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Singularity Analysis and Integrability of a Simplified Multistrain Model for the Transmission of Tuberculosis and Dengue Fever

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Abstract

We apply singularity analysis to a caricature of the simplified multistrain model of Castillo-Chavez and Feng (*J Math Biol* **35** (1997) 629–656) for the transmission of tuberculosis and the coupled two-stream vector-based model of Feng and Velasco-Hernández (*J Math Biol* **35** (1997) 523–544) to identify values of the parameters for which the system of nonlinear first-order ordinary differential equations describing the model are integrable. A number of combinations of parameters for which the system is integrable are identified. We compare them with the results we obtained by a symmetry analysis in an earlier paper (*J Math Anal Appl* **333** (2007) 430-449.

1 Introduction

We make a singularity analysis of a caricature of the simplified multistrain model of Castillo-Chavez and Feng [4] for the transmission of tuberculosis and to the coupled two-stream vector-based model of Feng and Velasco-Hernandez [7]. These models have been investigated in terms of their reproductive numbers and subthreshold epidemic equilibria by van den Driessche and Watmough [16]. This study complements our earlier study [13] of this model using symmetry methods. Singularity and symmetry analyses are not common for mathematical models of epidemiological phenomena. The usual approach, apart from direct numerical integration, is through the theory of dynamical systems. Since most systems of first-order ordinary differential equations are nonintegrable, the qualitative information provided by the theory of dynamic systems is about as much as one can expect to obtain. Our primary interest is in integrable systems. In the case of a model containing a number of parameters there exists the possibility of combinations of the parameters allowing the system to be integrable. In our symmetry analysis of this model we identified numerous combinations of the parameters which lead to the system being linearisable or for which one could obtain at least a first integral if not the general solution in closed form. In this paper we examine the model using the method of singularity analysis with the same purpose in mind, more precisely the identification of combinations of values of the parameters of the model for which the solution exists as an analytic function.

The model which we wish to discuss in detail is presented by van den Driessche and Watmough [16] in their equations (14a), (14b) and (14c), *videlicet*

$$\begin{aligned} \dot{I}_1 &= -(b+\gamma_1)I_1 + \nu I_1 I_2 + \beta_1 I_1 S \\ \dot{I}_2 &= -(b+\gamma_2)I_2 - \nu I_1 I_2 + \beta_2 I_2 S \\ \dot{S} &= -b(S-1) + \gamma_1 I_1 + \gamma_2 I_2 - (\beta_1 I_1 + \beta_2 I_2)S, \end{aligned} \tag{1.1}$$

where β_1 and β_2 represent the infection rates for the two strains in the case of the tuberculosis model and for the two vectors in the Dengue fever model, ν is the common contact rate of infection, b is the common birth and death rate and γ_1 and γ_2 are the recovery rates. This model does not represent the full system discussed by Castillo-Chavez and Feng [4], but is a caricature of it, presumably to be able to combine both models. The model has only a single susceptible compartment, but it has two infectious compartments corresponding to the two infectious agents. Although it is not stated in [16], the source papers [4, 7] are clear that the variables represent proportions of a constant population which has thereby been scaled to unity, *ie* $I_1 + I_2 + S = 1$. The addition of the three components of (1.1) reflects this in that, if $I_1 + I_2 + S = 1$ is an initial point of the system, then it is a singular point and provides a stationary solution. Consequently it is not correct to consider the system (1.1) as three-dimensional since it is subject to the constraint $I_1 + I_2 + S = 1$[1] and so (1.1) exists on a surface in the three-dimensional configuration space of (1.1). To obviate the necessity to consider the Lie analysis of (1.1) subject to this constraint we revert to a manifold of lower dimension by using the constancy of the total population to write

$$S = 1 - I_1 - I_2 \tag{1.2}$$

so that the three-dimensional system is reduced to the two-dimensional system

$$\begin{aligned} \dot{I}_1 &= (\beta_1 - b - \gamma_1)I_1 - \beta_1 I_1^2 - (\beta_1 - \nu)I_1 I_2 \\ \dot{I}_2 &= (\beta_2 - b - \gamma_2)I_2 - (\beta_2 + \nu)I_1 I_2 - \beta_2 I_2^2. \end{aligned} \tag{1.3}$$

As we mentioned above, in a previous paper [13] we considered this problem from the viewpoint of symmetry analysis. This approach has been demonstrated to be suggestively successful [15, 5] for some models of sexually transmitted diseases. Here we use the methods of singularity analysis to examine system (1.1) for its integrability in terms of functions analytic away from their polelike singularities. Although there are close connections between symmetry analysis and singularity analysis [6, 3], the correspondence between the

[1] The giving of a precise value to a first integral can make a considerable difference to the properties which the subset of the system enjoys. An integral having a precise numerical value was termed a ‘configurational invariant’ by Hall [9] and subsequently the correct meaning was explained by Sarlet *et al* [14].

possession of a suitable number of Lie point symmetries to ensure integrability and the presence of solutions which are analytic apart from movable poles is amply demonstrated in the fifty classes of second-order ordinary differential equations which have the latter property [10]. The number of Lie point symmetries possessed by these equations varies from zero to eight, with the latter being the maximal number of Lie point symmetries for a second-order ordinary differential equation. Consequently a full theoretical analysis of the system is incomplete without both the singularity and symmetry analyses being performed.

The paper is structured as follows. Due to the constraint on the total population we eliminated the variable S and replaced system (1.1) by a pair of first-order differential equations. Not only do we perform the singularity analysis on this system but also we consider the 'equivalent' second-order equation obtained by the elimination of one of the dependent variables. The reason for this is that the results of a singularity analysis are not automatically preserved – as is the case for symmetries under nonpoint transformations – by anything more than a Möbius transformation. In Section 2 we perform the singularity analysis of the reduced system (1.3). In Section 3 by making use of the fact that the system (1.3) is autonomous we reduce it to a single first-order equation and examine that. In Section 4 we replace the system of two first-order equations by a single second-order equation and analyse that equation. In the concluding section, Section 5, we summarise the results of the present investigation and compare them with the results of the symmetry analysis previously performed [13].

2 Painlevé analysis of the reduced system

To determine the leading-order behaviour of the reduced system (1.3) we make the substitutions

$$I_1 = \alpha\tau^p \quad \text{and} \quad I_2 = \delta\tau^q, \tag{2.1}$$

where $\tau = t - t_0$ and t_0 is the location of the putative singularity in the complex time plane, into the dominant terms of the system (1.3), obvious by virtue of their common self-similar symmetry $-t\partial_t + I_1\partial_{I_1} + I_2\partial_{I_2}$ [8], to find the pattern of exponents

$$\begin{array}{ccc} p-1 & 2p & p+q \\ q-1 & p+q & 2q \end{array} \tag{2.2}$$

for (1.3a) and (1.3b) respectively. The terms balance for $p = q = -1$. With these values of the exponents of the leading-order terms of I_1 and I_2 the coefficients α and δ satisfy the system

$$\begin{aligned} &\beta_1\alpha^2 + (\beta_1 - \nu)\alpha\delta = \alpha \\ &(\beta_2 + \nu)\alpha\delta + \beta_2\delta^2 = \delta \end{aligned} \tag{2.3}$$

which, since α and δ are by implication nonzero, is equivalent to the linear system

$$\beta_1\alpha + (\beta_1 - \nu)\delta = 1$$

$$(\beta_2 + \nu)\alpha + \beta_2\delta = 1. \tag{2.4}$$

There are three possible solutions. If $\nu \neq 0$ and $\beta_1 \neq \nu + \beta_2$,

$$\alpha = \frac{1}{\nu}, \qquad \delta = -\frac{1}{\nu}. \tag{2.5}$$

If $\nu = 0$, it is necessary that $\beta_2 = \beta_1$, but this is not a realistic case, as we showed in [13]. If $\beta_1 = \nu + \beta_2$,

$$\delta = \frac{1}{\beta_2}(1 - \beta_1\alpha). \tag{2.6}$$

For the second case the coefficient α is arbitrary and so the required second arbitrary constant enters the expansion at the leading-order terms. Consequently for these values of the parameters the system (1.3) passes the Painlevé Test. That the system (1.3) possesses the Painlevé Property and is integrable in terms of functions analytic away from isolated singularities may be demonstrated by explicit solution of the system and we do so towards the end of this section.

For the first case, *videlicet* $\nu \neq 0$ and $\beta_1 \neq \nu + \beta_2$, in which the coefficients of the leading-order terms are given by (2.5), we must determine the resonances. We substitute

$$I_1 = \alpha\tau^{-1} + \mu\tau^{r-1}, \qquad I_2 = \delta\tau^{-1} + \sigma\tau^{r-1} \tag{2.7}$$

into the dominant terms of (1.3) and require balancing of the terms linear in μ and σ, *ie*

$$\begin{aligned}(r-1)\mu &= -2\beta_1\alpha\mu - (\beta_1 - \nu)\delta\mu - (\beta_1 - \nu)\alpha\sigma \\ (r-1)\sigma &= -(\beta_2 + \nu)\delta\mu - (\beta_2 + \nu)\alpha\sigma - 2\beta_2\delta\sigma.\end{aligned} \tag{2.8}$$

The consistency of this homogeneous linear system requires that

$$\begin{vmatrix} r + \dfrac{\beta_1}{\nu} & -\left(1 - \dfrac{\beta_1}{\nu}\right) \\ -\left(1 + \dfrac{\beta_2}{\nu}\right) & r - \dfrac{\beta_2}{\nu} \end{vmatrix} = 0, \tag{2.9}$$

ie

$$r = -1,\ 1 - \frac{\beta_1 - \beta_2}{\nu}. \tag{2.10}$$

For the system (1.3) to pass the Painlevé Test in the case that $\beta_1 \neq \beta_2 + \nu$ it is necessary for the nongeneric resonance to be an integer, *ie* the parameters be related according to

$$\beta_1 = \beta_2 - n\nu,\ n \in Z/\{-1\}. \tag{2.11}$$

In the absence of nondominant terms n may be any nonzero integer, *ie* the Laurent expansion for the solution can be in terms of either a Left Painlevé Series or a Right Painlevé Series [6], but the presence of the nondominant terms precludes the possibility of the existence of a Left Painlevé Series. Hence n is a positive integer.

To determine the condition(s) for consistency we substitute

$$I_1 = \sum_{i=0} a_i \tau^{i-1}, \quad I_2 = \sum_{i=0} b_i \tau^{i-1} \tag{2.12}$$

into the full system (1.3) to obtain

$$\begin{aligned} (i-1)a_i\tau^{i-2} &= (\beta_1 - b - \gamma_1)a_i\tau^{i-1} - \left[\beta_1 a_i a_j + (\beta_1-\nu)a_i b_j\right]\tau^{i+j-2} \\ (i-1)b_i\tau^{i-2} &= (\beta_2 - b - \gamma_2)b_i\tau^{i-1} - [(\beta_2+\nu)a_i b_j + \beta_2 b_i b_j]\tau^{i+j-2}, \end{aligned} \tag{2.13}$$

where there is summation on both i and j, and require that this be an identity for all powers of $\tau > -2$. The identity for τ^{-2} has already been established above. The only difficulty that can occur is at the resonance. We illustrate the procedure in the case that the nongeneric resonance is $r = 1$, *ie* $\beta_1 = \beta_2$. The vanishing of the coefficients at the power τ^{-1} gives the system

$$\begin{bmatrix} 1+\dfrac{\beta_1}{\nu} & -\left(1-\dfrac{\beta_1}{\nu}\right) \\ -\left(1+\dfrac{\beta_1}{\nu}\right) & 1-\dfrac{\beta_1}{\nu} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \dfrac{\beta_1 - b - \gamma_1}{\nu} \\ -\dfrac{\beta_1 - b - \gamma_2}{\nu} \end{bmatrix} \tag{2.14}$$

which is manifestly consistent if $\gamma_1 = \gamma_2$. For greater integral values of the nongeneric resonance the computation is lengthier, but the essentials are the same. The rank of the coefficient matrix drops from two to one at the resonance and this imposes a single constraint on the nonhomogeneous terms. This leads to some relationship between γ_1 and γ_2. In general this involves the other parameters.

Thus the system (1.3) passes the Painlevé Test subject to the one constraint $\beta_1 = \beta_2 + \nu$ or the two constraints $\beta_1 = \beta_2 + (n+1)\nu$ and a relation connecting γ_1 and γ_2 for nonnegative n.

That the system (1.3) possesses the Painlevé Property and is integrable in terms of functions analytic away from the movable singularity at t_0 may be demonstrated by explicit solution of the system. There are two cases. We recall that Case I corresponds to one of the coefficients of the leading-order behaviour being arbitrary.

Case I: $\beta_1 = \nu + \beta_2$

$$I_1 = \frac{\exp\left[(\beta_2+\nu-b-\gamma_1)\,t\right]}{A_1 + A_2\exp\left[(\beta_2-b-\gamma_2)\,t\right] + \dfrac{\beta_2+\nu}{\beta_2+\nu-b-\gamma_1}\exp\left[(\beta_2+\nu-b-\gamma_1)\,t\right]} \tag{2.15}$$

$$I_2 = \frac{(\beta_2-b-\gamma_2)\,A_2\exp\left[(\beta_2+\nu-b-\gamma_2)\,t\right]}{\beta_2\left\{A_1 + A_2\exp\left[(\beta_2-b-\gamma_2)\,t\right] + \dfrac{\beta_2+\nu}{\beta_2+\nu-b-\gamma_1}\exp\left[(\beta_2+\nu-b-\gamma_1)\,t\right]\right\}} \tag{2.16}$$

in the general Case and for $\gamma_1 = \nu + \gamma_2$

$$I_1 = \frac{\exp\left[(\beta_2 - b - \gamma_2)\, t\right]}{A_1 + A_2 \exp\left[(\beta_2 - b - \gamma_2)\, t\right]} \tag{2.17}$$

$$I_2 = \frac{\left\{(\beta_2 - b - \gamma_2)\, A_2 - (\beta_2 + \nu)\right\} \exp\left[(\beta_2 + \nu - b - \gamma_2)\, t\right]}{\beta_2 \left\{A_1 + A_2 \exp\left[(\beta_2 - b - \gamma_2)\, t\right]\right\}} \tag{2.18}$$

with A_1 and A_2 being constants of integration.

Case II: $\nu \neq 0$ and $\beta_1 \neq \nu + \beta_2$

We illustrate the possession of the Painlevé Property in the particular case considered above, *ie* the nongeneric resonance is $r = 1$. If the nongeneric resonance is to be $r = 1$, we require that $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$. The system (1.3) is then

$$\dot{I}_1 = (\beta_1 - b - \gamma_1)\, I_1 - \beta_1 I_1^2 - (\beta_1 - \nu) I_1 I_2 \tag{2.19}$$

$$\dot{I}_2 = (\beta_1 - b - \gamma_1) I_2 - (\beta_1 + \nu) I_1 I_2 - \beta_1 I_2^2. \tag{2.20}$$

The system (2.19,2.20) is an example of a decomposed system [2] since the two equations may be added to give the single equation

$$\dot{I} = (\beta_1 - b - \gamma_1) I - \beta_1 I^2, \tag{2.21}$$

where $I = I_1 + I_2$. As a Riccati equation, a Bernoulli equation and a variables separable equation (2.21) is eminently solvable. We obtain

$$I = \frac{\exp\left[(\beta_1 - b - \gamma_1)\, t\right]}{A_1 + \dfrac{\beta_1}{\beta_1 - b - \gamma_1} \exp\left[(\beta_1 - b - \gamma_1)\, t\right]} \tag{2.22}$$

and using this can substitute for I_1 in (2.20) to obtain a nonautonomous equation for I_2. Thus we come to the solution

$$I_1 = \frac{\exp\left[(\beta_1 - b - \gamma_1)\, t\right]}{A_1 + \dfrac{\beta_1}{\beta_1 - b - \gamma_1} \exp\left[(\beta_1 - b - \gamma_1)\, t\right]} - I_2 \tag{2.23}$$

$$I_2 = \frac{\exp\left[Qt\right]}{A_2 \left(A_1 + \dfrac{\beta_1}{Q} \exp\left[Qt\right]\right)^{1+\nu/\beta_1} - \dfrac{\beta_1}{\nu} \left(A_1 + \dfrac{\beta_1}{Q} \exp\left[Qt\right]\right)}, \tag{2.24}$$

where $Q = \beta_1 - b - \gamma_1$, and so we have given an explicit illustration of the possession of the Painlevé Property in this case.

In the analysis above we have considered the case in which $p = q = -1$ for which all dominant terms are considered. However, there does exist the possibility of special cases. We now consider these two special cases. Both cases arise as a consequence of particular values of one of the coefficients of the leading-order terms. The first case is for $p = -1$ and $q > -1$. The second is for $p > -1$ and $q = -1$. We dispose of each in turn. In the case of the former, if the coefficient of the leading-order term for I_1 is $\alpha = 1/\beta_1$, the leading-order term for I_2 can be $\beta\tau^q$, where β is arbitrary and $q = -(\beta_2 + \nu)/\beta_1$ provided $\beta_2 + \nu < \beta_1$ which is necessary to maintain the dominance of I_1. Since the parameters are necessarily positive, $-1 < q < 0$ and the system (1.3) can only be a candidate for the possession of the so-called weak Painlevé Property.

In a similar fashion for the second case we have the coefficient of the leading-order term for I_2 to be $1/\beta_2$ and the coefficient of τ^p is arbitrary and $p = -(\beta_1 - \nu)/\beta_2$. Now there are two possibilities. The exponent, p, is in the interval $(-1, 0)$ if $0 < \beta_1 - \nu < \beta_2$ and again we have the situation of the possession of the weak Painlevé Property. However, if we remove the lower bound of the inequality, there is the possibility that p takes integral values. We exclude the case of zero since that corresponds to $\beta_1 = \nu$ and the separability of the system. When not all variables in the system are singular, one must proceed a little cautiously. When we substitute the series

$$I_1 = \sum_{i=0} A_i\tau^{m+i} \quad \text{and} \quad I_2 = \sum_{i=0} B_i^{-1+i}, \quad m \geq 1, \tag{2.25}$$

into system (1.3), we obtain

$$\begin{aligned} &\sum_{i=0} \big\{(m+i)A_i\tau^{m+i-1} - (\beta_1 - b - \gamma_1)A_i\tau^{m+i} + \beta_1 A_i A_j \tau^{2m+i+j} \\ &\qquad + m\beta_2 A_i B_j \tau^{m+i+j-1}\big\} \qquad (2.26) \\ &\sum_{i=0} \big\{(i-)B_i\tau^{i-2} - (\beta_2 - b - \gamma_2)B_i\tau^{i-1} + (\beta_2 + \nu)A_i B_j \tau^{m+i+j-1} \\ &\qquad + m\beta_2 B_i B_j \tau^{+i+j-2}\big\}. \qquad (2.27) \end{aligned}$$

From the coefficients of the two leading-order terms we find that $B_0 = 1/\beta_2$ and A_0 is arbitrary. The second, nontrivial, resonance is always zero. Thereafter the subsequent coefficients in the series expansions are determined by standard recursion relations.

3 Analysis of the reduced equation

Since the system (1.3) is autonomous, *ie* possesses the Lie point symmetry ∂_t, it may be reduced to a single first-order ordinary differential equation using this symmetry. The reduction is easily achieved by the simple expedient of dividing (1.3a) by (1.3b) to obtain

$$\frac{\mathrm{d}I_1}{\mathrm{d}I_2} = \frac{(\beta_1 - b - \gamma_1)I_1 - \beta_1 I_1^2 - (\beta_1 - \nu)I_1 I_2}{(\beta_2 - b - \gamma_2)I_2 - (\beta_2 + \nu)I_1 I_2 - \beta_2 I_2^2}. \tag{3.1}$$

Some simplification is achieved by the substitution $I_1 = uI_2$. One finds that

$$\frac{\mathrm{d}u}{\mathrm{d}I_2} = \frac{(\beta_1 - \beta_2 - \gamma_1 + \gamma_2)u - (\beta_1 - \beta_2 - \nu)u(u+1)I_2}{(\beta_2 - b - \gamma_2)I_2 - (\beta_2 + \nu)uI_2^2 - \beta_2 I_2^2}. \tag{3.2}$$

Equation (3.2) is an Abel's equation of the second kind for $u(I_2)$ or equally $I_2(u)$ and in general one can never expect much joy with the solution of an Abel's equation, be it of the first or of the second kind.

We observe in (3.2) the occurrence of the factor $(\beta_1 - \beta_2 - \nu)$ in the numerator and are reminded that for $\beta_1 = \nu + \beta_2$ the system (1.3) passes the Painlevé Test. If we put this factor to zero in (3.2), we obtain a linear first-order equation in I_2^{-1}, *videlicet*

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{I_2}\right) + \frac{\beta_2 - b - \gamma_2}{(\nu - \gamma_1 + \gamma_2)\, u}\left(\frac{1}{I_2}\right) = \frac{(\beta_2 + \nu)u + \beta_2}{\nu - \gamma_1 + \gamma_2} \tag{3.3}$$

which is trivially integrable. We obtain

$$I_2 = \frac{u^{\dfrac{\beta_2 - b - \gamma_2}{\nu - \gamma_1 + \gamma_2}}}{K + \dfrac{\beta_2 + \nu}{\beta_2 + 2\nu - b - 2\gamma_1 + \gamma_2}u^A + \dfrac{\beta_2}{\beta_2 + \nu - b - \gamma_1}u^B}, \tag{3.4}$$

where K is the constant of integration,

$$A = \frac{\beta_2 + 2\nu - b - 2\gamma_1 + \gamma_2}{\nu - \gamma_1 + \gamma_2}, \qquad B = \frac{\beta_2 + \nu - b - \gamma_1}{\nu - \gamma_1 + \gamma_2}$$

and we recall that $I_1 = uI_2$.

In the case that the Painlevé Property is possessed with the nongeneric resonance $r = 1$ we have the two constraints $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$. This makes the first bracket of the numerator of (3.2) zero and the equation again reduces to a linear first-order equation, this time in I_2, *videlicet*

$$\frac{\mathrm{d}I_2}{\mathrm{d}u} + \frac{(\beta_2 + \nu)u + \beta_2}{\nu u(u+1)} I_2 = \frac{\beta_2 - \gamma_2}{\nu u(u+1)} \tag{3.5}$$

which is also trivially integrable. The solution is

$$I_2 = \frac{1}{u+1}\left[K u^{-\beta_2/\nu} + \frac{\beta_2 - \gamma_2}{\beta_2}\right] \tag{3.6}$$

in the case that $\beta_2 \neq 0$ and

$$I_2 = \frac{1}{u+1}\left[K - \frac{\gamma_2}{\nu}\log u\right] \tag{3.7}$$

in the case that $\beta_2 = 0$ which is not really physical. In both cases K is the constant of integration and I_1 follows from $I_1 = uI_2$.

In fact without the double constraint imposed at the resonance $r = 1$ but simply the requirement that $\beta_1 + \gamma_2 = \beta_2 + \gamma_1$ we also obtain a simple linear first-order equation, *videlicet*

$$\frac{\mathrm{d}I_2}{\mathrm{d}u} - \frac{(\beta_2 + \nu)u + \beta_2}{(\beta_1 - \beta_2 - \nu)\, u(u+1)} I_2 = -\frac{\beta_2 - \gamma_2}{(\beta_1 - \beta_2 - \nu)\, u(u+1)}. \tag{3.8}$$

However, in this case we are left with a nontrivial integration to obtain the particular solution and have

$$I_2 = \left[u^{\beta_2}(u+1)^{\nu}\right]^{1/(\beta_1-\beta_2-\nu)} \left[K + \int \left[u^{\nu-\beta_1}(u+1)^{\beta_2-\beta_1}\right]^{1/(\beta_1-\beta_2-\nu)} \mathrm{d}u\right]. \tag{3.9}$$

We note that, if in the numerator of (3.2) we set both brackets at zero, we have u is a constant, *ie* $I_1 = KI_2$. This is a special case of the case for which the nongeneric resonance is zero.

For this system the improvement in the ease of integrability of the differential equation when the parameters are constrained to the values required by the values $r = 0, 1$ of the nongeneric resonance is quite dramatic.

4 Analysis of an equivalent second-order ordinary differential equation

We may solve (1.3a) for I_2 in the case that $\beta_1 \neq \nu$ as

$$I_2 = -\left[\frac{\dot{I}_1}{I_1} - (\beta_1 - b - \gamma_1] + \beta_1 I_1\right] \frac{1}{\beta_1 - \nu} \tag{4.1}$$

and substitute this into (1.3b) to obtain a second-order equation for I_1, *videlicet*

$$\begin{aligned}
& I_1\ddot{I}_1 - \frac{\beta_1+\beta_2-\nu}{\beta_1-\nu}\dot{I}_1^2 + \frac{(\beta_1+\nu)(\beta_1-\beta_2-\nu)}{\beta_1-\nu} I_1^2\dot{I}_1 \\
& + \frac{\beta_1\nu(\beta_1-\beta_2-\nu)}{\beta_1-\nu} I_1^4 + \frac{1}{\beta_1-\nu}\left[2\beta_2(\beta_1-b-\gamma_1) - (\beta_1-\nu)(\beta_2-b-\gamma_2)\right] I_1\dot{I}_1 \\
& + \frac{1}{\beta_1-\nu}\left\{(\beta_1-b-\gamma_1)\left[2\beta_1\beta_2 - (\beta_1-\nu)(\beta_1+\nu)\right] - \beta_1(\beta_1-\nu)(\beta_2-b-\gamma_2)\right\} I_1^3 \\
& + \frac{\beta_1-b-\gamma_1}{\beta_1-\nu}\left[(\beta_1-\nu)(\beta_2-b-\gamma_2) - \beta_2(\beta_1-b-\gamma_1)\right] I_1^2 = 0.
\end{aligned} \tag{4.2}$$

Equation (4.2) is rather complex, but it has a structure reminiscent of a second-order equation of the Riccati hierarchy [11] and so one does have some hope. Firstly we consider the particular case for which the Painlevé analysis gave the nongeneric resonance to be zero.

In this case we have $\beta_1 = \beta_2 + \nu$ and (4.2) becomes

$$\begin{aligned}
& \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{I_1}\right) - (2\nu - b + \beta_2 - 2\gamma_1 + \gamma_2)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{I_1}\right) \\
& + (\nu - \gamma_1 + \gamma_2)(\beta_2 + \nu - b - \gamma_1)\left(\frac{1}{I_1}\right) = -(\beta_2+\nu)\,\nu
\end{aligned} \tag{4.3}$$

which is also a linear second-order equation with constant coefficients but now with a constant nonhomogeneous term. The solution of (4.3),

$$I_1 = \frac{1}{A_1 \exp\left[(\gamma_1 - \gamma_2 - \nu)\, t\right] + A_2 \exp\left[(\beta_2 + \nu - b - \gamma_1)\, t\right] - \dfrac{\nu\,(\beta_2 + \nu)}{(\nu - \gamma_1 + \gamma_2)\,(\beta_2 - \nu - b - \gamma_1)}}, \tag{4.4}$$

is also trivial and its eight Lie point symmetries translate back to eight Lie symmetries of (1.3). (See Case II(A) of [13].) The solution is also analytic.

The order of equation (4.2) can be increased to a third-order equation by means of the Riccati transformation $I_1 = \alpha\dot{\omega}/\omega$. The resulting equation has the form

$$\begin{aligned} &\frac{1}{\omega^2}\left[\dot{\omega}\,\dddot{\omega} - a\ddot{\omega}^2 + d\dot{\omega}\ddot{\omega} + f\dot{\omega}^2\right] + \frac{1}{\omega^3}\left[e\alpha - d\right]\dot{\omega}^3 \\ &+\frac{1}{\omega^3}\left[b\alpha + 2d - 3\right]\dot{\omega}^2\ddot{\omega} + \frac{1}{\omega^4}\left[c\alpha^2 - b\alpha + 2 - a\right]\dot{\omega}^4 = 0, \end{aligned} \tag{4.5}$$

where a through f are the coefficients of the terms in I and its derivatives of (4.2) as the equation is written above. This equation may be reduced to one with the symmetry ∂_ω if the coefficients of $\dot{\omega}^4$, $\dot{\omega}^2\ddot{\omega}$ and $\dot{\omega}^3$ are set to zero [1]. The first is achieved by setting α to either $1/\beta_1$ or $1/\nu$. The former imposes a nonphysical constraint on the parameters when the coefficient of $\dot{\omega}^2\ddot{\omega}$ is set equal to zero. The latter works if one requires $\beta_2 = \beta_1$. Finally the coefficient of $\dot{\omega}^3$ is made zero if $(\beta_1 - \nu)\gamma_1 = \gamma_2(\beta_1 + \nu)$, implying that $\beta_1 > \nu$. There remains the equation

$$\dot{\omega}\,\dddot{\omega} - a\ddot{\omega}^2 + d\dot{\omega}\ddot{\omega} + f\dot{\omega}^2 = 0. \tag{4.6}$$

We use the symmetry ∂_ω to reduce the order with the transformation

$$v = \dot{\omega}^{-a+1} \tag{4.7}$$

to obtain the linear second-order equation

$$\ddot{v} + d\dot{v} + f(1-a)v = 0 \tag{4.8}$$

which is readily solved and which possesses eight Lie point symmetries. Note that $a \neq 1$ since the equality imposes the unacceptable constraint $\beta_1 = 0$.
Remark: This procedure is an instance of the application of the Jacobi Last Multiplier [12] in order to increase the order of system (1.3).

5 Discussion

In [13] we obtained the following results from the symmetry analysis of (1.3) and related equations. If $\beta_1 = \nu$, the equations are separable and closed-form solutions are easily obtained. The derived second-order equations are linearisable under the following circumstances.

1. $\beta_2 = \nu$ and $\beta_1 = 2\nu$.

2. $\beta_2 = \beta_1 - \nu$.
   There are a number of subcases, *videlicet* $\gamma_1 = \gamma_2 + \nu$, $\beta_1 = b + \gamma_1$, $\beta_1 = b + \gamma_2 + \nu$, $\beta_1 = \frac{1}{2}(\gamma_2 + \nu + \gamma_1 + 2b)$, $\beta_1 = b + 2\gamma_1 - \gamma_2 - \nu$, $\beta_1 = \frac{1}{4}(4b - \gamma_1 + 5\gamma_2 + 5\nu)$, $\beta_1 = b - \gamma_1 + 2\gamma_2 + 2\nu$, $\beta_1 = b - 4\gamma_1 + 5\gamma_2 + 5\nu$, for which the solutions take various forms.

3. $\beta_2 = \beta_1 = \frac{1}{2}\nu$ and $\gamma_2 = \gamma_1$.

When $\beta_2 = \beta_1 = \nu$ and $\gamma_1 = \gamma_2$ we obtain a variant of the Ermakov-Pinney equation with the three-element algebra, $sl(2, R)$.

In all of the above cases we are able to obtain solutions in closed form. There is a number of instances for which the derived second-order equation possesses two symmetries. In the first case the equation can be integrated to obtain a solution in implicit form. For the remaining cases a first integral is obtained. In general the integral cannot be written as a quadrature. The cases are

1. $\beta_2 = \nu$ and $\beta_1 = -\gamma_2 + \nu + \gamma_1$.

2. $\beta_1 = \nu - \gamma_1 + \gamma_2$ and $\beta_2 = -2\gamma_1 + 2\gamma_2 + \nu$.

3. $\beta_1 = 2(\gamma_1 - \gamma_2 - \nu)$ and $\beta_2 = \gamma_1 - \gamma_2 - 2\nu$.

4. $\beta_2 = \frac{1}{2}(\beta_1 - \nu)$ and $\beta_1 = 2\gamma_1 - 2\gamma_2 - \nu$.

5. $\beta_2 = 2(\beta_1 - \nu)$ and $\beta_1 = \gamma_2 - \gamma_1 + 2\nu$.

6. $\beta_2 = 2(\beta_1 - \nu)$ and $\gamma_2 = 2(\gamma_1 - \nu) + b, \quad \beta_1 = b + \gamma_1$.

7. $\beta_2 = b + \gamma_2$ and $\beta_1 = b + \gamma_1$.

8. $\beta_2 = \dfrac{\nu - \beta_1}{6\beta_1\nu}\left(2\beta_1^2 + \beta_1\nu + 2\nu^2\right)$ and $\gamma_1 = \dfrac{1}{6\beta_1\nu}\left(2\beta_1^3 + 5\beta_1^2\nu - 2\nu^3 + (6\gamma_2 + \nu)\beta_1\nu\right)$.

In terms of the singularity analysis presented above we found that for the case $\beta_2 = \beta_1 - \nu$ the system (1.3) possessed the Painlevé Property with the nongeneric resonance being $r = 0$. The coincidence with the linearisable cases above is noted. For $\beta_2 \neq \beta_1 - \nu$ with $\nu \neq 0$ the nongeneric resonance is $r = 1 + (\beta_2 - \beta_1)/\nu$. The Painlevé Property is found if $\beta_2 - \beta_1 = n\nu$, where n is a nonnegative integer provided there is a further constraint on the other parameters. When we compare the results of the singularity analysis with those of the symmetry analysis given in [13] and summarised above, we see that there is coincidence for the linearisable cases and the Ermakov-Pinney form found in the symmetry analysis. We saw that the linearising transformation was of Möbius type. In the cases for which the derived second-order equation possesses two Lie point symmetries we recognise the resonance, $r = 1$, in the first, second, third and fourth instances when the conditions $\beta_2 = \beta_1$ and $\gamma_1 = \gamma_2$ are imposed. Of course the presence of the two Lie point symmetries does not imply integrability in terms of analytic functions. We saw that generally it was not possible to go beyond the existence of a first integral. When there are many symmetries, the singularity and symmetry analyses give coincident results. When there are two symmetries, there can be coincidence, but the singularity analysis imposes further constraints upon the parameters. However, the singularity analysis reveals cases not indicated by the symmetry analysis. These cases come in two forms. Firstly there are the combinations of parameters for which the nongeneric resonance is greater than one. Secondly there is the case in which only one of the variables, I_2, has a polelike singularity. This requires that the parameters β_1, β_2 and ν be related according to $\nu = \beta_1 + m\beta_2$, where m is a positive integer. One can be quite certain that the solutions so obtained, albeit analytic, cannot be expressed in closed form.

In this paper and [13] we have presented the symmetry and singularity analyses of the system (1.3) which is the mathematical expression of the simplified multistrain model for the transmission of tuberculosis and the coupled two-stream vector-based model for dengue fever and have identified relationships between the parameters in the models for which the equations can be integrated, possessed at least a first integral or a solution which is analytic. This is the first part of our programme. The second is to relate our results to the statistics available for these diseases. There is some evidence [15, 5] that the relationships found between the parameters by means of mathematical analysis can be of relevance in reality.

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