\epsilon^2 - 2\,\epsilon\,\psi + \psi^2)}, \tag{46}$$

and from (44) one obtains:

$$s = \frac{1}{2}\,\frac{a_1\,\left(\cos\left(\dfrac{a_1\ a_2 - a_1\ t}{\epsilon - \psi}\right) - 1\right)}{\sin\left(\dfrac{a_1\ a_2 - a_1\ t}{\epsilon - \psi}\right)\,(\epsilon^2 - 2\,\epsilon\,\psi + \psi^2)}. \tag{47}$$

This general solution of system (41)-(42) is clearly periodic in apparent contrast with the findings in Ref. 34. Note that the functions (46)-(47) are neither bounded nor positive nor continuous, and do not belong to the feasibility region (40). In fact b must be positive for nonexistence of periodic solutions. However in Ref. 34 the condition $b = 0$ was allowed in order to show that system (37)-(39) has periodic solutions if $\Phi(s, i) = \phi s i$, and $\Psi(r, i) = 0$.

4. Determining Lagrangians from Symmetries

The inverse problem of calculus of variation has attracted a lot of interest since in the second half of the 18th century Euler[36] and then Lagrange[37] introduced the direct problem, namely the idea of linking the solution of a differential equation to the maximum/minimum of a functional, the celebrated problem of the brachistochrone being indeed the most famous classical example. It will take hundreds of pages to cite all the papers and books that have been published since. Most authors mark the birthdate of the inverse problem[f] with the 1887-papers by either Helmholtz[38] or Volterra.[39] Some other especially among the Russian speaking researchers pushes the date slightly back to the 1886-paper by Sonin.[40] Very few recognize the seminal work by Jacobi, namely his 1845-paper[41] and his 1842-1843 Dynamics Lectures published posthumously in 1884,[42] available in English[43] since 2009, where he links his last multiplier to the Lagrangian for any even-order ordinary differential equation (ODE). Actually both Volterra[g] and Sonin recognize the contribution of Jacobi last multiplier in their papers, Sonin more explicitly than Volterra since he showed that his own method involves the Jacobi last multiplier (p.10 in Ref. 40).

It was shown in Ref. 45 that Jacobi Last Multiplier yields the Lagrangian for any equation of even order[h]

$$u^{(2n)} = F(x, u, u', u'', \dots, u^{(2n-1)}), \tag{48}$$

since it can be derived from the following formula

$$M^{1/n} = \frac{\partial^2 L}{\partial (u^{(n)})^2}, \tag{49}$$

where M is the Jacobi Last Multiplier of equation (48) and L is its Lagrangian. This formula was given by Jacobi himself in Ref. 41 p. 364.

We recall that Fels has proved[46] that the Lagrangian is unique in the case of fourth-order equation, provided it exists. In the case of equations of sixth and eight order the uniqueness was proved by Juráš in Ref. 47.

The method of Jacobi last multiplier was enhanced when Lie determined the link with his symmetries,[48] a link very easy to implement that allows us to derive many multipliers and therefore Lagrangians.

[f] Roughly speaking, the problem of finding a Lagrangian if it exists.
[g] In his 1906 address at the Congress of Italian Naturalists[44] Volterra wrote *One of the most celebrated discovery by the mathematician Jacobi, that of the principle of the last multiplier.*
[h] We use a prime to indicate the derivative with respect to x.

As pointed out by Tonti[49] many authors have dealt with the inverse problem of calculus of variations by either using a formal approach or an operatorial approach following on the steps of either Helmholtz or Volterra: for example Refs. 50-55 and many others.

We have not underestimated the research of these very distinguished authors. Yet when possible we prefer to follow Jacobi since his Last Multiplier has a direct link to conservation laws and symmetries that are the essential elements that in our opinion make the difference between a mathematical abstraction and physical concreteness.[56]

We would like also to mention that many systems do not admit a Lagrangian. Nevertheless they may admit a Lagrangian if put in a different form as suggested by Bateman,[57] namely *finding a set of equations equal in number to a given set, compatible with it and derivable from a variational principle.* In Ref. 58 it was demonstrated how to construct many different Lagrangians for two famous examples which were deemed by Douglas[59] not to have a Lagrangian. Following Bateman's dictat different sets of equations compatible with those by Douglas and derivable from a variational principle were found in Ref. 58.

In Ref. 56 it was shown that a method presented by Trubatch and Franco[60] and later by Paine[61] for finding Lagrangians of classic models in biology, is actually based on finding the Jacobi Last Multiplier of such models. Using known properties of Jacobi Last Multiplier it was shown how to obtain linear Lagrangians of systems of two first-order ordinary differential equations and nonlinear Lagrangian of the corresponding single second-order equation that can be derived from them, even in the case where those authors failed such as the host-parasite model. Also it was shown that the Lagrangians of certain second-order ordinary differential equations derived by Volterra in Ref. 62 are particular cases of the Lagrangians that can be obtained by means of the Jacobi Last Multiplier and consequently more than one Lagrangian for those Volterra's equations were derived. Here after a survey on Jacobi Last Multiplier and its properties, we show two examples from Ref. 56.

4.1. *Jacobi last multiplier*

The method of the Jacobi Last Multiplier provides a means to determine all the solutions of the partial differential equation

$$\mathcal{A}f = \sum_{i=1}^{n} a_i(x_1, \dots, x_n)\frac{\partial f}{\partial x_i} = 0 \tag{50}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \dots = \frac{\mathrm{d}x_n}{a_n}. \tag{51}$$

In fact, if one knows the Jacobi Last Multiplier and all but one of the solutions, namely $n-2$ solutions, then the last solution can be obtained by a quadrature. The

Jacobi Last Multiplier M is given by

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = M\mathcal{A}f, \tag{52}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{53}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (50) or, equivalently, first integrals of (51) independent of each other. This means that M is a function of the variables $(x_1, \ldots, x_n)$ and depends on the chosen $n-1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi Last Multiplier are:

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \ldots, \eta_{n-1}$ of equation (50), then the corresponding last multiplier N is linked to M by the relationship:

$$N = M\frac{\partial(\eta_1, \ldots, \eta_{n-1})}{\partial(\omega_1, \ldots, \omega_{n-1})}.$$

(b) Given a non-singular transformation of variables

$$\tau: \quad (x_1, x_2, \ldots, x_n) \longrightarrow (x'_1, x'_2, \ldots, x'_n),$$

then the last multiplier M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x'_1, x'_2, \ldots, x'_n)},$$

where M obviously comes from the $n-1$ solutions of $\mathcal{A}F = 0$ which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1}.

(c) One can prove that each multiplier M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0; \tag{54}$$

viceversa every solution M of this equation is a Jacobi Last Multiplier.

(d) If one knows two Jacobi Last Multipliers M_1 and M_2 of equation (50), then their ratio is a solution ω of (50), or, equivalently, a first integral of (51). Naturally the ratio may be quite trivial, namely a constant. Viceversa the product of a multiplier M_1 times any solution ω yields another last multiplier $M_2 = M_1\omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was provided by Lie.[48,63] A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi.[64] If we know $n-1$ symmetries of (50)/(51), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \dots, x_n)\partial_{x_j}, \quad i = 1, n-1, \tag{55}$$

a Jacobi last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}. \tag{56}$$

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation of motion is missing the variable associated with that component, i.e., $\partial a_i/\partial x_i = 0$, the last multiplier is a constant, and any other Jacobi last multiplier is a first integral.

Another property of the Jacobi Last Multiplier is its relationship with the Lagrangian, namely Jacobi's formula (49), that for any second-order equation

$$\ddot{x} = \phi(t, x, \dot{x}) \tag{57}$$

becomes (see also Ref. 65)

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} \tag{58}$$

where $M = M(t, x, \dot{x})$ satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \frac{\partial \phi}{\partial \dot{x}} = 0. \tag{59}$$

Then equation (57) becomes the Euler-Lagrangian equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0. \tag{60}$$

The proof[65] is given by taking the derivative of (60) by $\dot{x}$ and showing that this yields (59). If one knows a Jacobi Last Multiplier, then L can be obtained by a double integration, i.e.:

$$L = \int \left(\int M \,\mathrm{d}\dot{x} \right) \mathrm{d}\dot{x} + \ell_1(t, x)\dot{x} + \ell_2(t, x), \tag{61}$$

where ℓ_1 and ℓ_2 are functions of t and x which have to satisfy a single partial differential equation related to (57), as it was shown in Ref. 66. However in Ref. 67,

ℓ_1, ℓ_2 were related to the gauge function $F = F(t,x)$. In fact, we may assume

$$\begin{aligned} \ell_1 &= \frac{\partial F}{\partial x} \\ \ell_2 &= \frac{\partial F}{\partial t} + \ell_3(t,x) \end{aligned} \tag{62}$$

where ℓ_3 has to satisfy the mentioned partial differential equation and F is obviously arbitrary.

In Ref. 60 it was shown that a system of two first-order ordinary differential equations

$$\begin{aligned} \dot{u}_1 &= \phi_1(t,u_1,u_2) \\ \dot{u}_2 &= \phi_2(t,u_1,u_2) \end{aligned} \tag{63}$$

always admits a linear Lagrangian of the form

$$L = U_1(t,u_1,u_2)\dot{u}_1 + U_2(t,u_1,u_2)\dot{u}_2 - V(t,u_1,u_2). \tag{64}$$

The key is a function W such that

$$W = -\frac{\partial U_1}{\partial u_2} = \frac{\partial U_2}{\partial u_1} \tag{65}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log W) + \frac{\partial \phi_1}{\partial u_1} + \frac{\partial \phi_2}{\partial u_2} = 0. \tag{66}$$

It is obvious that equation (66) is the equation (54) of the Jacobi Last Multiplier for system (63). Therefore once a Jacobi Last Multiplier $M(t,u_1,u_2)$ has been found, then a Lagrangian of system (63) can be obtained by two integrations, i.e.:

$$L = \left(\int M\,\mathrm{d}u_1\right)\dot{u}_2 - \left(\int M\,\mathrm{d}u_2\right)\dot{u}_1 + g(t,u_1,u_2) + \frac{\mathrm{d}}{\mathrm{d}t}G(t,u_1,u_2), \tag{67}$$

where $g(t,u_1,u_2)$ satisfies two linear differential equations of first order that can be always integrated, and $G(t,u_1,u_2)$ is the arbitrary gauge function that should be taken into consideration in order to correctly apply Noether's theorem.[68] If a Noether's symmetry

$$\Gamma = \xi(t,u_1,u_2)\partial_t + \eta_1(t,u_1,u_2)\partial_{u_1} + \eta_2(t,u_1,u_2)\partial_{u_2} \tag{68}$$

exists for the Lagrangian L in (67) then a first integral of system (63) is

$$-\xi L - \frac{\partial L}{\partial \dot{u}_1}(\eta_1 - \xi\dot{u}_1) - \frac{\partial L}{\partial \dot{u}_2}(\eta_2 - \xi\dot{u}_2) + G(t,u_1,u_2). \tag{69}$$

4.2. *Volterra-Lotka's model*

The Volterra-Lotka's model considered in Ref. 60 is the following:

$$\begin{aligned} \dot{w}_1 &= w_1(a + bw_2) \\ \dot{w}_2 &= w_2(A + Bw_1). \end{aligned} \tag{70}$$

In order to simplify system (70) we follow Ref. 60 and introduce the change of variables

$$w_1 = \exp(r_1), \quad w_2 = \exp(r_2) \tag{71}$$

and then system (70) becomes

$$\begin{aligned} \dot{r}_1 &= b\exp(r_2) + a \\ \dot{r}_2 &= B\exp(r_1) + A. \end{aligned} \tag{72}$$

An obvious Jacobi Last Multiplier of this system is a constant, say 1, and consequently by means of (67) a linear Lagrangian of system (72) is

$$L_{[r]} = r_1\dot{r}_2 - r_2\dot{r}_1 + 2(-B\exp(r_1) + b\exp(r_2) - Ar_1 + ar_2) + \frac{\mathrm{d}}{\mathrm{d}t}G(t, r_1, r_2) \tag{73}$$

which (minus the gauge function G) was found in Ref. 60. Moreover we can derive a Jacobi Last Multiplier for the Volterra-Lotka system (70) by using property (b). In fact we have to calculate the Jacobian of the transformation (71) between (w_1, w_2) and (r_1, r_2) and this yields a Jacobi Last Multiplier of system (70), i.e.

$$M_{[w]} = M_{[r]}\frac{\partial(r_1, r_2)}{\partial(w_1, w_2)} = \begin{vmatrix} \dfrac{1}{w_1} & 0 \\ 0 & \dfrac{1}{w_2} \end{vmatrix} = \frac{1}{w_1 w_2}. \tag{74}$$

Finally, formula (67) yields a linear Lagrangian of system (70)

$$L_{[w]} = \log(w_1)\frac{\dot{w}_2}{w_2} - \log(w_2)\frac{\dot{w}_1}{w_1} + 2(-A\log(w_1) + a\log(w_2) - Bw_1 + bw_2). \tag{75}$$

This Lagrangian was not obtained in Ref. 60. We note that (70) is autonomous and therefore invariant under time translation, namely ∂_t. It is easy to show that the Lagrangian $L_{[w]}$ in (75) yields a time-invariant first integral through Noether's theorem,[68] i.e.:

$$-L_{[w]} + \dot{w}_1\frac{\partial L_{[w]}}{\partial \dot{w}_1} + \dot{w}_2\frac{\partial L_{[w]}}{\partial \dot{w}_2} = A\log(w_1) - a\log(w_2) + Bw_1 - bw_2 = const. \tag{76}$$

Following Ref. 60 we can transform system (72) into an equivalent second-order ordinary differential equation by eliminating, say, r_1. In fact from the second equation in (72) one gets

$$r_1 = \log\left(\frac{\dot{r}_2 - A}{B}\right), \tag{77}$$

and the equivalent second-order equation in r_2 is the following

$$\ddot{r}_2 = -\Big(b\exp(r_2) + a\Big)(A - \dot{r}_2). \tag{78}$$

A Jacobi Last Multiplier for this equation has to satisfy equation (59), i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + b\exp(r_2) + a = 0 \tag{79}$$

namely

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \dot{r}_1 = 0, \tag{80}$$

by taking into account the first equation in (72), and consequently we get the following Jacobi Last Multiplier for equation (78):

$$M_1 = \exp(-r_1) = \frac{B}{\dot{r}_2 - A}, \tag{81}$$

the last equality thanks to (77). Then a Lagrangian can be obtained by a double integration as in (61), i.e.

$$L_1 = B\Big((\dot{r}_2 - A)\log(A - \dot{r}_2) - \dot{r}_2 + b\exp(r_2) + ar_2\Big) + \frac{\mathrm{d}}{\mathrm{d}t}F(t, r_2). \tag{82}$$

The same Lagrangian (minus the gauge function F) was obtained in Ref. 60. We note that (78) is autonomous and therefore invariant under time translation. It is easy to show that the Lagrangian L_1 in (82) yields a time-invariant first integral, through Noether's theorem,[68] i.e.:

$$I_1 = -ar_2 + \dot{r}_2 + A\log(A - \dot{r}_2) - b\exp(r_2) = const. \tag{83}$$

4.3. *Volterra-Verhulst-Pearl equation*

In 1939 Volterra wrote:[62] *I have been able to show that the equations of the struggle for existence depend on a question of Calculus of Variations (omissis). In order to obtain this result, I have replaced the notion of* population *by that of* quantity of life *[69]. In this manner I have also obtained some results by which dynamics is brought into relation to problems of the struggle for existence.* The quantity of life X and the population N of a species are connected by the relation

$$N = \frac{\mathrm{d}X}{\mathrm{d}t}. \tag{84}$$

It is immediately obvious that this idea of raising the order of each equation is totally different from that by Trubach and Franco who provided a method for finding a linear Lagrangian for systems of first-order equations. Also Volterra's method is different from that of deriving a single second-order equation from a system of two first-order equations: indeed Volterra takes a system of first-order equations and transform it into a system of second-order equations.

One of the two equations considered by Volterra is the Verhulst-Pearl equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N(\varepsilon - \lambda N) \tag{85}$$

that through (84) becomes

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2}=\frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right). \tag{86}$$

At glance it admits a two-dimensional Lie symmetry algebra generated by the operators ∂_t and ∂_X. Then a Jacobi Last multiplier for (86) can be obtained by means of (56), i.e.

$$\Delta_{(86)}=\det\begin{bmatrix}1 & \frac{\mathrm{d}X}{\mathrm{d}t} & \frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\Longrightarrow M_{(86)}=\frac{1}{\Delta_{(86)}}=\frac{1}{\frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)} \tag{87}$$

and consequently from Jacobi's formula (58) we obtain the Lagrangian

$$L_{(86)}=\frac{1}{\varepsilon}\frac{\mathrm{d}X}{\mathrm{d}t}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)+\frac{1}{\varepsilon\lambda}\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)\log\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)+X \tag{88}$$

which is indeed the Volterra's Lagrangian in Ref. 62.

Actually equation (86) admits an eight-dimensional Lie symmetry algebra generated by the following operators:

$$\begin{aligned}&\Gamma_1=\exp(\lambda X-\varepsilon t)\partial_t,\quad \Gamma_2=\exp(\lambda X)\left(\partial_t+\frac{\varepsilon}{\lambda}\,\partial_X\right),\quad \Gamma_3=\exp(-\lambda X+\varepsilon t)\partial_X,\\ &\Gamma_4=\exp(-\lambda X)\partial_X,\quad \Gamma_5=\exp(\varepsilon t)\left(\frac{\lambda}{\varepsilon}\,\partial_t+\partial_X\right),\Gamma_6=\partial_X,\\ &\Gamma_7=\exp(-\varepsilon t)\partial_t,\quad \Gamma_8=\partial_t.\end{aligned} \tag{89}$$

Therefore equation (86) is linearizable by means of a point transformation. In order to find the linearizing transformation we have to look for a two-dimensional abelian intransitive subalgebra, and, following Lie's classification of two-dimensional algebras in the real plane,[63] we have to transform it into the canonical form

$$\partial_u,\qquad y\partial_u$$

with u and y the new dependent and independent variables, respectively. We found that one such subalgebra is that generated by X_3 and X_4. Then it is easy to derive that

$$y=\exp(-\varepsilon t),\quad u=\frac{1}{\lambda}\exp(\lambda X-\varepsilon t) \tag{90}$$

and equation (86) becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}y^2}=0. \tag{91}$$

As we have shown above the Volterra's Lagrangian (88) of the equation (86) comes from the Jacobi Last Multiplier that can be obtained by means of (56) with the two symmetries Γ_8 and Γ_6 in (89). This Lagrangian (88) admits two Noether symmetries and therefore two first integrals of equation (86) can be derived by Noether's theorem,[68] i.e.

$$\begin{aligned}\Gamma_6 &\Longrightarrow In_6 = \log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) - \log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \varepsilon t\\ \Gamma_8 &\Longrightarrow In_8 = \log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \lambda X.\end{aligned} \tag{92}$$

Other nine Jacobi Last Multipliers and therefore Lagrangians can be obtained by means of (56) and any other combination of two symmetries in (89). The nine Jacobi Last Multipliers are:

$$\begin{aligned}
&JLM_{14} = \frac{\exp(\varepsilon t)}{\lambda\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2},\ JLM_{15} = -\frac{\varepsilon\exp(-\lambda X)}{\frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2},\ JLM_{17} = -\frac{\exp(2\varepsilon t - \lambda X)}{\lambda\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^3},\\
&JLM_{18} = \frac{\exp(\varepsilon t - \lambda X)}{\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)},\ JLM_{23} = \frac{\lambda\exp(-\varepsilon t)}{\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2},\\
&JLM_{25} = \frac{\varepsilon\lambda\exp(-\varepsilon t - \lambda X)}{\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)^3}, JLM_{34} = -\frac{\exp(-\varepsilon t - 2\lambda X)}{\varepsilon},\\
&JLM_{36} = -\frac{\exp(-\varepsilon t + \lambda X)}{\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}},\ JLM_{37} = \frac{\exp(\lambda X)}{\frac{\mathrm{d}X}{\mathrm{d}t}},
\end{aligned} \tag{93}$$

and the indices indicate which two of the symmetries in (89) have been used. Consequently the nine Lagrangians are:

$$\begin{aligned}
Lag_{14} &= -\exp(\varepsilon t)\left(\frac{1}{\lambda}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) + X\right),\\
Lag_{15} &= \exp(-\lambda X)\left(\frac{1}{\varepsilon}\frac{\mathrm{d}X}{\mathrm{d}t}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \frac{1}{\varepsilon}\log\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t} - \varepsilon\right)\frac{\mathrm{d}X}{\mathrm{d}t} + \frac{1}{\lambda}\right),\\
Lag_{17} &= -\frac{1}{2\lambda\frac{\mathrm{d}X}{\mathrm{d}t}}\exp(2\varepsilon t - \lambda X),\\
Lag_{18} &= \frac{1}{\varepsilon^2}\exp(\varepsilon t - \lambda X)\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t} - \varepsilon\right)\left(\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) - \varepsilon\log\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t} - \varepsilon\right)\right),\\
Lag_{23} &= -\frac{1}{\lambda}\exp(-\varepsilon X)\left(\log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \lambda X\right),\\
Lag_{25} &= \frac{\varepsilon\exp(-\varepsilon t - \lambda X)}{2\lambda\left(\varepsilon t - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)},\\
Lag_{34} &= -\frac{1}{2\varepsilon}\exp(-\varepsilon t + 2\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2,\\
Lag_{36} &= \frac{1}{\lambda^2}\exp(-\varepsilon t + \lambda X)\left(\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t} - \varepsilon\right)\log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right),\\
Lag_{37} &= \frac{1}{\varepsilon}\exp(\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) - \frac{\mathrm{d}X}{\mathrm{d}t} + \frac{\varepsilon}{\lambda}\right).
\end{aligned} \tag{94}$$

These Lagrangians admit a different number of Noether symmetries. The Lagrangians $Lag_{17}, Lag_{25}, Lag_{34}$ admit five Noether symmetries, the possible higher number. For example the Lagrangian Lag_{34} in (94) yields the following five Noether symmetries and corresponding first integrals of equation (86)

$$\begin{aligned} \Gamma_3 &\Longrightarrow Int_3 = \exp(\lambda X)\left(-\varepsilon + \lambda \frac{\mathrm{d}X}{\mathrm{d}t}\right), \\ \Gamma_4 &\Longrightarrow Int_4 = \exp(-\varepsilon t + \lambda X)\frac{\mathrm{d}X}{\mathrm{d}t}, \\ \Gamma_5 &\Longrightarrow Int_5 = \exp(2\lambda X)\left(\varepsilon - \lambda \frac{\mathrm{d}X}{\mathrm{d}t}\right)^2, \\ \Gamma_6 + 2\frac{\lambda}{\varepsilon}\Gamma_8 &\Longrightarrow Int_6 = \exp(-\varepsilon t + 2\lambda X)\frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon - \lambda \frac{\mathrm{d}X}{\mathrm{d}t}\right), \\ \Gamma_7 &\Longrightarrow Int_7 = \exp(-2\varepsilon t + 2\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2. \end{aligned} \tag{95}$$

5. Quantizing with Noether Symmetries

Different methods exist to describe the dynamics of quantum systems, namely

(i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;
(ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function;
(iii) the time-dependent Wigner function.

All three methods were shown to be linked via a dynamical invariant, the so-called Ermakov invariant, in Ref. 70. Therefore we pursue the quantization of classical problems by searching for a time-dependent Schrödinger equation.

In Ref. 71 it was inferred that Lie symmetries should be preserved if a consistent quantization is desired. In Ref. 72 [ex. 18, p. 433] an alternative Hamiltonian for the simple harmonic oscillator was presented. It is obtained by applying a nonlinear canonical transformation to the classical Hamiltonian of the harmonic oscillator. That alternative Hamiltonian was used in Ref. 73 to demonstrate what nonsense the usual quantization schemes[i] produce. In Ref. 77 a quantization scheme that preserves the Noether symmetries was proposed and applied to Goldstein's example in order to derive the correct Schrödinger equation. In Ref. 78 the same quantization scheme was applied in order to quantize the second-order Riccati equation, while in Ref. 79 the quantization of the dynamics of a charged particle in a uniform magnetic field in the plane and Calogero's goldfish system were achieved. In Ref. 80 the same method yielded the Schrödinger equation of an equation related to a

[i]Such as normal-ordering[74,75] and Weyl quantization.[76]

Calogero's goldfish, and Ref. 81 that of two nonlinear equations somewhat related to the Riemann problem.[82] In Ref. 83, and Ref. 84 it was shown that the preservation of the Noether symmetries straightforwardly yields the Schrödinger equation of a Liénard I nonlinear oscillator in the momentum space,[85] and that of a family of Liénard II nonlinear oscillators,[86] respectively.

If a system of second-order equations is considered, i.e.

$$\ddot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}}), \quad \mathbf{x} \in I\!R^N, \tag{96}$$

that comes from a variational principle with a Lagrangian of first order, then the quantization method that was first proposed in Ref. 77 consists of the following steps:

Step I. Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \mathbf{x})\partial_t + \sum_{k=1}^{N} W_k(t, \mathbf{x})\partial_{x_k}.$$

Step II. Among them find the Noether symmetries

$$\Gamma = V(t, \mathbf{x})\partial_t + \sum_{k=1}^{N} V_k(t, \mathbf{x})\partial_{x_k}.$$

This may require searching for the Lagrangian yielding the maximum possible number of Noether symmetries.[66, 87–89]

Step III. Construct the Schrödinger equation[j] admitting these Noether symmetries as Lie symmetries, namely

$$2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(\mathbf{x})\psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\mathbf{x})\psi_{x_k} + f_0(\mathbf{x})\psi = 0 \tag{97}$$

admitting the Lie symmetries

$$\Omega = V(t, \mathbf{x})\partial_t + \sum_{k=1}^{N} V_k(t, \mathbf{x})\partial_{x_k} + G(t, \mathbf{x}, \psi)\partial_\psi,$$

without adding any other symmetries apart from the two symmetries that are present in any linear homogeneous partial differential equation, namely

$$\psi\partial_\psi, \qquad \alpha(t, \mathbf{x})\partial_\psi,$$

where $\alpha = \alpha(t, \mathbf{x})$ is any solution of the Schrödinger equation (97).

Here we present the quantization with Noether symmetries of a charged particle in a uniform magnetic field in the plane as derived in Ref. 79. The corresponding classical Lagrangian is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \omega(y\dot{x} - x\dot{y}) \tag{98}$$

[j] We assume $\hbar = 1$ without loss of generality.

and consequently the Lagrangian equations are

$$\ddot{x} = -\omega\dot{y}, \quad \ddot{y} = \omega\dot{x}. \tag{99}$$

In Ref. 90 the following Schrödinger equation was determined

$$2i\psi_t + \psi_{xx} + \psi_{yy} - i\omega(y\psi_x - x\psi_y) - \frac{\omega^2}{4}(x^2 + y^2)\psi = 0. \tag{100}$$

The Lie symmetry algebra admitted by the linear system (99) has dimension fifteen,[91] and the classical Lagrangian (98) admits eight Noether symmetries generated by the following operators:

$$\begin{aligned}
X_1 &= \cos(\omega t)\partial_t - \frac{1}{2}\left(\sin(\omega t)\omega x + \cos(\omega t)\omega y\right)\partial_x + \frac{1}{2}\left(\cos(\omega t)\omega x - \sin(\omega t)\omega y\right)\partial_y,\\
X_2 &= -\sin(\omega t)\partial_t - \frac{1}{2}\left(\cos(\omega t)\omega x - \sin(\omega t)\omega y\right)\partial_x - \frac{1}{2}(\sin(\omega t)\omega x + \cos(\omega t)\omega y)\partial_y,\\
X_3 &= \partial_t,\\
X_4 &= -y\partial_x + x\partial_y,\\
X_5 &= -\sin(\omega t)\partial_x + \cos(\omega t)\partial_y,\\
X_6 &= -\cos(\omega t)\partial_x - \sin(\omega t)\partial_y,\\
X_7 &= \partial_y,\\
X_8 &= \partial_x.
\end{aligned} \tag{101}$$

The Schrödinger equation (100) admits an infinite Lie symmetry algebra generated by the operator $\alpha(t,x,y)\partial_\psi$, where α is any solution of the equation itself, and also a finite dimensional Lie symmetry algebra generated by the following operators:

$$\begin{aligned}
Y_1 &= X_1 + \frac{1}{4}\left(2\sin(\omega t)\omega - i\cos(\omega t)\omega^2(x^2+y^2)\right)\partial_\psi,\\
Y_2 &= X_2 + \frac{1}{4}\left(2\cos(\omega t)\omega + i\sin(\omega t)\omega^2(x^2+y^2)\right)\partial_\psi,\\
Y_3 &= X_3,\\
Y_4 &= X_4,\\
Y_5 &= X_5 - \frac{1}{2}\omega\left(x\cos(\omega t) + y\sin(\omega t)\right)\partial_\psi,\\
Y_6 &= X_6 + \frac{1}{2}\omega\left(x\sin(\omega t) - y\cos(\omega t)\right)\partial_\psi,\\
Y_7 &= X_7 + \frac{i}{2}\omega x\partial_\psi,\\
Y_8 &= X_8 - \frac{i}{2}\omega y\partial_\psi,\\
Y_9 &= \psi\partial_\psi.
\end{aligned} \tag{102}$$

This known example supports the method introduced here, namely that the Schrödinger equation admits a finite Lie symmetry algebra that corresponds to the Noether symmetries admitted by the classical Lagrangian plus the symmetry Y_9 admitted by any homogeneous linear partial differential equation.

6. Nonclassical Symmetries and Heir-Equations

Nonclassical symmetries were introduced in 1969 in a seminal paper by Bluman and Cole.[92] They obtained new exact solutions of the linear heat equation, i.e. solutions not deducible from the classical symmetry method. (Some authors call them Q-conditional symmetries[k] of the second type, e.g. Ref. 94, while others call them reduction operators, e.g. Ref. 95.)

The nonclassical symmetry method can be viewed as a particular instance of the more general differential constraint method that, as stated by Kruglikov,[96] *dates back at least to the time of Lagrange... and was introduced into practice by Yanenko*, see Ref. 97. The method was set forth in details in Yanenko's monograph[98] that was not published until after his death.[99] A more recent account and generalization of Yanenko's work can be found in Ref. 100.

The nonclassical symmetry method consists of adding the invariant surface condition to the given equation, and then apply the classical symmetry method. The main difficulty of this approach is that the determining equations are no longer linear. On the other hand, the nonclassical symmetry method may give more solutions than the classical symmetry method.

Galaktionov[101] and King[102] had found exact solutions of certain evolution equations which are not derived by either the classical or nonclassical symmetry method. However, in Ref. 103 it was shown that these solutions can be obtained by iterating the nonclassical symmetry method. Successive iterations generate new equations, that were named heir-equations because, although more complex than the original equations, they inherit the same Lie symmetry algebra. Invariant solutions of these heir-equations are just the solutions found by Galaktionov and by King.

We recall that the heir-equations are just some of the many possible n-extended equations as defined by Guthrie in Ref. 104.

In Ref. 103 it was also shown that this iterating method yields both partial symmetries as given by Vorobev in Ref. 105, and differential constraints as given by Olver.[106]

Fokas and Liu[107] and Zhdanov[108] independently introduced the method of generalised conditional symmetries, i.e., conditional Lie-Bäcklund symmetries. In Ref. 109 it was shown that the heir-equations can retrieve all the conditional Lie-Bäcklund symmetries found by Zhdanov. In Ref. 110 Goard has shown that Nucci's method

[k] In Ref. 93 this name was introduced for the first time.

of constructing heir equations by iterating the nonclassical symmetries method is equivalent to the generalised conditional symmetries method.

Moreover in Ref. 111 it was shown that one can find the nonclassical symmetries of any evolution equations of any order by using a suitable heir-equation and searching for a given particular solution among all its solutions, thus avoiding any complicated calculations.

In Ref. 112 a class of reaction-diffusion equations, i.e.

$$u_t = u_{xx} + cu_x + R(u, x), \tag{103}$$

with nonlinear source $R(u, x)$, was introduced as a model that incorporates climate shift, population dynamics, and migration for a population of individuals $u(t, x)$ that reproduce, disperse, and die within a patch of favorable habitat surrounded by unfavorable habitat. It is assumed that due to a shifting climate, the patch moves with a fixed speed $c > 0$ in a one-dimensional universe. In Ref. 113 nonclassical symmetries of (103) were searched for. Several cases were obtained by using suitable solutions of the heir-equations. Here after recalling the concept of heir-equations[103] and their link to nonclassical symmetries,[111] we outline its application to (103) and present one of the cases determined in Ref. 113, where one can find more details.

6.1. *Heir-equations and nonclassical symmetries*

For the sake of simplicity let us consider an evolution equation in two independent variables and one dependent variable of second order:

$$u_t = H(t, x, u, u_x, u_{xx}). \tag{104}$$

If

$$\Gamma = V_1(t, x, u)\partial_t + V_2(t, x, u)\partial_x - F(t, x, u)\partial_u \tag{105}$$

is a generator of a Lie point symmetry of equation (104) then the invariant surface condition is given by:

$$V_1(t, x, u)u_t + V_2(t, x, u)u_x = F(t, x, u). \tag{106}$$

Let us take the case with $V_1 = 0$ and $V_2 = 1$, so that (106) becomes[l]:

$$u_x = G(t, x, u) \tag{107}$$

Then, an equation for G is easily obtained. We call this the G-equation.[114] Its invariant surface condition is given by:

$$\xi_1(t, x, u, G)G_t + \xi_2(t, x, u, G)G_x + \xi_3(t, x, u, G)G_u = \eta(t, x, u, G). \tag{108}$$

Let us consider the case $\xi_1 = 0$, $\xi_2 = 1$, and $\xi_3 = G$, so that (108) becomes:

$$G_x + GG_u = \eta(t, x, u, G), \tag{109}$$

[l]We have replaced $F(t, x, u)$ with $G(t, x, u)$ in order to avoid any ambiguity in the following discussion.

which yields equation the η-equation. Clearly:

$$G_x + GG_u \equiv u_{xx} \equiv \eta. \tag{110}$$

We could keep iterating to obtain the Ω-equation, which corresponds to:

$$\eta_x + G\eta_u + \eta\eta_G \equiv u_{xxx} \equiv \Omega(t, x, u, G, \eta), \tag{111}$$

the ρ-equation, which corresponds to:

$$\Omega_x + G\Omega_u + \eta\Omega_G + \Omega\Omega_\eta \equiv u_{xxxx} \equiv \rho(t, x, u, G, \eta, \Omega) \tag{112}$$

and so on. Each of these equations inherits the symmetry algebra of the original equation, with the right prolongation: first prolongation for the G-equation, second prolongation for the η-equation, and so on. Therefore, these equations were named heir-equations in Ref. 103. This implies that even in the case of few Lie point symmetries many more Lie symmetry reductions can be performed by using the invariant symmetry solution of any of the possible heir-equations, as it was shown in Refs. 103, 115, 116.

6.2. *Nonclassical symmetries of equation (103)*

The $G-$equation of (103) is:

$$G_t + RG_u - G_{xx} - 2GG_{xu} - G^2G_{uu} - cG_x - R_uG - R_x = 0. \tag{113}$$

and the $\eta-$equation is

$$\begin{aligned} &\eta_t + R\eta_u + R_uG\eta_G - \eta_{xx} - 2G\eta_{xu} - 2\eta\eta_{xG} - G^2\eta_{uu} - 2G\eta\eta_{uG} \\ &\quad -\eta^2\eta_{GG} - c\eta_x - R_{uu}G^2 - R_u\eta - 2GR_{xu} + R_x\eta_G - R_{xx} = 0. \end{aligned} \tag{114}$$

The particular solution of the η-equation that we are looking for is

$$\eta(t, x, u, G) = -R(u, x) - cG + F(t, x, u) - V_2(t, x, u)G, \tag{115}$$

that replaced into (114) yields an overdetermined system in the unknowns F, V_2 and $R(u, x)$. We obtain a polynomial of third degree in G which four coefficients we call d_i, $\;(i = 0, 1, 2, 3)$ where i stands for the corresponding power of G. We impose all of them to be zero. From d_3, we obtain

$$V_2(t, x, u) = ss_1(t, x)u + ss_2(t, x), \tag{116}$$

while d_2 yields

$$F(t, x, u) = -\frac{1}{3}ss_1^2u^3 + \frac{1}{2}\left(\frac{\partial ss_1}{\partial x} - 2css_1 - 2ss_1ss_2\right) + ss_3(t, x)u + ss_4(t, x), \tag{117}$$

with $ss_j(t, x)$, $\;(j = 1, 2, 3, 4)$ arbitrary functions of t and x. Then after differentiating d_1 four times with respect to u we obtain

$$\frac{\partial^4 R(u, x)}{\partial u^4} = 0, \tag{118}$$

which implies that $R(u, x)$ must be a polynomial in u of third degree at most, i.e.

$$R(u, x) = -\frac{a_3^2(x)}{6}u^3 + \frac{a_2(x)}{2}u^2 + a_1(x)u + a_0(x), \tag{119}$$

where $a_i(x)$, $(i = 0, 1, 2, 3)$ are arbitrary functions of x. Since none of the remaining arbitrary functions depends on u, and d_1 has now become a polynomial of degree 3 in u, we have to annihilate all the four coefficients, i.e. $d_{1,i}$, $(i = 0, 1, 2, 3)$. From $d_{1,3}$ we have that $ss_1(t, x)$ must be a constant, and two cases raise:

Case 1. $ss_1 = \pm\frac{\sqrt{3}}{2}\, a_3(x)$,

Case 2. $ss_1 = 0$.

In Case 1, from coefficients $d_{1,2}$, $d_{1,1}$, $d_{1,0}$ we obtain ss_2, ss_3, and ss_4, respectively. All of them are functions of x only, e.g.

$$ss_2 = -\frac{1}{2a_3(x)}\left(-4a_3'(x) + \sqrt{3}a_2(x) + 2ca_3(x)\right), \tag{120}$$

where $'$ denotes differentiation with respect to x. Now the only remaining coefficient is d_0 which has become a linear polynomial in u. Therefore we are left with two expressions to annihilate, namely an underdetermined system of two equations that contain the derivative of $a_3(x)$ up to fifth order, and fourth order, respectively, and lower derivatives of the other three functions $a_2(x)$, $a_1(x)$, and $a_0(x)$. If we assume $a_3(x) = \sqrt{3}x$, and $a_2(x)$, $a_1(x), a_0(x)$ to be constants then we obtain that

$$R(u, x) = -\frac{1}{2}x^2u^3 + 3u^2 + \frac{1}{2}c^2u, \tag{121}$$

and

$$ss_1(t, x) = \frac{3x}{2}, \quad ss_2(t, x) = -\frac{1 + cx}{x}, \quad ss_3(t, x) = c\frac{-2 + 3cx}{4x}, \quad ss_4(t, x) = 0. \tag{122}$$

Thus, (115) becomes

$$\eta = -\frac{x^3u^3 + 2cu - c^2xu + 6x^2uG - 4G}{4x}, \tag{123}$$

namely

$$u_{xx} = -\frac{x^3u^3 + 2cu - c^2xu + 6x^2uu_x - 4u_x}{4x}, \tag{124}$$

that can be solved in closed form, i.e.

$$u(t, x) = \frac{c^2R_2(t)e^{\frac{cx}{2}} - c^2(1 + cx)e^{\frac{-cx}{2}}}{R_1(t) + (cx - 2)R_2(t)e^{\frac{cx}{2}} + (10 + 5cx + c^2x^2)e^{\frac{-cx}{2}}}, \tag{125}$$

with $R_k(t), k = 1, 2$ arbitrary functions of t. Substituting (125) into (103) yields the following nonclassical symmetry solution

$$u(t, x) = \frac{c^2c_1e^{c^2t+\frac{cx}{2}} - c^2(1 + cx)e^{\frac{-cx}{2}}}{c_2e^{\frac{-c^2t}{4}} + c_1(cx - 2)e^{c^2t+\frac{cx}{2}} + (10 + 5cx + c^2x^2)e^{\frac{-cx}{2}}}, \tag{126}$$

with $c_k, k = 1, 2$ arbitrary constants. We observe that

$$\lim_{t\to\infty} u(t, x) = \frac{c^2}{cx - 2}, \qquad \lim_{x\to\pm\infty} u(t, x) = 0 \tag{127}$$

and that $u(t, x) < 0$ for $t > 0, x < 0$. This means that the solution (126) is not defined at $x = 2/c$ and is positive[m] if $x \geq 0$.

More cases can be found in Ref. 113.

7. What Can Symmetries Do for You?

Symmetries can

- transform a nonlinear problem into a linear one;
- determine exact solutions, and/or conservation laws;
- transform a butterfly into a tornado;
- determine periodic solution for non-periodic problems;
- determine Lagrangians for (nearly) all problems;
- quantize a classical problem;
- and much more since we do not claim to have exhausted all the possibilities.

Last but not least, symmetries may also inspire beauty and peace in you
https://www.flickr.com/photos/17667265@N07/3192922037/in/photostream/

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[m]It depends also on the values given to the arbitrary constants.

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