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Quantizing preserving Noether symmetries

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A procedure which obviates the constraint imposed by the conflict between consistent quantization and the invariance of the Hamiltonian description under nonlinear canonical transformation is proposed. This new quantization scheme preserves the Noether point symmetries of the underlying Lagrangian in order to construct the Schrödinger equation. Two examples are given, one known and one new: the quantization of a charged particle in a uniform magnetic field in the plane, and that of the 'goldfish' many-body problem extensively studied by Calogero et al.

Keywords: Classical quantization; Lie symmetries; Noether symmetries; Calogero's goldfish many-body problem

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1. Introduction

It has been known for over fifty years that quantization and nonlinear canonical transformations have no guarantee of consistency [32]. As recently stated by Brodlie in [4] there is a never-ending interest about "the passage of canonical transformations from classical mechanics to quantum mechanics". In [1], and reiterated in [4], it was said that *canonical transformations have three important roles in both quantum and classical mechanics:*

(i) time evolution
(ii) physical equivalence of two theories, and
(iii) solving a system.

For a more recent perspective on the canonical transformations in quantum mechanics see [3] where an up to date account of the various approaches to tackle canonical transformation is also provided.

In this paper we propose a procedure which obviates the constraint imposed by the conflict between consistent quantization and the invariance of the Hamiltonian description under nonlinear canonical transformation.

It should be noted that nonlinear canonical transformations do not commute with quantization; thus e.g. they affect the indeterminacy relations, and Planck cells are affected by such a transformation. This issue belongs to the realm of the semiclassical approach to quantum mechanics, which

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is not considered here. Examples exist where its ambiguities related to different regularizations, including a Planck cell regularization, have been solved thanks to a consistent quantum version [30].

As far as we know nobody has ever proposed[a] a quantization scheme that preserves the Noether point symmetries of the underlying Lagrangian in order to construct the Schrödinger equation.

In [13] (ex. 18, p. 433) an alternative Hamiltonian for the simple harmonic oscillator was presented. It is obtained by applying a nonlinear canonical transformation to the classical Hamiltonian of the harmonic oscillator. That alternative Hamiltonian was used in [24] to demonstrate what nonsense the usual quantization schemes[b] produce. In [20] the quantization scheme that preserves the Noether symmetries was proposed and applied to this example in order to derive the correct Schrödinger equation for the alternative Hamiltonian. We have already inferred that Lie symmetries should be preserved if a consistent quantization is desired [21].

Our method quantizes nonlinear Lagrangian equations – i.e., any system of equations that comes from a variational principle with a Lagrangian of first order –

$$\ddot{\underline{x}} = \underline{f}(\underline{x}, \dot{\underline{x}}) \tag{1.1}$$

that can be linearized through nonlinear canonical transformations.
It yields the Schrödinger equation and can be summarized as follows

(1) Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

(2) Among them find the Noether symmetries

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}$$

This may require searching for the Lagrangian yielding the maximum possible number of Noether symmetries [22], [23], [25], [26]

(3) Construct the Schrödinger equation admitting these Noether symmetries as Lie symmetries

$$2iu_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})u_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})u_{x_k} + f_3(\underline{x})u = 0$$

$$\Omega = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k} + G(t,\underline{x},u)\partial_u$$

without adding any other symmetries apart from the two symmetries that are present in any linear homogeneous partial differential equation, namely

$$u\partial_u, \qquad \alpha(t,\underline{x})\partial_u,$$

where $\alpha = \alpha(t,\underline{x})$ is any solution of the Schrödinger equation.

[a] Actually we aim to show here and in future work that many successful quantization schemes leading to a Schrödinger equation do indeed preserve the Noether point symmetries of the underlying Lagrangian.
[b] Such as normal-ordering [2,17] and Weyl quantization [33].

For example, let us consider the well-known problem of a charged particle in a uniform magnetic field in the plane. The corresponding classical Lagrangian is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \omega(y\dot{x} - x\dot{y}) \tag{1.2}$$

and consequently the Lagrangian equations are

$$\ddot{x} = -\omega\dot{y}, \quad \ddot{y} = \omega\dot{x}. \tag{1.3}$$

The Schrödinger equation was determined by Darwin in [9] to be

$$2i\psi_t + \psi_{xx} + \psi_{yy} - i\omega(y\psi_x - x\psi_y) - \frac{\omega^2}{4}(x^2 + y^2)\psi = 0. \tag{1.4}$$

The Lie symmetry algebra admitted by the linear system (1.3) has dimension fifteen [12], and the classical Lagrangian (1.2) admits eight Noether symmetries generated by the following operators[c]:

$$\begin{aligned}
X_1 &= \cos(\omega t)\partial_t - \frac{1}{2}\left(\sin(\omega t)\omega x + \cos(\omega t)\omega y\right)\partial_x + \frac{1}{2}\left(\cos(\omega t)\omega x - \sin(\omega t)\omega y\right)\partial_y,\\
X_2 &= -\sin(\omega t)\partial_t - \frac{1}{2}\left(\cos(\omega t)\omega x - \sin(\omega t)\omega y\right)\partial_x - \frac{1}{2}(\sin(\omega t)\omega x + \cos(\omega t)\omega y)\partial_y,\\
X_3 &= \partial_t,\\
X_4 &= -y\partial_x + x\partial_y,\\
X_5 &= -\sin(\omega t)\partial_x + \cos(\omega t)\partial_y,\\
X_6 &= -\cos(\omega t)\partial_x - \sin(\omega t)\partial_y,\\
X_7 &= \partial_y,\\
X_8 &= \partial_x.
\end{aligned} \tag{1.5}$$

The Schrödinger equation (1.4) admits an infinite Lie symmetry algebra[d] generated by the operator $\alpha(t,x,y)\partial_\psi$, where α is any solution of the equation itself, and also a finite dimensional Lie

[c]The addition of a total derivative $\frac{\mathrm{d}g}{\mathrm{d}t}$ to the Lagrangian (1.2) with $g = g(t,x,y)$ is a fundamental element if one wants to apply Noether theorem correctly.

[d]This is true for any linear partial differential equations.

symmetry algebra generated by the following operators:

$$
\begin{aligned}
Y_1 &= X_1 + \frac{1}{4}\left(2\sin(\omega t)\omega - i\cos(\omega t)\omega^2(x^2+y^2)\right)\partial_\psi,\\
Y_2 &= X_2 + \frac{1}{4}\left(2\cos(\omega t)\omega + i\sin(\omega t)\omega^2(x^2+y^2)\right)\partial_\psi,\\
Y_3 &= X_3,\\
Y_4 &= X_4,\\
Y_5 &= X_5 - \frac{1}{2}\omega\left(x\cos(\omega t) + y\sin(\omega t)\right)\partial_\psi,\\
Y_6 &= X_6 + \frac{1}{2}\omega\left(x\sin(\omega t) - y\cos(\omega t)\right)\partial_\psi,\\
Y_7 &= X_7 + \frac{i}{2}\omega x\partial_\psi,\\
Y_8 &= X_8 - \frac{i}{2}\omega y\partial_\psi,\\
Y_9 &= \psi\partial_\psi.
\end{aligned} \tag{1.6}
$$

This known example supports the method introduced here, namely that the Schrödinger equation admits a finite Lie symmetry algebra that corresponds to the Noether symmetries admitted by the classical Lagrangian plus the symmetry Y_9 admitted by any homogeneous linear partial differential equation.

As a second example, the quantization of the 'goldfish' many-body problem extensively studied by Calogero et al is presented in detail: in Section 2 we find the Lie and Noether symmetries of the two-body problem; in Section 3 we derive the Schrödinger equation of the two-body problem and the general formula yielding the Schrödinger equation of the 'goldfish' many-body problem. The last Section contains a discussion and final remarks.

In [5] Calogero derived a solvable many-body problem, i.e.

$$\ddot{x}_n = 2\sum_{m=1,\, m\neq n}^{N} \frac{\dot{x}_n\dot{x}_m}{x_n - x_m}, \quad (n = 1,\ldots,N) \tag{1.7}$$

by considering the following solvable nonlinear partial differential equation:

$$\varphi_t + \varphi_x + \varphi^2 = 0, \qquad \varphi \equiv \varphi(x,t)$$

and looking at the behavior of the poles of its solution. In [6] the same system (1.7) was presented, its properties were further studied and its solution was given in terms of the roots of the following algebraic equation in x:

$$\sum_{m=1}^{N} \frac{\dot{x}_m(0)}{[x - x_m(0)]} = \frac{1}{t} \tag{1.8}$$

In that paper, Calogero called system (1.7) "a goldfish" following a statement by Zakharov [34] [p. 622], namely *A mathematician, using the dressing method to find a new integrable system, could be compared with a fisherman, plunging his net into the sea. He does not know what a fish he will pull out. He hopes to catch a goldfish, of course. But too often his catch is something that could not be*

used for any known purpose to him. He invents more and more sophisticated nets and equipments, and plunges all that deeper and deeper. As a result, he pulls on the shore after a hard work more and more strange creatures. He should not despair, nevertheless. The strange creatures may be interesting enough if you are not too pragmatic, and who knows how deep in the sea do goldfishes live?
Calogero and others have extensively studied system (1.7), e.g. [7], [19], [14], [15].

2. Lie and Noether symmetries of the "goldfish" two-body problem

In the case $N = 2$ system (1.7) reduces to

$$\begin{aligned}\ddot{x}_1 &= 2\frac{\dot{x}_1\dot{x}_2}{x_1 - x_2}\\ \ddot{x}_2 &= -2\frac{\dot{x}_1\dot{x}_2}{x_1 - x_2}.\end{aligned} \tag{2.1}$$

Using the interactive REDUCE programs [18], we obtain a fifteen-dimensional Lie point symmetry algebra – that is isomorphic to $sl(4,\mathbb{R})$ [11], [12] – generated by the following fifteen operators:

$$\begin{aligned}
\Gamma_1 &= \frac{x_1x_2}{x_1 - x_2}\left(t(x_1 - x_2)\partial_t + x_1^2\partial_{x_1} - x_2^2\partial_{x_2}\right)\\
\Gamma_2 &= x_1x_2\partial_t\\
\Gamma_3 &= t(x_1 + x_2)\partial_t + x_1^2\partial_{x_1} + x_2^2\partial_{x_2}\\
\Gamma_4 &= (x_1 + x_2)\partial_t\\
\Gamma_5 &= -\frac{x_1x_2}{x_1 - x_2}(x_1\partial_{x_1} - x_2\partial_{x_2})\\
\Gamma_6 &= \frac{1}{2(x_1 - x_2)}\left(2t(x_1 - x_2)\partial_t + x_1^2\partial_{x_1} - x_2^2\partial_{x_2}\right)\\
\Gamma_7 &= \partial_t\\
\Gamma_8 &= -\frac{t}{x_1 - x_2}(x_1\partial_{x_1} - x_2\partial_{x_2})\\
\Gamma_9 &= -\frac{1}{x_1 - x_2}(x_1\partial_{x_1} - x_2\partial_{x_2})\\
\Gamma_{10} &= -\frac{t}{x_1 - x_2}(\partial_{x_1} - \partial_{x_2})\\
\Gamma_{11} &= -\frac{1}{x_1 - x_2}(\partial_{x_1} - \partial_{x_2})\\
\Gamma_{12} &= \frac{t}{x_1 - x_2}\left(t(x_1 - x_2)\partial_t + x_1^2\partial_{x_1} - x_2^2\partial_{x_2}\right)\\
\Gamma_{13} &= -\frac{1}{3}(x_1\partial_{x_1} + x_2\partial_{x_2})\\
\Gamma_{14} &= -\frac{1}{3(x_1 - x_2)}\Big((2x_1 + x_2)\partial_{x_1} - (x_1 + 2x_2)\partial_{x_2}\Big)\\
\Gamma_{15} &= -\frac{1}{3(x_1 - x_2)}\left((x_1^2 + 2x_1x_2)\partial_{x_1} - (x_2^2 + 2x_1x_2)\partial_{x_2}\right)
\end{aligned} \tag{2.2}$$

which means that system (2.1) is linearizable [10], [31]. In order to find the linearising transformation we look for a four-dimensional abelian subalgebra $L_{4,2}$ of rank 1 and have to transform it into the canonical form [31]

$$\partial_y, \quad y_1\partial_y, \quad y_2\partial_y, \quad y\partial_y,$$

with y, y_1 and y_2 the new independent and dependent variables, respectively. We find that one such subalgebra is that generated by

$$\Gamma_7 = \partial_t, \quad \Gamma_4 = (x_1+x_2)\partial_t, \quad \Gamma_2 = x_1x_2\partial_t, \quad \Gamma_6+\Gamma_{13}+\tfrac{1}{2}\Gamma_{15} = t\partial_t\,. \tag{2.3}$$

Then, it is easy to derive that the linearizing transformation is

$$y = t, \qquad y_1 = x_1+x_2, \qquad y_2 = x_1x_2 \tag{2.4}$$

and system (2.1) becomes

$$\ddot{y}_1 = 0, \qquad \ddot{y}_2 = 0 \tag{2.5}$$

which may be interpreted as the equations of motion of a free particle on a plane.

The hidden linearity of system (2.1) is already known [6].

Since the kinetic energy of a free particle on a plane is

$$T = \tfrac{1}{2}(\dot{y}_1^2+\dot{y}_2^2), \tag{2.6}$$

then transformation (2.4) yields the following Lagrangian for system (2.1):

$$L = \tfrac{1}{2}\left((\dot{x}_1+\dot{x}_2)^2+(x_2\dot{x}_1+x_1\dot{x}_2)^2\right)+\frac{\mathrm{d}g}{\mathrm{d}t}\,, \tag{2.7}$$

where $g = g(t,x_1,x_2)$ is the gauge function, a fundamental element if one wants to apply Noether theorem correctly.

This Lagrangian admits eight Noether point symmetries [12] out of the fifteen Lie point symmetries in (2.2), i.e.:

$$\begin{aligned}
\Gamma_5+3\Gamma_{14} &= -\frac{1}{x_1-x_2}\left((x_1^2x_2+2x_1+x_2)\partial_{x_1}-(x_1x_2^2+x_1+2x_2)\partial_{x_2}\right)\\
\Gamma_6 &= \frac{1}{2(x_1-x_2)}\left(2t(x_1-x_2)\partial_t+x_1^2\partial_{x_1}-x_2^2\partial_{x_2}\right)\\
\Gamma_7 &= \partial_t\\
\Gamma_8 &= -\frac{t}{x_1-x_2}\left(x_1\partial_{x_1}-x_2\partial_{x_2}\right)\\
\Gamma_9 &= -\frac{1}{x_1-x_2}\left(x_1\partial_{x_1}-x_2\partial_{x_2}\right)\\
\Gamma_{10} &= -\frac{t}{x_1-x_2}\left(\partial_{x_1}-\partial_{x_2}\right)\\
\Gamma_{11} &= -\frac{1}{x_1-x_2}\left(\partial_{x_1}-\partial_{x_2}\right)\\
\Gamma_{12} &= \frac{t}{x_1-x_2}\left(t(x_1-x_2)\partial_t+x_1^2\partial_{x_1}-x_2^2\partial_{x_2}\right)
\end{aligned} \tag{2.8}$$

To each Noether symmetry corresponds a first integral of system (2.1). For example Γ_7 yields the Lagrangian (2.7) itself as a conserved quantity.

It was proven in [12] that the n^2+4n+3-dimensional (i.e., of maximal dimension) Lie symmetry algebra of a system of n equations of second order is isomorphic to $sl(n+2,\mathbb{R})$, and the corresponding Noether symmetries generate a $(n^2+3n+6)/2$-dimensional Lie algebra g^V whose structure (Levi-Malćev decomposition and realization by means of a matrix algebra) was determined. Recently the Lie and Noether symmetries of a non autonomous linear Lagrangian system of two second-order equations, i.e.

$$\ddot{q}_1 = -\frac{k}{m}q_1 + \frac{t}{m}, \qquad \ddot{q}_2 = -\frac{k}{m}q_2. \tag{2.9}$$

were determined [27].

3. Quantization of the "goldfish"

The Hamiltonian corresponding to the Lagrangian (2.7) is:

$$H = \frac{1}{2(x_1-x_2)^2}\left((p_1x_1-p_2x_2)^2+(p_1-p_2)^2\right). \tag{3.1}$$

One may try to quantize this Hamiltonian by using the various classical methods. Neither the normal ordering method nor the Weyl quantisation procedure lead to a result which is physical. This is due to the nonlinearity of the canonical transformation (2.4) between system (2.1) and system (2.5).

Instead we assume that the Schrödinger equation corresponding to system (2.1) be of the following type:

$$2iu_t + \sum_{k,j=1}^{2} f_{kj}(x_1,x_2)u_{x_jx_k} + \sum_{k=1}^{2} h_k(x_1,x_2)u_{x_k} + h_0(x_1,x_2)u = 0 \tag{3.2}$$

with f_{kj}, h_k, h_0 functions of x_1, x_2 to be determined in such a way that equation (3.2) admits the following eight Lie symmetries:

$$\begin{aligned}
\Gamma_5+3\Gamma_{14} \Rightarrow \Omega_1 &= -\frac{1}{x_1-x_2}\left((x_1^2x_2+2x_1+x_2)\partial_{x_1}-(x_1x_2^2+x_1+2x_2)\partial_{x_2}\right)+\omega_1\partial_u\\
\Gamma_6 \Rightarrow \Omega_2 &= \frac{1}{2(x_1-x_2)}\left(2t(x_1-x_2)\partial_t+x_1^2\partial_{x_1}-x_2^2\partial_{x_2}\right)+\omega_2\partial_u\\
\Gamma_7 \Rightarrow \Omega_3 &= \partial_t+\omega_3\partial_u\\
\Gamma_8 \Rightarrow \Omega_4 &= -\frac{t}{x_1-x_2}\left(x_1\partial_{x_1}-x_2\partial_{x_2}\right)+\omega_4\partial_u\\
\Gamma_9 \Rightarrow \Omega_5 &= -\frac{1}{x_1-x_2}\left(x_1\partial_{x_1}-x_2\partial_{x_2}\right)+\omega_5\partial_u\\
\Gamma_{10} \Rightarrow \Omega_6 &= -\frac{t}{x_1-x_2}\left(\partial_{x_1}-\partial_{x_2}\right)+\omega_6\partial_u\\
\Gamma_{11} \Rightarrow \Omega_7 &= -\frac{1}{x_1-x_2}\left(\partial_{x_1}-\partial_{x_2}\right)+\omega_7\partial_u\\
\Gamma_{12} \Rightarrow \Omega_8 &= \frac{t}{x_1-x_2}\left(t(x_1-x_2)\partial_t+x_1^2\partial_{x_1}-x_2^2\partial_{x_2}\right)+\omega_8\partial_u
\end{aligned} \tag{3.3}$$

where $\omega_i = \omega_i(t, x_1, x_2, u), (i = 1, 8)$ are functions of t, x_1, x_2, u that have to be determined. Equation (3.2) also admits the following two symmetries

$$\Omega_9 = u\partial_u, \qquad \Omega_\alpha = \alpha(t, x_1, x_2)\partial_u \tag{3.4}$$

with α any solution of equation (3.2) itself, since any linear homogeneous partial differential equation possesses these two symmetries.

Using the interactive REDUCE programs [18], we obtain[e] that

$$\begin{gathered} f_{11} = \frac{x_1^2+1}{(x_1-x_2)^2}, \quad f_{12} = f_{21} = -\frac{x_1x_2+1}{(x_1-x_2)^2}, \quad f_{22} = \frac{x_2^2+1}{(x_1-x_2)^2}, \\ h_1 = \frac{\partial f_{11}}{\partial x_1}, \quad h_2 = \frac{\partial f_{22}}{\partial x_2}, \quad h_0 = -E_0^2 \end{gathered} \tag{3.5}$$

and

$$\begin{gathered} \omega_1 = 0, \quad \omega_2 = -\tfrac{1}{2} i E_0^2 t u, \quad \omega_3 = 0, \quad \omega_4 = -i(x_1+x_2)u, \quad \omega_5 = 0, \\ \omega_6 = i x_1 x_2 u, \quad \omega_7 = 0, \quad \omega_8 = (i x_1 x_2 - t)u + \frac{iu}{2}\left(x_1^2x_2^2 + x_1^2 + x_2^2 - t^2E_0^2\right), \end{gathered} \tag{3.6}$$

with E_0 an arbitrary constant. Therefore the Schrödinger equation of system (2.1) is

$$\begin{gathered} 2iu_t + \frac{x_1^2+1}{(x_1-x_2)^2}u_{x_1x_1} - 2\frac{x_1x_2+1}{(x_1-x_2)^2}u_{x_1x_2} + \frac{x_2^2+1}{(x_1-x_2)^2}u_{x_2x_2} \\ + \frac{\partial}{\partial x_1}\left(\frac{x_1^2+1}{(x_1-x_2)^2}\right)u_{x_1} + \frac{\partial}{\partial x_2}\left(\frac{x_2^2+1}{(x_1-x_2)^2}\right)u_{x_2} - E_0^2 u = 0. \end{gathered} \tag{3.7}$$

In fact if we assume $u = \psi(t, y_1, y_2)$ with $y_1 = x_1 + x_2, y_2 = x_1x_2$ as given in (2.4) then equation (3.7) becomes the well-known Schrödinger equation for the two-dimensional free particle, i.e.:

$$2i\psi_t + \psi_{y_1y_1} + \psi_{y_2y_2} - E_0^2\psi = 0. \tag{3.8}$$

It is now obvious that if the Schrödinger equation for the N-dimensional free particle is considered, i.e.

$$2i\psi_t(t,\mathbf{y}) + \triangle\psi(t,\mathbf{y}) - E_0^2\psi(t,\mathbf{y}) = 0, \quad \mathbf{y} = (y_1,\ldots,y_N), \tag{3.9}$$

then the transformation

$$u = \psi(t,\mathbf{y}), \quad \mathbf{y} = \left(\sum_{i=1}^N x_i, \sum_{i,j=1,\, i<j}^N x_i x_j, \sum_{i,j,k=1,\, i<j<k}^N x_i x_j x_k, \ldots, \prod_{i=1}^N x_i\right) \tag{3.10}$$

yields the Schrödinger equation of system (1.7).

[e]We recall that finding the Lie symmetries admitted by a linear homogeneous partial differential equation consists of solving its determining equations that are linear and overdetermined, e.g. [28].

4. Discussion and final remarks

There exist many Hamiltonians for system (1.7). For example the following one was considered in [6]:

$$H_C = \sum_{n=1}^{N} \exp(p_n) \prod_{m=1, m\neq n}^{N} (x_n - x_m)^{-1}. \tag{4.1}$$

For $N = 2$, this Hamiltonian yields the following Lagrangian

$$L_{C_2} = \dot{x}_1 \log(\dot{x}_1(x_1 - x_2)) + \dot{x}_2 \log(\dot{x}_2(x_2 - x_1)) - \dot{x}_1 - \dot{x}_2, \tag{4.2}$$

that admits three Noether point symmetries only. Therefore this Lagrangian (Hamiltonian) is not the right one to quantize with Noether symmetries. Of course, one can always devise some trick to be able to quantize system (1.7) using the Hamiltonian (4.1). Here we have proposed a straightforward method that does not require any trick, just the preservation of Noether symmetries.

Since the difference among gauge-independent[f] Lagrangians of the same system is not only the number but also which Noether symmetries they admit, we would like to conclude this paper with the following two observations.

The first observation is about the N-dimensional free-particle Schrödinger equation[g], i.e.:

$$
\begin{aligned}
&N = 1 && 2i\psi_t + \psi_{qq} = 0, && (4.3)\\
&N = 2 && 2i\psi_t + \psi_{q_1q_1} + \psi_{q_2q_2} = 0, && (4.4)\\
&N = 3 && 2i\psi_t + \psi_{q_1q_1} + \psi_{q_2q_2} + \psi_{q_3q_3} = 0, && (4.5)\\
&\ \ \vdots && \ \ \vdots && \\
&\text{any } N && 2i\psi_t + \Delta_N \psi = 0 && (4.6)
\end{aligned}
$$

The finite dimensional Lie symmetry algebra admitted by the one-dimensional free-particle Schrödinger equation (4.3) is generated by $\psi\partial_\psi$ and the five Noether symmetries admitted by the physical Lagrangian (i.e., the kinetic energy) $L = \frac{1}{2}\dot{q}^2$ of the one-dimensional free-particle classical equation

$$\ddot{q} = 0, \tag{4.7}$$

namely

$$
\begin{aligned}
X_1 = \partial_t,\ X_2 = \partial_q,\ X_3 = t\partial_q + iq\psi\partial_\psi,\ X_4 = 2t\partial_t + q\partial_q,&\\
X_5 = t^2\partial_t + tq\partial_q + \tfrac{1}{2}(iq^2 - t)\psi\partial_\psi.&
\end{aligned} \tag{4.8}
$$

The same happenstance was observed in the case of the Schrödinger equation for the linear harmonic oscillator [16]. This is also true for any N-dimensional free-particle Schrödinger equation. In particular the finite dimensional Lie symmetry algebra admitted by the two-dimensional Schrödinger equation (4.4) is generated by $\psi\partial_\psi$ and the eight Noether symmetries admitted by the physical

[f] Namely Lagrangians that do not differ by a total derivative.
[g] For the sake of simplicity we omit the linear term in ψ.

Lagrangian $L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2)$ of the two-dimensional free-particle classical equations

$$\ddot{q}_1=0, \qquad \ddot{q}_2=0, \tag{4.9}$$

namely

$$\begin{aligned} X_1=\partial_t,\, X_2=\partial_{q_1},\, X_3=\partial_{q_2},\, X_4=q_1\partial_{q_2}-q_2\partial_{q_1},\\ X_5=t\partial_{q_1}+iq_1\psi\partial_\psi,\, X_6=t\partial_{q_2}+iq_2\psi\partial_\psi,\, X_7=2t\partial_t+q_1\partial_{q_1}+q_2\partial_{q_2},\\ X_8=t^2\partial_t+tq_1\partial_{q_1}+tq_2\partial_{q_2}+\tfrac{1}{2}(iq_1^2+iq_2^2-2t)\psi\partial_\psi. \end{aligned} \tag{4.10}$$

Also the finite dimensional Lie symmetry algebra admitted by the three-dimensional free-particle Schrödinger equation (4.5) is generated by $\psi\partial_\psi$ and the twelve Noether symmetries admitted by the physical Lagrangian $L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)$ of the three-dimensional free-particle classical equations

$$\ddot{q}_1=0, \qquad \ddot{q}_2=0, \qquad \ddot{q}_3=0, \tag{4.11}$$

namely

$$\begin{aligned} X_1=\partial_t,\ X_2=\partial_{q_1},\ X_3=\partial_{q_2},\ X_4=\partial_{q_3},\\ X_5=q_1\partial_{q_2}-q_2\partial_{q_1},\ X_6=q_1\partial_{q_3}-q_3\partial_{q_1},\ X_7=q_3\partial_{q_2}-q_3\partial_{q_2},\\ X_8=t\partial_{q_1}+iq_1\psi\partial_\psi,\ X_9=t\partial_{q_2}+iq_2\psi\partial_\psi,\ X_{10}=t\partial_{q_3}+iq_3\psi\partial_\psi,\\ X_{11}=2t\partial_t+q_1\partial_{q_1}+q_2\partial_{q_2}+q_3\partial_{q_3},\\ X_{12}=t^2\partial_t+tq_1\partial_{q_1}+tq_2\partial_{q_2}+tq_3\partial_{q_3}+\tfrac{1}{2}(iq_1^2+iq_2^2+iq_3^2-3t)\psi\partial_\psi. \end{aligned} \tag{4.12}$$

Finally the finite dimensional Lie symmetry algebra of the N-dimensional free-particle Schrödinger equation has dimension $(N^2+3N+8)/2$ [8] while the Noether symmetries admitted by the physical Lagrangian $L=\frac{1}{2}\sum_{k=1}^N\dot{q}_k^2$ of the N-dimensional free-particle classical equations

$$\ddot{q}_k=0, \qquad (k=1,\ldots,N) \tag{4.13}$$

have indeed dimension $(N^2+3N+6)/2$ [12]. The Lie symmetry algebra of (4.6) is therefore generated by the following operators[h]:

$$\begin{gathered} \partial_t,\ \partial_{q_k},\ q_j\partial_{q_k}-q_k\partial_{q_j},\ t\partial_{q_k}+iq_k\psi\partial_\psi,\\ 2t\partial_t+\sum_{k=1}^N q_k\partial_{q_k},\ t^2\partial_t+\sum_{k=1}^N tq_k\partial_{q_k}+\tfrac{1}{2}\left(\sum_{k=1}^N iq_k^2-Nt\right)\psi\partial_\psi, \quad (k,j=1,\ldots,N). \end{gathered} \tag{4.14}$$

The fact that there is a correspondence between the Noether symmetries admitted by the Lagrangian of the N-dimensional free-particle classical equations and the finite Lie symmetry algebra of the N-dimensional free-particle Schrödinger equation gives support to the method presented here.

[h] We omit $\psi\partial_\psi$.

The second observation that we would like to make concerns the equations of motions of two uncoupled harmonic oscillators, i.e.:

$$\ddot{q}_1 = -\omega^2 q_1, \qquad \ddot{q}_2 = -\omega^2 q_2, \tag{4.15}$$

It is known [29] that this system possesses two Lagrangians[i] (at least): one is the usual well-known physical Lagrangian, i.e.:

$$L_1 = \tfrac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right) \tag{4.16}$$

that admits the following eight Noether symmetries:

$$\begin{aligned} Y_1 = -q_2\partial_{q_1} + q_1\partial_{q_2},\ Y_2 = \cos(2\omega t)\partial_t - \sin(2\omega t)\omega q_1\partial_{q_1} - \sin(2\omega t)\omega q_2\partial_{q_2},\\ Y_3 = -\sin(2\omega t)\partial_t - \cos(2\omega t)\omega q_1\partial_{q_1} - \cos(2\omega t)\omega q_2\partial_{q_2},\ Y_4 = \partial_t,\ Y_5 = \cos(\omega t)\partial_{q_2},\\ Y_6 = -\sin(\omega t)\partial_{q_2},\ Y_7 = \cos(\omega t)\partial_{q_1},\ Y_8 = -\sin(\omega t)\partial_{q_1}, \end{aligned} \tag{4.17}$$

and another nonphysical Lagrangian that can be found in [29] p.122:

$$L_2 = \dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2 \tag{4.18}$$

that admits the following eight Noether symmetries:

$$\tilde{Y}_1 = -q_1\partial_{q_1} + q_2\partial_{q_2}, \quad Y_2, \quad Y_3, \quad Y_4, \quad Y_5, \quad Y_6, \quad Y_7, \quad Y_8. \tag{4.19}$$

Both Lagrangians admit the maximal number of Noether symmetries, albeit they differ just by one, namely Y_1 instead of $\tilde{Y}_1$. Indeed the rotational symmetry Y_1 is an essential physical property of two uncoupled harmonic oscillators since it yields the conservation of angular momentum. From this example we can infer the obvious conjecture that the physical Lagrangian admits the maximum number of Noether symmetries that also lead to the essential physical conservation laws[j].
Indeed if we apply the method described in this paper, i.e using the Noether symmetries (4.17) admitted by the physical Lagrangian L_1, then the known Schrödinger equation for the two-dimensional oscillator is obtained[k], i.e.:

$$2i\psi_t + \psi_{q_1q_1} + \psi_{q_2q_2} - \omega^2(q_1^2 + q_2^2)\psi = 0. \tag{4.20}$$

Instead no linear partial differential equation in the dependent variable $\psi(t, q_1, q_2)$ of the three independent variables t, q_1, q_2, i.e.:

$$f_{11}\psi_{tt} + f_{12}\psi_{tq_1} + f_{13}\psi_{tq_2} + f_{22}\psi_{q_1q_1} + f_{23}\psi_{q_1q_2} + f_{33}\psi_{q_2q_2} + f_1\psi_t + f_2\psi_{q_1} + f_3\psi_{q_2} + f_0\psi = 0 \tag{4.21}$$

where $f_{ij}(i, j = 1,2,3), f_k(k = 0,1,2,3)$ are arbitrary functions of t, q_1, q_2, admits as Lie finite symmetries the eight Noether symmetries (4.19) of the nonphysical Lagrangian L_2.
Thus the physical Lagrangian is indeed the one that directly yields to quantization without any further trick.

[i]Both Lagrangians could be obtained by means of the Jacobi Last Multiplier [23].
[j]We remark that these conservation laws are obviously too many – and therefore some are functional depending on the others – for the integration of the classical system but they are just right in number and physical quality for the corresponding Schrödinger equation.
[k]Obviously, this equation admits also the symmetry $\psi\partial_\psi$.

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