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COMPARISON BETWEEN SOLUTIONS OF A TWO-DIMENSIONAL TIME-FRACTIONAL DIFFUSION–REACTION EQUATION THROUGH LIE SYMMETRIES

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ABSTRACT. In this paper, exact and numerical solutions of two dimensional time-fractional diffusion-reaction equation involving the Riemann-Liouville derivative are determined, by applying a procedure that combines the Lie symmetry analysis with the numerical methods. Two new reduced fractional differential equations are obtained by using the Lie symmetry theory. Applying only one Lie transformation, we get a new time-fractional partial differential equation and, applying a further Lie transformation, we get an ordinary differential equation. Numerical solutions of the reduced differential equations are computed separately by implicit numerical methods. A comparative study between numerical solutions is performed.

1. Introduction

In the recent literature, the fractional calculus is greatly used in the description of nonlinear phenomena such as diffusion processes, solid mechanics, wave propagation problems, as well as the population dynamics and the combustion theory. The fractional differential equations are considered as the general form of differential equations, as they are involved with the derivatives of any real or complex order. They are valuable for describing reaction of anomalous diffusion problems in dispersive transport media, for example in viscoelasticity when the material damage occurs or when localised deformation develops, typical phenomena of fractured and porous media.

Several methods are used to solve fractional differential equations; from the classical Laplace transform for linear fractional equations (Miller and Ross 1993; Samko *et al.* 1993; Podlubny 1999; Kilbas *et al.* 2006) to Adomian decomposition (Daftardar-Geji and Jafari 2005; Cheng and Chu 2011), from operational methods for determining analytic solutions (Hilfer *et al.* 2009; Garra 2012; Garra and Polito 2012) to the homotopy perturbation methods (He 2000, 2006; Momani and Odibat 2007; Gómez-Aguilar *et al.* 2016a). Analytical solutions of FPDE can be found in (Gómez-Aguilar and Hernández 2014; Gómez-Aguilar *et al.* 2016b,c; Inc *et al.* 2020; Pandey *et al.* 2020). Also efficient numerical methods have been developed, including finite element schemes (Liu *et al.* 2018; Yin *et al.* 2019), finite volume (Fu *et al.* 2019; Jiang and Xu 2019; Li *et al.* 2021), finite difference methods (Huang *et al.* 2020; Jannelli 2020; Zhou *et al.* 2020; Rajeev *et al.* 2021) and spectral ones (Zhao *et al.* 2019; Dwivedi *et al.* 2020; Zaky *et al.* 2020). Most known methods lead to get only approximate solutions and not exact ones. As well known, Lie symmetries of a differential equation are a powerful tool for the determination of exact solutions of partial and ordinary differential equations (Ovsiannikov 1982; Olver 1986; Bluman and Kumei 1989; Ibragimov 1993, 1994, 1995). In particular, in the case of partial differential equations with n independent variables the Lie symmetries allow to reduce a partial differential equation to a new one involving $n-1$ independent variables and by iterating the procedure through the study of Lie symmetries admitted by new reduced equation it is possible, in some cases, to get an ordinary differential equation. The solutions of the reduced equation, by means of invertible Lie transformations, lead to obtain solutions to the target equation. For this reason, an extension of the Lie symmetry method to fractional differential equations (FPDEs) has been proposed by Buckwar and Luchko (1998), Gazizov *et al.* (2007, 2009, 2011), and Leo *et al.* (2014).

Recently, we proposed a procedure that combines the Lie symmetry analysis with the numerical methods to get exact and numerical solutions of FPDEs (Jannelli *et al.* 2018, 2019a,b, 2020). We applied this procedure to the two-dimensional time-fractional diffusion–reaction model (Jannelli and Speciale 2021), governed by the following FPDE:

$$\partial_t^\alpha u(t,x,y) - k_1 \partial_{xx} u(t,x,y) - k_2 \partial_{yy} u(t,x,y) = f(t,x,y,u) \tag{1}$$

with $0 < \alpha < 1$, where ∂_t^α is the Riemann-Liouville fractional derivative operator

$$\partial_t^\alpha u(t,x,y) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(s,x,y)}{(t-s)^\alpha} ds,$$

$u(t,x,y)$ is the field variable with t, x and y independent variables; $k_1 > 0$, $k_2 > 0$ are the diffusion coefficients and the reaction term f is assumed as $f = f_1(t,x,y)u + f_2(t,x,y)$, with f_1 and f_2 arbitrary functions of their arguments. For instance, in chemistry context the model describes the phenomena of so called anomalous sub-diffusion of material competes with the production of that material by some form of chemical reaction, when f(t,x,y,u) is the net rate, produced in a chemical reaction.

Jannelli and Speciale (2021), upon applying the extended Lie symmetry theory (Buckwar and Luchko 1998; Gazizov *et al.* 2007, 2009, 2011; Leo *et al.* 2014), obtained transformations that map Eq. (1) into a new FPDE involving two new independent variables instead of (t,x,y) and, therefore, reducing the dimension of the space. Furthermore, numerical solutions of this reduced FPDE were computed by applying an implicit unconditionally stable finite difference method (Jannelli 2020).

In this paper, we apply our procedure in order to find solutions of the model (1). In particular, we analyze the Lie symmetries of the reduced FPDE obtained by Jannelli and Speciale (2021) and get a Lie transformation that leads to a fractional ordinary equation (FODE). Exact and numerical solutions of the FODE are found, the numerical ones are obtained by using the implicit unconditionally stable trapezoidal method (Jannelli *et al.* 2019a, 2020). A comparison between the numerical solutions of the model (1), obtained by solving separately the reduced FODE and the reduced FPDE found by Jannelli and Speciale (2021), is reported pointing out the good performance of these two proposed approaches that reveal to be efficient and reliable for solutions of two-dimensional fractional-time differential equations. In particular, the numerical solutions are found by implementing two numerical methods, based on integral rules and finite difference formula, to the reduced equations by introducing the Caputo derivative.

The paper plan is as follows. In Section 2, we briefly recall the main concepts of Lie symmetry theory and its extension to FPDEs. In Section 3, we report the Lie transformation admitted by the two dimensional model (1), that reduces it into a new one dimensional FPDE. In Section 4, we compute the Lie symmetry analysis that allows to map the reduced FPDE into a FODE and exact solutions are presented. In Section 5, we show the numerical results and report the comparisons between numerical solutions obtained by applying two different numerical schemes. The errors and the convergence order of the proposed methods are presented. Concluding remarks on the obtained numerical results are presented.

2. Lie symmetry method

In this section, we briefly recall the main definitions and properties of Lie Symmetry theory and its extension to FDEs according to the theory developed by (Gazizov *et al.* 2007, 2011).

Invertible transformations of the variables t,x,y,u

$$T=T(t,x,y,u,a),\quad X=X(t,x,y,u,a),\quad Y=Y(t,x,y,u,a),\quad U=U(t,x,y,u,a),\tag{2}$$

depending on a continuous parameter a, are said to be *one-parameter Lie point symmetry transformations* of Eq. (1) if Eq. (1) has the same form in the new variables T,X,Y,U. The set G of all such transformations forms a *continuous group*, also known as the group admitted by Eq. (1).

According to the Lie theory, by expanding (2) in Taylor's series around $a=0$, we get the infinitesimal transformations

$$\begin{aligned} T &= t+a\xi_1(t,x,y,u)+o(a), \quad X=x+a\xi_2(t,x,y,u)+o(a),\\ Y &= y+a\xi_3(t,x,y,u)+o(a), \quad U=u+a\eta(t,x,y,u)+o(a) \end{aligned}$$

where their infinitesimals ξ_1, ξ_2. ξ_3 and η are given by

$$\xi_1(t,x,y,u)=\left.\frac{\partial T}{\partial a}\right|_{a=0},\quad \xi_2(t,x,y,u)=\left.\frac{\partial X}{\partial a}\right|_{a=0},$$
$$\xi_3(t,x,y,u)=\left.\frac{\partial Y}{\partial a}\right|_{a=0},\quad \eta(t,x,y,u)=\left.\frac{\partial U}{\partial a}\right|_{a=0}.$$

The corresponding operator

$$\Xi=\xi_1(t,x,y,u)\partial_t+\xi_2(t,x,y,u)\partial_x+\xi_3(t,x,y,u)\partial_y+\eta(t,x,y,u)\partial_u\tag{3}$$

is known in the literature as the infinitesimal operator or *generator* of the group G.

We get the point transformations that leave Eq. (1) invariant, applying the Lie's algorithm, that requires the k-order prolongation of the operator (3) acting on (1), identified by Δ, to be zero along the solutions, *i.e.*

$$\Xi^k\Delta=0|_{\Delta=0}.\tag{4}$$

The invariance condition (4) leads to get an overdetermined set of linear differential equations (determining equations) for the infinitesimals that (by integration) allows to find the generators of Lie point symmetries admitted by Eq. (1).

In the extension of Lie symmetry method to a given FPDE (Buckwar and Luchko 1998; Gazizov *et al.* 2009, 2011; Leo *et al.* 2014), for the presence of Riemann-Liouville fractional derivative, a new infinitesimal has been introduced and, in order to conserve the structure of the fractional derivative operator, the following invariance condition is also required

$$\xi_1(t,x,u)|_{t=0} = 0.$$

The new infinitesimal ζ_α^1 is given by prolongation formula (Gazizov *et al.* 2007)

$$\begin{aligned}\zeta_\alpha^1 = {} & D_t^\alpha(\eta) + \xi_2 D_t^\alpha(u_x) - D_t^\alpha(\xi_2 u_x) + \xi_3 D_t^\alpha(u_y) - D_t^\alpha(\xi_3 u_y) \\ & + D_t^\alpha(D_t(\xi_1)u) - D_t^{\alpha+1}(\xi_1 u) + \xi_1 D_t^{\alpha+1}(u),\end{aligned}$$

where D_t denotes the total derivative. Its presence leads to get that, when we require that the invariance condition (4) must be satisfied, the coefficients of the determining equations depend on all derivatives of variable u and $D_t^\alpha u$.

The fractional symmetries of Eq. (1) are obtained by using an algorithm implemented in the MAPLE package (FracSym (Jefferson and Carminati 2013)). This algorithm uses some routines of the MAPLE symmetry packages DESOLVII (Vu *et al.* 2012) and ASP (Jefferson and Carminati 2013); these routines automate the method of finding symmetries for FDEs as proposed by Buckwar and Luchko (1998), Gazizov *et al.* (2011), and Leo *et al.* (2014).

3. The reduction into a FPDE

In this Section, we report the Lie transformation and the reduced FPDE, in terms of two independent variables (T,Z) instead of the three (t,x,y), found by Jannelli and Speciale (2021).

The Lie transformation, obtained by the Lie rotation symmetry admitted by Eq. (1), is

$$\begin{aligned} & T = t, \qquad Z = r^2 = k_2 x^2 + k_1 y^2, \\ & U = e^{\frac{-a_5\theta}{a_4\sqrt{k_1 k_2}}} u(t,x,y) + \int \frac{e^{\frac{-a_5\theta}{a_4\sqrt{k_1 k_2}}}}{a_4 k_2 x} \chi dy \end{aligned} \tag{5}$$

being $\theta = \arctan\sqrt{\frac{k_1}{k_2}}\frac{y}{x}$ and $\chi = \chi(t,x)$ an arbitrary function.

The functions f_1 and f_2, according to the previous transformation, assume the form

$$\begin{aligned} & f_1 = \phi_1, \\ & f_2 = e^{\frac{-a_5\theta}{a_4\sqrt{k_1 k_2}}} \left(\phi_2 - \int \frac{e^{\frac{a_5\theta}{a_4\sqrt{k_1 k_2}}}}{a_4 k_2 x} (\partial_t^\alpha \chi - k_1 \partial_{xx}\chi - k_2 \partial_{yy}\chi + \phi_1 \chi) dy \right) \end{aligned} \tag{6}$$

where $\phi_1 = \phi_1(T,Z)$ and $\phi_2 = \phi_2(T,Z)$ are arbitrary functions of their arguments. By means of the transformation (5) and inserting previous forms of f_1 and f_2 in Eq. (1), we get the following FPDE involving two independent variables T and Z,

$$D_T^\alpha U - 4k_1k_2(ZD_{ZZ}U + D_ZU) + \phi_1 U - \frac{a_5^2}{a_4^2 Z}U + \phi_2 = 0. \tag{7}$$

Numerical and exact solutions of the above FPDE were obtained by Jannelli and Speciale (2021), where the efficiency of the proposed procedure, implemented by using the classical L1 implicit finite difference scheme to numerically solve the reduced FPDE, was shown.

4. The reduction into a FODE and exact solutions

In this Section, we study the Lie symmetries admitted by the reduced FPDE (7). We are able to get a Lie transformation that reduces the FPDE (7) into a FODE. So that, applying the inverse transformation, and, then, applying the transformation (5), we obtain solutions of the original Eq. (1).

We get that Eq. (7) is left invariant by the operator

$$\overline{\Xi} = \overline{\xi}_1(T,Z,U)\partial_T + \overline{\xi}_2(T,Z,U)\partial_Z + \overline{\eta}(T,Z,U)\partial_U \tag{8}$$

with

$$\overline{\xi}_1 = 2b_1T, \quad \overline{\xi}_2 = 2(b_1\alpha Z + 2b_2\sqrt{Z}), \quad \overline{\eta} = \overline{\chi} + (b_1(\alpha-1) - \frac{b_2}{\sqrt{Z}} + b_3)U$$

where $\overline{\chi} = \overline{\chi}(T,Z)$ and the following constraints must be satisfied

$$\begin{aligned}
&2b_1(T\partial_T\phi_1 + \alpha Z\partial_Z\phi_1 + \alpha\phi_1) + \frac{b_2}{Z\sqrt{Z}}(4Z^2\partial_Z\phi_1 + k_1k_2 + 4\frac{a_5^2}{a_4^2}) = 0,\\
&2b_1(\alpha Z\partial_Z\phi_2 + T\partial_T\phi_2 + (\alpha+1)\phi_2) + \frac{b_2}{\sqrt{Z}}(4Z\partial_Z\phi_2 + \phi_2) - b_3\phi_2\\
&\qquad + \partial_T^\alpha\overline{\chi} - 4k_1k_2(Z\partial_{ZZ}\overline{\chi} + \partial_Z\overline{\chi}) - \frac{a_5^2}{4a_4^2Z}\overline{\chi} = 0.
\end{aligned} \tag{9}$$

We omit the analysis of stretching symmetry because it leads to get a transformation that reduces Eq. (1) in a new FODE involving the Erdelyi-Kober fractional differential operator whose resolution, as it is well known, is not immediate. So that, in the following, we focus on the symmetries identified by the parameters b_2 and b_3. We get

$$T = T, \qquad W = e^{-\frac{b_3\sqrt{Z}}{2b_2}} Z^{\frac{1}{4}} \left(U - \int \frac{e^{-\frac{b_3\sqrt{Z}}{2b_2}}\overline{\chi}}{4b_2 Z^{\frac{1}{4}}} dZ \right). \tag{10}$$

In this case, by the constraints (9), we get

$$\begin{aligned}
&\phi_1 = \left(\frac{k_1k_2}{4} + \frac{a_5^2}{a_4^2}\right)\frac{1}{Z} + \psi_1,\\
&\phi_2 = e^{\frac{b_3\sqrt{z}}{2b_2}} Z^{-\frac{1}{4}} \left(\psi_2 - \int \frac{e^{-\frac{b_3\sqrt{Z}}{2b_2}} (\partial_t^\alpha\overline{\chi} - 4k_1k_2(Z\partial_{ZZ}\overline{\chi} + \partial_Z\overline{\chi}) + (\frac{k_1k_2}{4Z} + \psi_1)\overline{\chi})}{4b_2 Z^{\frac{1}{4}}} dZ \right),
\end{aligned} \tag{11}$$

with $\psi_1 = \psi_1(T)$ and $\psi_2 = \psi_2(T)$ arbitrary functions of their argument. We get the reduced FODE

$$D_T^\alpha W(T) + \left(\psi_1(T) - \frac{b_3^2}{4b_2^2} k_1 k_2\right) W(T) + \psi_2(T) = 0. \tag{12}$$

An exact solution. Here, an example of exact solution is obtained by assuming

$$\psi_1 = c_1, \qquad \psi_2 = c_2 e^{c_3 T}, \tag{13}$$

with c_1, c_2 and c_3 arbitrary constants. Equation (12) reads

$$D_T^\alpha W(T) + \lambda W(T) + c_2 e^{c_3 T} = 0$$

being $\lambda = c_1 - \frac{b_3^2}{4b_2^2} k_1 k_2$. As well known, under non-vanishing initial condition

$$[D_T^{\alpha-1} W(T)]_{T=0} = a_0,$$

using the Laplace transform (Podlubny 1999), the following exact solution is obtained

$$W(T) = a_0 T^{\alpha-1} E_{\alpha,\alpha}(\lambda T^\alpha) - c_2 \int_0^T e^{c_3 T} (T-S)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (T-S)^\alpha) dS \tag{14}$$

where $E_{\alpha,\alpha}(t)$ is the Mittag-Leffler function (Podlubny 1999). In particular, when $\lambda = 0$ (i.e. $c_1 = \frac{b_3^2}{4b_2^2} k_1 k_2$), the previous solution (14) reads

$$W(T) = \frac{a_0}{\Gamma(\alpha)} T^{\alpha-1} - \frac{c_2 e^{c_3 t}}{c_3^\alpha} \left(1 - \frac{\Gamma(\alpha, c_3 t)}{\Gamma(\alpha)}\right). \tag{15}$$

Now, through the inverse transformation (10) we get

$$U(T,Z) = W(T) e^{\frac{b_3 \sqrt{Z}}{2b_2}} Z^{-\frac{1}{4}} + \int \frac{e^{-\frac{b_3\sqrt{Z}}{2b_2}} \overline{\chi}}{4b_2 Z^{\frac{1}{4}}} dZ \tag{16}$$

and then through inverse transformation (5), we get the exact solution of Eq. (1)

$$\begin{aligned} u(t,x,y) = W(t) \frac{e^{\frac{-a_5\theta}{a_4\sqrt{k_1k_2}} + \frac{b_3}{2b_2}\sqrt{k_2x^2+k_1y^2}}}{\sqrt[4]{k_2x^2+k_1y^2}} \\ - e^{\frac{-a_5\theta}{a_4\sqrt{k_1k_2}}} \left(\int \frac{e^{\frac{-a_5\theta}{a_4\sqrt{k_1k_2}}}}{a_4 k_2 x} \chi dy - \int \frac{e^{-\frac{b_3\sqrt{Z}}{2b_2}}\overline{\chi}}{4b_2 Z^{\frac{1}{4}}} dZ \right). \end{aligned} \tag{17}$$

In the cylindrical coordinates r and θ, setting $\chi = \overline{\chi} = 0$, the previous solution reads

$$u(t,r,\theta) = \left(\frac{a_0}{\Gamma(\alpha)} T^{\alpha-1} - \frac{c_2 e^{c_3 t}}{c_3^\alpha} \left(1 - \frac{\Gamma(\alpha, c_3 t)}{\Gamma(\alpha)}\right)\right) \frac{e^{\frac{-a_5\theta}{a_4\sqrt{k_1k_2}} + \frac{b_3}{2b_2} r}}{\sqrt{r}} \tag{18}$$

with

$$f(t,r,\theta,u) = \left(\left(\frac{k_1k_2}{4} + \frac{a_5^2}{a_4^2} \right) \frac{1}{r^2} + c_1 \right) u + \frac{c_2 e^{c_3 t - \frac{a_5\theta}{a_4\sqrt{k_1k_2}} + \frac{b_3 r}{2b_2}}}{\sqrt{r}}. \tag{19}$$

5. Numerical results

In this Section, we perform a comparative study of the numerical solutions of the model (1) obtained by solving the reduced FPDE (7) and FODE (12), by means of only one Lie transformation (5) applied to the FPDE (1) and by means of two successive Lie ones (5) and (10), respectively. The numerical solutions are computed following two different approaches: in the first case, we solve the reduced FPDE (7) by the implicit L1 finite difference method and compute the solution of the FPDE (1) by means of the inverse transformation (5); in the second case, we solve the reduced FODE (12) by the implicit trapezoidal method and compute the solution of the FPDE (1) by means of the inverse transformations (10) and (5). The results confirm the efficiency and the reliability of the procedure that can be also used for the solutions of two dimensional fractional-time differential equations, as reported by Jannelli and Speciale (2021). We start from solving the FODE because the assumptions on the involved functions allow us to find exact solutions of the model (1). In fact, by the exact solution of FODEs obtained in Section 4, we can assign the exact boundary conditions in order to solve the reduced FPDEs, so that a numerical comparison is available.

All numerical simulations are performed on Intel Core i7 by using Matlab 2020 software. Furthermore, for sake of simplicity, in the following numerical example, for the assumption of the involved functions, we set $\chi = 0$ and $\overline{\chi} = 0$.

From FODE in (T) **to FPDE in** (T,Z) **to FPDE in** (t,x,y)**.** Starting from the FODE (12) we find the numerical solution of the FPDE (1).
We consider the FODE (12) and recall the definition of the Caputo fractional derivative of the function $W(T)$

$$^{*}D_T^{\alpha} W(T) = \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{1}{(T-S)^{\alpha}} \frac{d}{dS} W(S) dS$$

and its link with the Riemann-Liouville fractional derivative

$$^{*}D_T^{\alpha} W(T) \quad = \quad D_T^{\alpha} \left(W(T) - W(0)\right)$$

where $W(0)$ is the initial condition imposed on the solution. Assuming vanishing initial condition we obtain the following FIVP in terms of Caputo derivative

$$\begin{aligned} & ^{*}D_T^{\alpha} W(T) + \left(\psi_1(T) - \frac{b_3^2}{4b_2^2} k_1 k_2\right) W(T) + \psi_2(T) = 0 \qquad T > 0 \\ & W(0) = 0. \end{aligned} \tag{20}$$

The Caputo formulation has the advantage that the initial conditions take the same form as the one for integer-order differential equations, *i.e.*, they contain the limit values of integer order derivatives of unknown functions at the lower point. So that the initial conditions assume a physical meaning in agreement with many natural phenomena.

The functions ψ_1 and ψ_2 are chosen in such a way that the exact solution of the IVP (20) is known so that a comparison is available with the aim of showing the applicability and efficiency of the proposed procedure. Then, by assuming ψ_1 and ψ_2 according to (13), with $c_1 = \frac{b_3^2}{4b_2^2} k_1 k_2$ and c_2 and c_3 arbitrary constants, we obtain the following exact solution of the IVP (20) that is the solution (15) with $a_0 = 0$

$$W(T) = -c_2 \frac{e^{c_3 T}}{c_3^{\alpha}} \left(1 - \frac{\Gamma(\alpha, c_3 T)}{\Gamma(\alpha)}\right). \tag{21}$$

In order to solve numerically the FIVP (20), we construct a uniform computational grid choosing the time step size of the mesh ΔT. We define the mesh points T^n with $T^n = n\Delta T$, for $n = 0, \cdots, N$, with N positive integer. We denote by W^n the numerical approximation of the exact solution $W(T^n)$ at the mesh points T^n, for $n = 0, \cdots, N$. We propose the classical fractional trapezoidal method, that is a generalization to the FODEs of the classical trapezoidal method, a widely used numerical scheme for solving the linear and nonlinear ordinary differential equations. It is an unconditionally stable method with convergence order $O((\Delta T)^{min(1+\alpha,2)})$, for $\Delta T \to 0$. Generally, the convergence order of the fractional trapezoidal method is $1+\alpha$ when $0 < \alpha < 1$, and only when the solution is sufficiently smooth or when $\alpha > 1$, the expected order two is reached (see Diethelm 2004; Garrappa 2015, for details). The fractional trapezoidal method assures a high accuracy and has good stability property but it is implicit: at each step of the integration of the fractional ordinary differential equation by the numerical method, a nonlinear equation must be solved and therefore an algorithm for the solution of nonlinear equations is required. In this paper, the considered model is linear, then no root-finding solver is needed.

In Fig. 1, the comparison among the exact solutions (21) of the FIVP (20) and the numerical ones, obtained by using the trapezoidal method, is depicted for different values of α, $\alpha = 0.5, 0.7, 0.9$. The results are performed on the interval $[0,1]$ with $N = 100$ grid points by setting $k_1 = k_2 = 0.5$ and $b_2 = b_3 = 1$, $c_2 = -2$, $c_3 = 2$. By means of the transformation

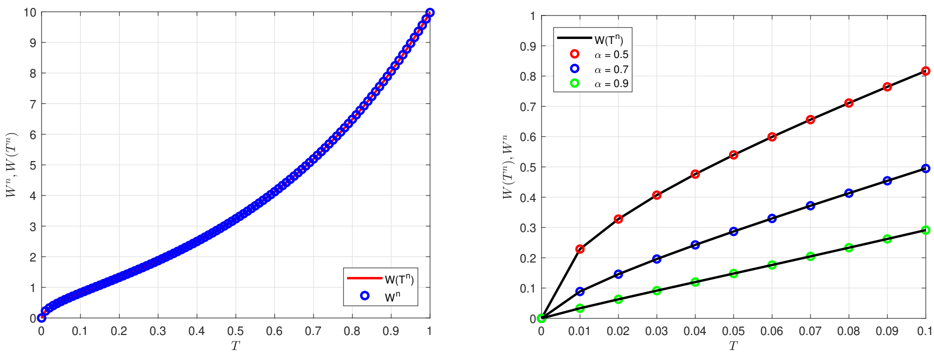

FIGURE 1. Numerical solution W^n and exact one $W(T^n)$ of the IVP (20) for different values of α. Right frame: detail of the solution.

(10), the exact solution $W(T)$ (21) is used in order to compute an exact solution $U(T,Z)$ (16), with $\overline{\chi} = 0$, of the problem (7)

$$U(T,Z) = W(T) e^{\frac{b_3 \sqrt{Z}}{2b_2}} Z^{-\frac{1}{4}}. \tag{22}$$

By means of the same transformation and using the computed numerical solution W^n, we compute the approximation of the exact one $U(T,Z)$, reported in the left frame of Fig. 2 on a computational domain $[0,1] \times [1,2]$ with $J = N = 100$ grid points. Now, by applying the transformation (5) to the exact solution (22), we are able to compute the exact solution (18), with $a_0 = 0$, of the original FPDE model in two dimensional space in cylindrical coordinates $r = \sqrt{Z}$ and $\theta = \arctan \sqrt{\frac{k_1}{k_2}} \frac{y}{x}$

$$u(t,r,\theta) = U(T,Z)\, e^{\frac{a_5 \theta}{a_4 \sqrt{k_1 k_2}}}$$

with a linear reaction term given by (19). By means of the same transformation, by using the computed numerical solution U_j^n, we obtain the approximate solution $u_{j,m}^n$ of the two dimensional model (1) with $a_4 = a_5 = 1$. In the right frame of Fig. 2, the numerical solution in reported on a computational domain $[0,1] \times [1,\sqrt{2}] \times [-\pi/2, \pi/2]$ of grid points (t^n, r_j, θ_m) with $N = J = M = 100$. In Fig. 3, we report (left frame) the exact solution (17)

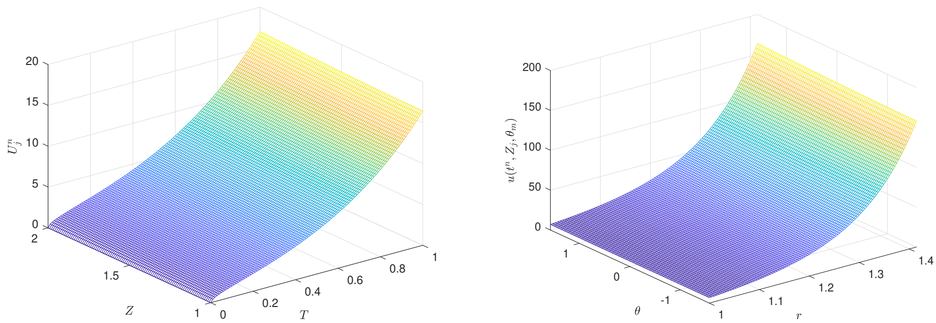

FIGURE 2. Numerical results. Left frame: solution U_j^n of the FPDE model. Right frame: solution $u_{j,m}^n$ of the two dimensional model (1) at the final time $t^N = 1$.

with $\chi = 0$ and $a_0 = 0$

$$u(t,x,y) = -\frac{c_2 e^{c_3 t}}{c_3^\alpha} \left(1 - \frac{\Gamma(\alpha, c_3 t)}{\Gamma(\alpha)}\right) \frac{e^{\frac{-a_5 \theta}{a_4 \sqrt{k_1 k_2}} + \frac{b_3}{2 b_2} \sqrt{k_2 x^2 + k_1 y^2}}}{4 \sqrt{k_2 x^2 + k_1 y^2}} \tag{23}$$

and numerical solution (right frame) of the two dimensional model (1) in (t,x,y) variables at the final time $t^N = 1$. In the solution (23), the term $\frac{1}{c_3^\alpha}\left(1 - \frac{\Gamma(\alpha, c_3 t)}{\Gamma(\alpha)}\right)$ is a damping factor and, due to the presence of the fractional parameter α, affects the solution. As the value of α increases toward to one, this term decreases, is always less than one for $t \in [0,1]$ and, then, the solution is damped when compared with solution obtained with $\alpha = 1$.

From FPDE in (T,Z) to FPDE in (t,x,y). Starting from the FPDE (7), we find the numerical solution of (1). We recall the results found by Jannelli and Speciale (2021) with the aim to perform a comparison with the numerical results found above.

We consider the FPDE (7) written in terms of the Caputo derivative (see Jannelli and Speciale 2021, for more details):

$$\begin{aligned} &{}^*D_T^\alpha U(T,Z) - 4 k_1 k_2 Z D_{ZZ} U(T,Z) - 4 k_1 k_2 D_Z U(T,Z) = F(U(T,Z)) \\ &U(0,Z) = 0 \end{aligned} \tag{24}$$

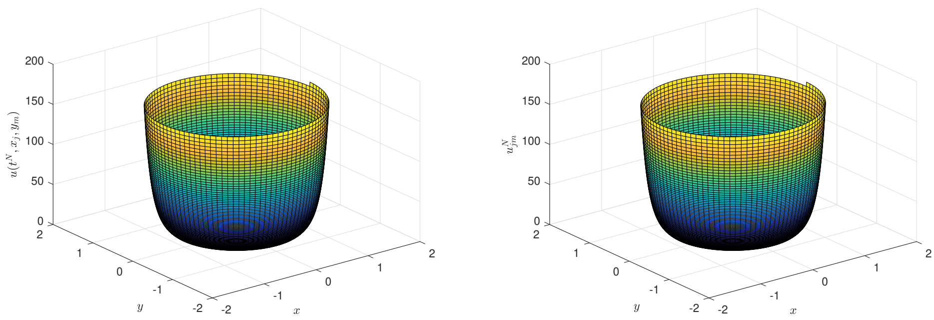

FIGURE 3. Exact (left frame) and numerical (right frame) solutions of the model (1) in (x,y) variables.

where

$$F(U(T,Z)) = -U(T,Z)\left(\frac{k_1k_2}{4Z}+c_1\right) - c_2 e^{\frac{b_3\sqrt{Z}}{2b_2}+c_3T} Z^{-\frac{1}{4}}.$$

obtained according to the (11) with (13) and subject to the boundary conditions obtained by the exact solution (22).

We denote by U_j^n the numerical approximation of the exact solution $U(Z_j,T^n)$ at the mesh points (Z_j,T^n), where $Z_j = Z_0 + j\Delta Z$ and $T^n = T^0 + n\Delta T$, for $j = 0,\cdots,J$ and $n = 0,\cdots,N$, with J and N positive integers. We propose the classical implicit L1 finite difference method

$$\begin{aligned}&(-K_1 - K_2 Z_j)\, U_{j-1}^n + (1+2\,K_2 Z_j)\, U_j^n + (K_1 - K_2 Z_j)\, U_{j+1}^n \\ &= U_j^{n-1} - \Delta T^\alpha \sum_{k=1}^{n-1} \bar{T}_{n,k}(U_j^k - U_j^{k-1}) + \bar{F}_j^n\,, \qquad 1 \le n \le N, \qquad 1 \le j \le J-1\,,\end{aligned} \tag{25}$$

where

$$K_1 = -4k_1\,k_2\Gamma(2-\alpha)\frac{\Delta T^\alpha}{2\Delta Z}\,, \quad K_2 = 4k_1\,k_2\Gamma(2-\alpha)\frac{\Delta T^\alpha}{\Delta Z^2}\,, \quad \bar{F}_j^n = \Delta T^\alpha \Gamma(2-\alpha) F_j^n\,,$$

obtained by using the following approximate formula for the Caputo fractional derivative

$$\begin{aligned}{}^*D_T^\alpha U(T^n,Z_j) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{T^n} (T^n-S)^{-\alpha}\frac{\partial U}{\partial S}(S,Z_j)dS \\ &= \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\frac{U(T^k,Z_j)-U(T^{k-1},Z_j)}{\Delta T}\left[(T^n-T^{k-1})^{1-\alpha} - (T^n-T^k)^{1-\alpha}\right] \\ &+ O(\Delta T^{2-\alpha})\,,\end{aligned} \tag{26}$$

with

$$\frac{1}{\Gamma(1-\alpha)}\int_{T^{k-1}}^{T^k}(T^n-S)^{-\alpha}dS = \frac{1}{\Gamma(2-\alpha)}[(T^n-T^{k-1})^{1-\alpha} - (T^n-T^k)^{1-\alpha}]\,.$$

Moreover, by assuming that the solution is sufficiently smooth, we approximate its first D_ZU and second order $D_{ZZ}U$ spatial derivatives by the second order three-points central finite difference formula

$$D_Z U(T^n, Z_j) = \frac{U(T^n, Z_{j+1}) - U(T^n, Z_{j-1})}{2\Delta Z} + O(\Delta Z^2)$$

$$D_{ZZ} U(T^n, Z_j) = \frac{U(T^n, Z_{j+1}) - 2U(T^n, Z_j) + U(T^n, Z_{j-1})}{\Delta Z^2} + O(\Delta Z^2) \,.$$

It is an unconditionally stable method with convergence order $O(\Delta T^{2-\alpha} + \Delta Z^2)$. For more details about the consistency, stability and convergence properties of the proposed numerical method, we refer to the papers by Jannelli (2020) and Jannelli and Speciale (2021).

In the left frame of Fig. 4, we report the numerical solution of the model (24) obtained by the L1 implicit finite difference method with $\alpha = 0.5$. The results are performed on the interval $[0,1] \times [1,2]$ with $N = J = 100$ grid points. In the right frame, the approximate solution $u^n_{j,m}$ of the two dimensional model (1), obtained by means of the transformation (5), is shown. The results are reported at the final time $t^N = 1$ on a computational domain $[0,1] \times [1, \sqrt{2}] \times [-\pi/2, \pi/2]$ defined by (t^n, r_j, θ_m) grid points, with $N = J = M = 100$. For the computations, we set the parameters values as the previous FODE. The results reported in Fig. 4 agree with ones reported in Fig. 2.

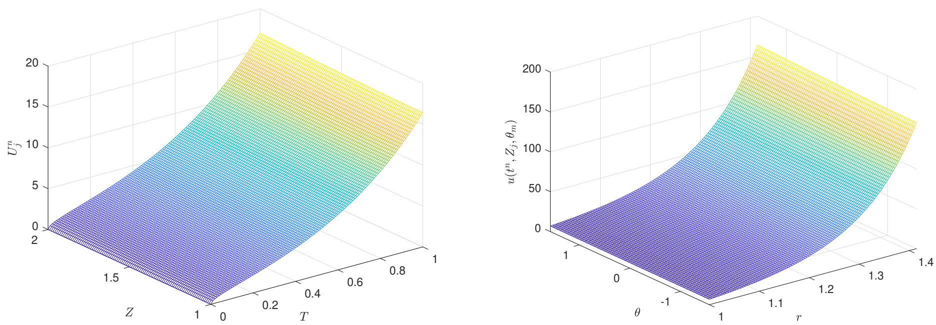

FIGURE 4. Numerical results. Left frame: solution U^n_j of the FPDE model (24). Right frame: solution $u^n_{j,m}$ of the two dimensional model (1) at the final time $t^N = 1$ for $\alpha = 0.5$.

Convergence analysis and comparison. Now, in order to validate the accuracy and efficiency of both approaches, we report a comparison of the numerical results. At this end, we investigate the errors and the convergence order of the numerical results obtained by used numerical methods, namely the trapezoidal method and L1 one.

We define the maximum error between the exact solution $W(T^n)$ and the numerical solution W^n, obtained by using the trapezoidal method, and the convergence order follows

$$E_\infty(N) \quad = \quad max_{0 \leq n \leq N} |W(T^n) - W^n|$$

and

$$\text{Order} = \log_2 \left(\frac{E_\infty(N)}{E_\infty(2N)} \right) \,.$$

TABLE 1. E_∞ and convergence order for different values of fractional order α.

α	N	E_∞	Observed Order
0.5	10	$1.513739e-02$	
	20	$3.989737e-03$	1.923751
	40	$1.019351e-03$	1.968643
	80	$2.573607e-04$	1.985787
	160	$6.464255e-05$	1.993236
	320	$1.619764e-05$	1.996701
	640	$4.053985e-06$	1.998371
0.7	10	$1.753866e-02$	
	20	$4.663763e-03$	1.910972
	40	$1.196676e-03$	1.962462
	80	$3.028049e-04$	1.982572
	160	$7.614678e-05$	1.991534
	320	$1.909219e-05$	1.995800
	640	$4.780008e-06$	1.997898
0.9	10	$1.830513e-02$	
	20	$4.926758e-03$	1.893537
	40	$1.272776e-03$	1.952660
	80	$3.233407e-04$	1.976852
	160	$8.149916e-05$	1.988198
	320	$2.046143e-05$	1.993878
	640	$5.126690e-06$	1.996807

Table 1 reports the maximum norm error and the observed convergence order for increasing values of α. The observed convergence order approaches 2. The numerical results confirm the theoretical convergence order of the trapezoidal method for a sufficiently smooth solution. Obviously, for the solution of the FPDE (7) by the transformation (10), we obtain the same order of convergence reported in Table 1. Similar results are found for the solution of the two dimensional FPDE (1) by the transformation (5). The rounding errors do not affect the order of convergence. Then, we can conclude that the numerical solution of the original model (1) by means of trapezoidal method has an accuracy of order 2.

In order to investigate the temporal error and the convergence order of the L1 numerical finite difference method, we define the maximum error between the exact solution $U(Z_j,T^N)$, obtained by (22) with $W(T^n)$ given by (21), and the numerical solution U_j^N at the final time T^N

$$E_\infty(N,J) \quad = \quad \max_{1\leq j\leq J} |U(Z_j,T^N)-U_j^N| \tag{27}$$

and the convergence order as follows

$$Order = \log_2 \left(\frac{E_\infty(N,J)}{E_\infty(2N,J)} \right) . \tag{28}$$

In this test, we fix $J = 400$, a value large enough such that the spatial error is negligible as compared with the temporal error. Table 2 shows the values of $E_\infty(N,J)$ and the corresponding numerical convergence orders for $\alpha = 0.5, 0.7$ and 0.9. It can be seen that the method is stable and convergent for solving the fractional problem (24). The numerical results agree well with the theoretical results. Obviously, for the solution of the two dimensional FPDE (1) by the transformation (5), we obtain the same order of convergence reported in Table 2. Rounding errors do not affect the order of convergence. Thus, we can conclude that the numerical solution of the original model (1) by means of L1 numerical method has an accuracy of order $2-\alpha$.

TABLE 2. $E_\infty(N,J)$ and convergence order for different values of fractional order α.

α	N	$E_\infty(N,J)$	Observed Order
0.5	10	$3.174254e-02$	
	20	$1.198749e-02$	1.404888
	40	$4.410641e-03$	1.442469
	80	$1.595545e-03$	1.466939
	160	$5.702411e-04$	1.484406
	320	$2.018238e-04$	1.498475
	640	$7.079317e-05$	1.511414
0.7	10	$6.718469e-02$	
	20	$2.861184e-02$	1.231520
	40	$1.193735e-02$	1.261130
	80	$4.922049e-03$	1.278152
	160	$2.015429e-03$	1.288172
	320	$8.217867e-04$	1.294251
	640	$3.341944e-04$	1.298077
0.9	10	$1.289254e-01$	
	20	$6.215841e-02$	1.052515
	40	$2.950356e-02$	1.075060
	80	$1.388923e-02$	1.086922
	160	$6.510333e-03$	1.093164
	320	$3.044666e-03$	1.096447
	640	$1.422192e-03$	1.098168

By the comparison of the numerical results presented in Tables 1 and 2, we can conclude that both approaches lead to compute highly accurate numerical solutions. We observe that the trapezoidal scheme is more accurate than the L1 one. As expected, we obtain an accuracy of order 2 with the trapezoidal method and an accuracy of order $2-\alpha$ with the L1 one, in line with their theoretical properties. In the first case, two successive reductions are required in order to reduce the original FPDE into a FODE that, after, it is solved by the classical trapezoidal method. In the second case, only one reduction is required to map the original FPDE into a new FPDE that is solved by the L1 numerical method, It is important to note that, by numerical point of view, the L1 method has a higher computational cost than the numerical method used for FODE. For example, for computing the numerical solutions with $\alpha = 0.5$ and $N = J = 100$, 2.316152 is the consumed CPU time by L1 scheme, versus 0.020939 consumed CPU time by trapezoidal method plus the transformation. By an analytical point of view, the L1 method has a minor effort because only one transformation is needed.

6. Concluding remarks

In this paper, we report a comparative study among numerical solutions of two dimensional FPDEs obtained by a procedure that combines the Lie symmetry analysis with the numerical methods. We obtain solutions that, as usual in the context of the fractional calculus, are affected by the presence of the fractional parameter α. We proceed by following two different approaches: in the first case, by a Lie transformation, we reduce the original FPDE into a new FPDE that is solved analytically and numerically by the L1 numerical method (Jannelli and Speciale 2021); in the second case, we apply another successive Lie transformation in order to map the reduced FPDE into a FODE that, after, it is solved analytically and numerically by the classical trapezoidal method (Jannelli *et al.* 2019a). The error estimates are provided and the orders of convergence of the schemes are demonstrated computationally in order to investigate the performance and to show the reliability and robustness of the proposed procedure for solving the two dimensional FPDEs. By the comparison of the presented results, we conclude that the computed numerical solutions are highly accurate. In particular, we observe that the numerical solution computed by the trapezoidal scheme is more accurate than the numerical one obtained by L1 scheme. Moreover, it is important to note that, the L1 method, by the numerical point of view, has an higher computational cost than the numerical method used for the FODE, but by the analytical point of view, a minor effort because only one transformation is needed. In the light of the obtained results, the future reserach is to extend this procedure for solving systems of FDEs and to perform a comparison with existing literature.

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