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MIRAGE IN A BOX: A CLASSROOM ACTIVITY ON THE REFRACTIVE INDEX GRADIENT

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Abstract. The presentation will illustrate how to realize, in a transparent tank, a stable refractive index gradient by using sucrose solutions and a gelling agent. The apparatus will be used in a classroom for hand-on experiments on the refraction of light and on related topics, with particular emphasis on atmospheric effects such as mirages and Fata Morgana. Besides, in the preparation of this activity thermodynamical concepts, such as solubility, phase transitions and thermal stability are also involved. The ingredients and tools used in the preparation are easily available in household stores and do not present any risk of serious injuries. The potential applications of a refractive index gradient for optical fibres will be illustrated as well.

1. Introduction

Optical natural phenomena in the atmosphere have fascinated human beings since early times, therefore they can be used to bring young students closer to a "physical" observation of the natural world. Indeed, most of these phenomena can be observed almost every day and everywhere. Common examples are the colours changing in the sky at sunset and dawn, rainbows, halos, mirages, coronas, glories, green flashes and many more (Greenler 1980; Mahoney 1993; Lynch and Livingston 2001; Cowley *et al.* 2004).

All these phenomena are due to the more or less complex interaction between light and matter that occurs in the atmosphere, such as water droplets, ice crystals, dust or aerosols. In many cases Rayleigh or Mie scattering theory, together with Snell laws provide a clear physical explanation of the observed phenomenon.

Among this fascinating atmospheric optical phenomena, mirages are the simplest ones, since they only take place in plain air when thermal gradients occur in the atmosphere and the associated refractive index change makes light rays bend. As a consequence an object can be seen floating higher than it actually is or reflected on the ground, thus creating an optical illusion. Simple examples can be found in everyday life, such as the "wet road" effect, that one can observe when driving on a hot summer day or the illusion of floating islands and ships when looking at the sea horizon. A well known example is "Fata

Morgana"[1] which being a combination of superior and inferior mirages requires specific atmospheric conditions and so it is more difficult to be seen.

Moreover, mirages provide an excellent opportunity to experimentally address several aspects of optics, such as light propagation, the reversibility of light rays, the refractive index and the physical realization of an optical image, including how it is processed by our brain. Moreover Snell laws provide a connection with the principle of stationary action through Fermat's principle (1662)(Finch and Hand 2008).

2. Refraction of light and Snell law

The refraction of light consists in the bending of rays of light at the interface between two media. Early documented investigation on refraction phenomena dates back to the first century (A.D.) when Ptolemy of Alexandria, the Greek astronomer and geographer, tabulated with remarkable accuracy the refraction angles at air-water and air-glass boundaries. He found an empirical law working only at small angles but he was not able to propose a reliable explanation at the time.
The correct sine law was first documented by Ibn Sahl a Persian mathematician and physicist of the Islamic Golden Age. In his manuscript "On Burning Mirrors and Lenses", written around 984, Ibn Sahl uses the sine law to obtain the profiles of aspherical lenses, that focus light with no geometric aberrations(Rashed 1990).

It was only several centuries later, as it is shown in a 1621 manuscript, that the Dutch mathematician and astronomer Willebrørd Snellius (Snell) succeeded in developing a law that defined a value related to the ratio of the sine of incidence and the refraction angles (subsequently named as the relative refractive index), but what he discovered remained unpublished during his lifetime. Some years later, in 1637, René Descartes published the correct sine law of refraction(Descartes 1994) in his "Discours de la méthode", using heuristic momentum conservation arguments. Although nowadays Snell and Descartes share credit for the discovery, Thomas Harriot(Lohne 1959), another physicist who carried out many experiments on refraction and prism, had actually discovered the sine law as early as 1602(Dudley and Kwan 1997). However he did not publish his notes which remained confined to his correspondence with Kepler.

Although at the time it was possible to predict the change of direction of ray light at interface, the physical reason behind that phenomenon was still unknown. A first explanation was given in 1662 by Pierre de Fermat (Born and Wolf 1970), who rejected Descartes' solution, and expressed his "Fermat's principle" stating that the path taken by a ray between two given points is the path that can be traversed in the least time. Taking account for the different speeds of light in the two media the Snell law is easily demonstrated. Initially this idea was not well received because it seemed to attribute knowledge and intent to nature. It was only in 1678 that the Dutch scientist Christiaan Huygens, using his own concept of wavefront, derived a mathematical relationship that explained Snell's observations. He

[1] Fata Morgana is the Italian version of Morgan le Fay: a sorceress in medieval beliefs, sister of the legendary King Arthur. According to the legend these mirages, often seen in the Straits of Messina, as of cliffs and buildings, that are distorted and magnified to resemble elaborate fairy castles in the air or false land, were created by her witchcraft to lure sailors to their deaths.

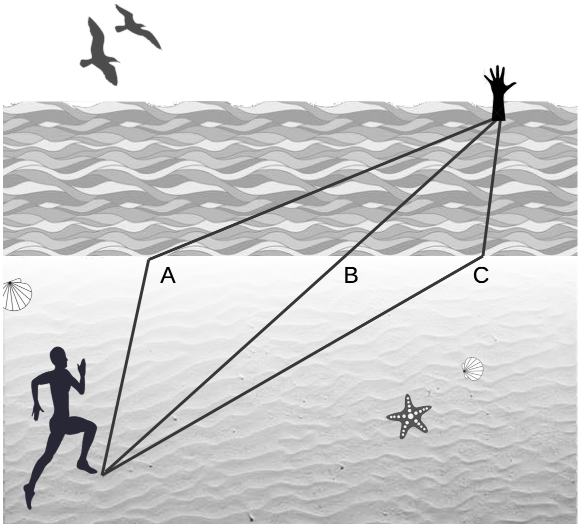

FIGURE 1. Least time principle (lifeguard's dilemma)

proposed that the refractive index of a material is related to the speed of light inside the substance.

2.1. Derivation from Fermat principle. In Optics Fermat's principle, or the principle of least time, states that "the path taken by a ray of light between two points is the path that can be traversed in the least time". Fermat made this statement in 1657, but did not submit his dissertation "Synthèse pour les réfractions" until 1662(Minnaert 1993).
In a sense Fermat's principle is the precursor of the principle of least action later developed, in the first half of 1700, by G. Leibniz, P.L. Maupertuis, L. Euler and J.L. Lagrange. Principle that has a central role in classical and modern physics.

Feynman provided a good analogy (Feynman *et al.* 2006) of the path taken by a light ray passing across media where it has different velocities, by considering the "lifeguard dilemma". As shown in fig.1, a lifeguard on a beach spots a swimmer in trouble some distance away, in a diagonal direction. He can run three times faster than he can swim. What is the quickest path to the swimmer? It is easy to realize that the fastest path is when one travels a greater distance on land in order to decrease the distance in water, since we swim much slower than we can run.

We then present the Feynman derivation of Snell law from Fermat principle, also because it implies an elementary application of the variational principle in physics, therefore it can be presented to undergraduate classes. With reference to Fig. 2b Feynman showed that the final solution to the problem is the path *ACB*, and that this path takes the shortest time of all possible ones.
If it is the shortest path, that means that if we take any other, it will be longer. So, if we were to plot the time it takes against the position of point X, we would get a curve something like

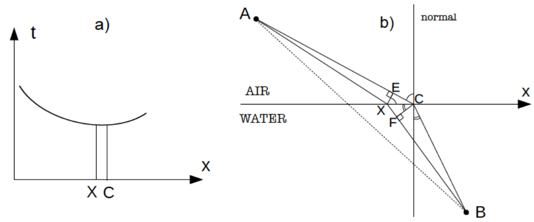

FIGURE 2. Left side: The minimum time corresponds to point C, but nearby points correspond to nearly the same time. Right side: Illustration of Fermat's principle for refraction.

similar to that shown in Fig.2a, where point C corresponds to the shortest of all possible times.

This means that if we move the point X to points near C, in the first approximation there is essentially no change in time because the slope at the bottom of the curve is zero. Therefore, our way of finding the law will be to consider that we move the place X by a very small amount, and to demand that there be is essentially no change in time. (Of course there is an infinitesimal change of a second order; we ought to have a positive increase for displacements in either direction from C.)

So we consider a nearby point X and we calculate how long it would take to go from A to B by along the two paths, and compare the new path with the old path. It is very easy to do. Of course, we want the difference to be nearly zero if the distance $\overline{XC}$ *is short. First, look at the path on land. If we draw a perpendicular* $\overline{XE}$*, we see that this path is shortened by the amount* $\overline{EC}$*.*

On the other hand, in the water, by drawing a corresponding perpendicular, $\overline{CF}$*, we find that we have to go the extra distance* $\overline{XF}$*, and that is what we lose. Or, in time, we gain the time it would have taken to travel the distance* $\overline{EC}$*, but we lose the time it would have taken to go the distance* $\overline{XF}$*. Those times must be equal, since in the first approximation there is to be no change in time. Assuming that the speed in the upper medium is* v_1 *and it changes to* v_2 *in the lower medium, then we must have* $\overline{EC}/v_1 = \overline{XF}/v_2$*. Therefore we see that when we have the right point,* $\overline{XC}\sin(\widehat{EXC})/v_1 = \overline{XC}\sin(\widehat{XCF})/v_2$ *or, cancelling the common hypotenuse length* $\overline{EXC}$ *and noting that* $\widehat{EXC} = \widehat{ECN} = \theta_i$ *and* $\widehat{XCF} \sim \widehat{BCN'} = \theta_r$ *(when X is near C), we have* $sin(\theta_i)/v_1 = \sin(\theta_r)v_2$*, or*

$$\frac{sin(\theta_i)}{\sin(\theta_r)} = \frac{v_1}{v_2} \tag{1}$$

So we see that to get from one point to another in the least time when the speeds in the two media are v_1 *and* v_2 *respectively, the light should enter at such an angle that the ratio of the sines of the angles* θ_i *and* θ_r *is the ratio of the speeds in the two media.*

It is interesting to observe that also ants do follow the principle of least time when forced to travel across two surfaces that differentially affect the ants' walking speed, as is was observed by Oettler *et al.* (2013). The trail followed by the ant row deviated from the most direct path, and on average it is not different from the path predicted by Fermat's principle.

2.2. Derivation from Huygens principle. Huygens in 1678 proposed the idea that waves propagate in the form of wave fronts, *i.e.*, the locus of all points where the wave has the same phase[2], which are perpendicular to the direction of propagation. Wavefronts propagate through space since any point where the light disturbance arrives acts as a source of spherical waves; the sum of these secondary waves determines the shape of the wavefront at a successive time.

By using his principle, Huygens was able to demonstrate that the ratio relating the angles of light rays across two materials, with differing refractive indexes, should be equal to the ratio of the light velocities in each material, thus providing a physical explanation to Snell-Descartes law.
In figure 3, the wavefront $\overline{AB}$ carries the incident rays and wavefront $\overline{A'B'}$ carries the refracted rays. In order to maintain the correct phase relation between the two wavefronts it is necessary that time intervals Δt spent by light travelling on $\overline{AB}$ and $\overline{A'B'}$ are equal, then:

$$\frac{\overline{BB'}}{v_1} = \frac{\overline{AA'}}{v_2} \tag{2}$$

where v_1 and v_2 are the light velocity in the upper an in the lower medium. Considering

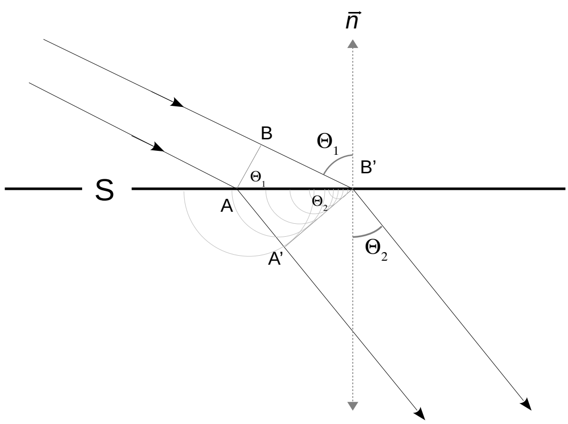

FIGURE 3. Wavefront crossing the surface separating two medium

that $\overline{AA'} = \overline{AB'}\sin(\theta_2)$ and $\overline{BB'} = \overline{AB'}\sin(\theta_1)$ one obtain

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2} \tag{3}$$

where θ_1 and θ_2 are the angles of incident end refracted beams with respect to surface normal. The ratio of v_1/v_2 has been then named "relative refractive index" of medium 2 ith respect to medium 1.

[2]This generally applies only to monochromatic fields, otherwise the phase is not well defined.

As a rule of thumb the direction of a light beam at the intersection of an interface approaches the normal if it goes more slowly after the interface; while it departs from normal if it goes faster after the interface.

The bending of wavefronts when passing across two media where it has different velocities, presents an interesting analogy in mechanics. Consider a platoon, like that shown on the left side of Fig.4, marching across terrains at different speeds, like firm ground and mud where soldiers move slowly. When they approach the separation line, the soldiers on the left side of the platoon start to slow down earlier then those on the right side, causing a change in the marching direction. The same reasoning applies to the wheels of a car, connected by an axle, as shown on the right side of Fig.4. The direction of motion changes when passing from a low friction to a higher friction road.

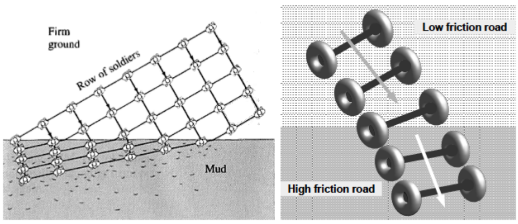

FIGURE 4. Left side: A platoon of soldiers marching across two media at different speeds. Right side: Wheels with axle crossing a line where the friction of the road changes.

2.3. Total internal reflection and gradient of refractive index. An interesting consequence of Snell Law is the total internal reflection of a ray of light when it passes from an optically denser medium (higher n, lower velocity) to an optically rarer medium (lower n, higher velocity), as it happens with light travelling from water to air. In fact, in such a case the direction of the outcoming ray will depart more and more from the normal and will eventually emerge with an angle of 90° (parallel to the interface) while the incident angle will be still less then 90°, as illustrated in right side of Fig.5 for direction B_4. The limiting angle where this condition takes place is called "critical angle" θ_c, then for angles $> \theta_c$ the ray is internally reflected.

The value of the critical angle can be easily found from eq.3 by putting $\theta_1 = 90°$ and obtaining $\theta_c = \arcsin(n_1/n_2)$.

Another interesting consequence of the Snell law is realized by the presence of an index of refraction gradually changing its value across the medium, along a given direction,thus realizing a gradient of refractive index. In such a case a ray of light undergoes a progressive deviation in its direction tracing a curved path, as illustrated on the left side of Fig.5, This effect is widely used in the manufacture of optical fibres so that for a certain cone of incidence of the light in an optical fibre, rays can be confined inside it, and can thus travel all along the fibre. The same effect is also responsible for mirages and other various atmospheric phenomena like mock sun, green flash and atmospheric duct.

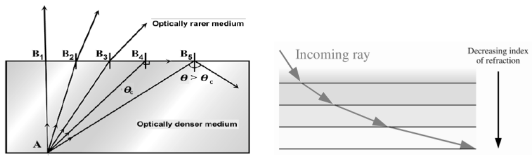

FIGURE 5. Left side: Optical paths for rays, at increasing incident angle, passing from an higher to a lower refractive index medium. The critical angle is reached when the angle of the refracted ray is 90°
Right side: Optical path for ray passing trough several staked layers of media with increasing (from bottom to top) refractive index. At a given depth the critical angle will be eventually achieved.

3. Mirage in a box

Mirages are a popular phenomena but their meaning is often misunderstood. Nevertheless they are based on simple physical laws and could be effectively used in teaching optics in undergraduate classrooms, thanks to their unexpected properties. The natural conditions for the formation of a mirage are quite peculiar and require both specific atmospheric conditions and terrain conformation. Of course this prevents a direct experimental demonstration. The presence of a gradient of refractive index in the atmosphere is related to the formation of a temperature gradient in air, which generates an air density gradient which in turn results in a varying index of refraction in the vertical direction.

On the other hand it is possible to recreate similar conditions on a small scale in laboratory so as to allow classroom demonstrations and hand-on activities, exploiting the fascinating nature of mirages.

Below we report the common classification of mirages and then illustrate a novel method aimed at realizing a stable gradient of refractive index for indoor experiments on mirages.

3.1. Classification of mirages. Mirages are generally observed over flat and relatively large areas, such as deserts, sea surface or arctic expanses but also in the common case of long straight roads, where for some reason the air above the surface has a temperature different from the surrounding atmosphere. The index of refraction for air doesn't change very much with temperature, it is around 1.000295 at 0°C and 1.000265 at 30°C. Given such a small change ($\Delta n \sim 3 \cdot 10^{-5}$), very large distances are needed for effective deflection of light rays.

In general one can observe three types of mirages depending on the direction of the gradient of refractive index, as shown in Fig.6 one can distinguish among:

Inferior mirages. Whenever the ground is much warmer than the air above it, the index of refraction is lower close to the ground. Therefore light rays will follow curved paths, as shown in the upper sketch of Fig.6, and when they reach the observer they are interpreted by the brain as a mirror-inverted image of the real object.
This is very common in desert areas or in tarmacked roads during sunny days. This kind of mirages are very often distorted and flickering due to the convective motion occurring above the hot surface.

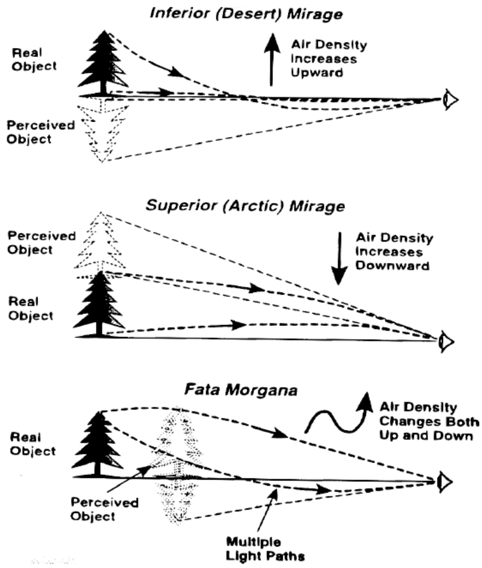

FIGURE 6. How Mirages Are Produced, from Davis (1982)

Superior mirages. Sometimes the air above the surface is cooler than in the surroundings and the index of refraction is higher close to the ground. This situation is frequently reported as the formation of an "inversion layer"[3]. In this situation the light rays will follow a different curved path, as shown in the middle sketch of Fig.6, and when they reach the observer they are interpreted by the brain as if the object were in a raised position.
This kind of mirage can be observed on offshore sea when water is much colder than air and in cold Arctic territories. Superior mirages, although blurred by atmospheric dust and distance, are much neater and stable then the former, since convective motions are not present in this case.

Fata Morgana. When both conditions are present simultaneously, giving rise to an alternating thermal gradient, a complex mix of both effects is observed, with a significant distortion of the observed objects. Alternating layers of hot and cold air create several different bands where superior and inferior mirages meet.
This mirage comprises several inverted and erect, often stretched, images, stacked on top of one another, as seen in Fig.8 taken on the Straits of Messina during the 2017 cold winter . Fata Morgana mirage moreover is subject to rapid changes, following the instability and fluctuations of the alternating thermal gradient, generating the illusory scenery that gave birth to the many legends connected to this natural phenomenon.

[3]A thermal inversion is an atmospheric condition where warmer air exists in a well-defined layer above a layer of significantly cooler air. This vertical temperature distribution is the opposite of what is normally the case; air is usually warmer close to the surface, and cooler higher up.

FIGURE 7. Left: Inferior mirage from a camel caravan in the desert. Right: Superior mirage of a ship on the horizon at Dubrovnik, Croatia (June 2006).

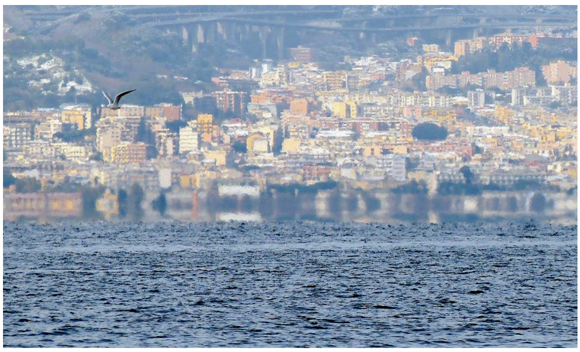

FIGURE 8. Fata Morgana taken from the Calabrian side, Messina coast appears to have moved forward, on the waters of the strait (December 2017).

In Fig.7 two photos of classical inferior and superior mirages are shown, along with a sketch of the path taken by some rays of light.

3.2. Build a stable refractive index gradient. In order to reproduce a mirage at laboratory scale it is necessary to use substances that allow to realize a high gradient of refractive index, recurring not to thermal gradients(Fabri *et al.* 1982) (like Nature does) but rather to density gradients. Usually water solutions of sucrose at different concentrations are used(Blanco-García and Vazquez-Dorrío 2014), but also sodium chloride(Greenler 1987; Vollmer 2009) or alcohol (López-Arias *et al.* 2009) can be used as solutes. Sucrose, thanks to its high solubility in water, is the most commonly used. In this case the variation of refractive index between saturate solutions and pure solvent is around 13%, enough to produce a strong deviation of ray paths over half a meter distance, see Table 1. The desired gradient of refractive index is attained by carefully pouring layers of solutions with decreasing solute concentration into an appropriate transparent container. The different density of the layers makes the whole structure of the exhibit relatively stable with respect to small mechanical perturbations. One obvious drawback of such a procedure is that the tank is not easy to

%C by wt.	n	ρ g/cm^3
0	1.3330	0.9982
15	1.3557	1.0592
30	1.3812	1.1270
45	1.4098	1.202
60	1.4419	1.2864
75	1.4779	1.3786
85	1.5040	1.4450

TABLE 1. Index of refraction and density of sucrose water solution at different concentrations.

be moved around since it is very likely that the layered solution mixes up. Furthermore, only gradients with maximum density at the bottom can be achieved. Finally, the procedure requires a long preparation time, so as to let the freshly poured layers to stabilize.

In order to overcome this problems we have added to the water solutions a gelling agent: the common gelatin sheet (collagen) used in cooking recipes and therefore harmless and easily found in grocery shops. Gelatin was dissolved in warm water according to the preparation instructions and part of it was used to prepare a stock solution at the highest sucrose concentration. Successive solutions, at lower sugar concentrations, were then prepared by dilution with the proper amount of the remaining water-gelatin warm solution.

At temperatures higher than 35°C the solution remains liquid and when poured on a surface at room temperature it solidifies into a gelly state, the process being fully reversible. In our experience, from six to eight layers stacked in) a transparent parallelepiped box were enough to get a smooth gradient since a new warm layer partially melts with the surface of the previous one. Some care must be taken to avoid an excessive inclusion of air bubbles when mixing the ingredients, since those will remain trapped during gelification, thus compromising the final transparency of the gel.

The obtained exhibit can be easily moved around and stored at fridge temperature for several days. Furthermore such a type of procedure allows for the preparation of positive or negative gradients of index of refraction and also of mixed structures aiming at reproducing the natural mechanism of mirage formation.

In Fig.9 two examples are shown illustrating the light path that gives rise to inferior and superior mirages, by using an ordinary laser pointer.

FIGURE 9. Two example of the exhibit with opposite gradient directions

Through these experimental steps, students have been introduced to laboratory procedures such as managing concentrations, making precision weightings, observing the effect of phase transitions. Moreover it is possible to use the prepared solutions to realize different flat shapes such as triangle, circle, semicircle, long bar and lens. This can be done by using proper moulds or by cutting them from a large slab of firm gelatin. It will be then possible to observe the light path through various shapes under different incident angles, by means of a laser pointer. It is also very feasible measuring the index of refraction of the various prepared solutions and verifying it against the expected value. The observation of total internal reflection as well as of double internal reflection, like that occurring in water droplets in a rainbow, is also possible, which makes the whole experience complete and entertaining.

The unedited experiment, which in 2016 was awarded the second Prize of "Student Chapter Competition - Frontiers in Optics and Laser Science " on the occasion of the 100th OSA Annual Meeting at Rochester (New York), turned out to be highly effective and stimulating in teaching concepts of Optics.

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