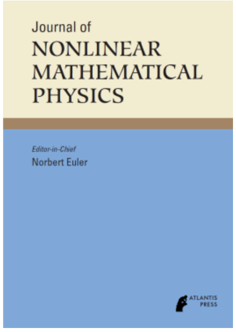

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LAGRANGIANS FOR DISSIPATIVE NONLINEAR OSCILLATORS: THE METHOD OF JACOBI LAST MULTIPLIER

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We present a method devised by Jacobi to derive Lagrangians of any second-order differential equation: it consists in finding a Jacobi Last Multiplier. We illustrate the easiness and the power of Jacobi's method by applying it to several equations, including a class of equations recently studied by Musielak with his own method [Z. E. Musielak, Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients *J. Phys. A: Math. Theor.* **41** (2008) 055205], and in particular a Liènard type nonlinear oscillator and a second-order Riccati equation. Also, we derive more than one Lagrangian for each equation.

Keywords: Ordinary differential equations; Lie symmetry algebra; Lagrangian.

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1. Introduction

In a 1957-paper suitably entitled *The range of application of the Lagrange formalism*, Havas wrote [12] "It has been found in many branches of physics that the solution of a variety of problems can be greatly simplified if the basic equations can be expressed in the form of a variational principle" (omissis). "In addition to the well-known usefulness of Lagrange's method for the integration of the equations of motions there are two further advantages: a knowledge of the Lagrangian and of its invariance properties enables one to obtain all the conservation laws of the system [23], and it forms the basis for the quantization of classical (discrete or continuous) systems. It is therefore of great importance to know which systems of forces or fields can be treated by Lagrange's method".

In the context of quantization we may refer to the seminal paper by Dirac [6, 11], and Feynman's recently published thesis [11] and his 1965 Nobel Lecture [10].

It should be well-known that the knowledge of a Jacobi Last Multiplier always yields a Lagrangian of any second-order ordinary differential equation [17, 39]. Yet many distinguished scientists seem to be unaware of this classical result. Havas himself cited the book by Whittaker [39] but only in connection with the formulation of Lagrangian equations [12].

In this paper, we present the method of the Jacobi Last Multiplier in order to compare the easiness and the power of Jacobi's method with that proposed by Museliak *et al.* [21,22] for the same purpose. The papers [25–38] and the references within may give an idea of the many fields of applications yielded by Jacobi Last Multiplier.

In [22] the authors searched for a Lagrangian of the following second-order ordinary differential equation

$$\ddot{x} + b(x)\dot{x}^2 + c(x)x = 0 \tag{1.1}$$

with $b(x), c(x)$ arbitrary functions of the dependent variable $x = x(t)$. After some lengthy calculations they found one Lagrangian. In [38] we showed that (1.1) is a subcase of a more general class of equations studied by Jacobi [16], i.e.

$$\ddot{x} + \frac{1}{2}\frac{\partial\varphi}{\partial x}\dot{x}^2 + \frac{\partial\varphi}{\partial t}\dot{x} + B = 0 \tag{1.2}$$

with φ and B arbitrary functions of t and x. We applied Jacobi's method to Eq. (1.1) and easily derived many (an infinite number of) different Lagrangians.

In the present paper, we show how to obtain many (an infinite number of) different Lagrangians for the class of equations

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.3}$$

with $f(x)$ and $g(x)$ arbitrary functions of the dependent variable $x(t)$. In [21] Musielak applied his lengthy method to (1.3) in order to obtain at least one Lagrangian. Here we show that many standard and nonstandard Lagrangians of (1.3) can be derived without much effort by using Jacobi's method.

This paper is organized in the following way. In Sec. 2, we illustrate the Jacobi Last Multiplier and its properties [13–17], its connection to Lie symmetries [18,19], and its link to the Lagrangian of any second-order differential equation [17,39]. We also exemplify Jacobi's method with an equation, described in 1974 [20], of the class of Eq. (1.1), i.e.:

$$\ddot{x} = x\frac{-a + \lambda\dot{x}^2}{\lambda x^2 + 1} \tag{1.4}$$

which is still of great physical interest as it can be seen in a 2007-paper by Cariñena *et al.* [4], and a nonautonomous equation [9] of the more general class of Eq. (1.2), i.e.:

$$\ddot{x} = -\frac{\dot{x}^2}{x} + \frac{\dot{x}}{t}. \tag{1.5}$$

Both examples were not included in [38]. In Sec. 3, we apply Jacobi's method to the class of Eq. (1.3), show some particular examples such as a Liènard type nonlinear oscillator, which has been recently studied for its asymptotic behavior in [3], and a second-order Riccati equation, which has been studied in [5] and also derived in the discussion on differential sequences [8]. In Sec. 4, we conclude with some final remarks.

In this paper we employ ad hoc interactive programs [24] written in the language REDUCE to calculate the Lie symmetry algebra of the equations which we study.

2. The Method by Jacobi

The method of the Jacobi Last Multiplier [13–17] provides a means to determine all the solutions of the partial differential equation

$$\mathcal{A}f = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n)\frac{\partial f}{\partial x_i} = 0 \tag{2.1}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \cdots = \frac{\mathrm{d}x_n}{a_n}. \tag{2.2}$$

In fact, if one knows the Jacobi Last Multiplier and all but one of the solutions, then the last solution can be obtained by a quadrature. The Jacobi Last Multiplier M is given by

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = M\mathcal{A}f, \tag{2.3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial \omega_1}{\partial x_1} & & \frac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \frac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{2.4}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (2.1) or, equivalently, first integrals of (2.2) independent of each other. This means that M is a function of the variables $(x_1, \ldots, x_n)$ and depends on the chosen $n-1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi Last Multiplier are:

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \ldots, \eta_{n-1}$ of Eq. (2.1), then the corresponding last multiplier N is linked to M by the relationship:

$$N = M\frac{\partial(\eta_1, \ldots, \eta_{n-1})}{\partial(\omega_1, \ldots, \omega_{n-1})}.$$

(b) Given a nonsingular transformation of variables

$$\tau : (x_1, x_2, \ldots, x_n) \to (x'_1, x'_2, \ldots, x'_n),$$

the last multiplier M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x'_1, x'_2, \ldots, x'_n)},$$

where M obviously comes from the $n-1$ solutions of $\mathcal{A}F = 0$ which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1}.

(c) One can prove that each multiplier M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0; \tag{2.5}$$

vice versa every solution M of this equation is a Jacobi Last Multiplier.

(d) If one knows two Jacobi Last Multipliers, M_1 and M_2, of Eq. (2.1), then their ratio is a solution ω of (2.1) or, equivalently, a first integral of (2.2). Naturally the ratio may be quite trivial, namely a constant. Vice versa the product of a multiplier M_1 times any solution ω yields another last multiplier $M_2 = M_1\omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternate formulation in terms of symmetries was provided by Lie [19]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [2]. If we know $n-1$ symmetries of (2.1)/(2.2), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \dots, x_n)\partial_{x_j}, \quad i = 1, \dots, n-1, \tag{2.6}$$

Jacobi's last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}. \tag{2.7}$$

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation of motion is missing the variable associated with that component, i.e. $\partial a_i/\partial x_i = 0$, the last multiplier is a constant and any other Jacobi Last Multiplier is a first integral.

Another property of the Jacobi Last Multiplier is its (almost forgotten) relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation

$$\ddot{x} = F(t, x, \dot{x}) \tag{2.8}$$

namely [17, 39]

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}, \tag{2.9}$$

where $M = M(t, x, \dot{x})$ satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \frac{\partial F}{\partial \dot{x}} = 0. \tag{2.10}$$

Equation (2.8) becomes the Euler–Lagrangian equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)+\frac{\partial L}{\partial x}=0. \tag{2.11}$$

The proof is given by taking the derivative of (2.11) by $\dot{x}$ and showing that this yields (2.10). If one knows a Jacobi Last Multiplier, then L can be easily obtained by a double integration, i.e.

$$L=\int\left(\int M\,\mathrm{d}\dot{x}\right)\mathrm{d}\dot{x}+f_1(t,x)\dot{x}+f_2(t,x), \tag{2.12}$$

where f_1 and f_2 are functions of t and x which have to satisfy a single partial differential equation related to (2.8) [33]. As it was shown in [33], f_1 and f_2 are related to the gauge function $G=G(t,x)$. In fact we may assume

$$\begin{aligned} f_1 &= \frac{\partial G}{\partial x} \\ f_2 &= \frac{\partial G}{\partial t}+f_3(t,x), \end{aligned} \tag{2.13}$$

where f_3 has to satisfy the partial differential equation mentioned and G is obviously arbitrary.

In [16] Jacobi himself found his "new multiplier" for the class of second-order ordinary differential equations studied by Euler [7] [Sect. I, Ch. VI, §§915 ff.] (1.2). Indeed Jacobi derived that the multiplier of Eq. (1.2) is given by:

$$M=e^{\varphi(t,x)}, \tag{2.14}$$

as it is obvious from (2.10). Consequently in our previous paper [38] we derived a Lagrangian of the class of Eq. (1.2) by means of (2.9), i.e.

$$L=\frac{1}{2}e^{\varphi(t,x)}\dot{x}^2+f_3(t,x)+\frac{\mathrm{d}}{\mathrm{d}t}G(t,x) \tag{2.15}$$

with f_3 a function of t and x satisfying the following equation:

$$\frac{\partial f_3}{\partial x}+e^{\varphi(t,x)}B(t,x)=0. \tag{2.16}$$

Equation (1.4) is a particular example of the equation considered by Jacobi. In fact from (2.14) and (2.15) we derive:

$$M=\frac{1}{\lambda x^2+1}, \tag{2.17}$$

and consequently

$$L=\frac{\dot{x}^2}{2(\lambda x^2+1)}-\frac{ax^2}{2(\lambda x^2+1)}+\frac{\mathrm{d}}{\mathrm{d}t}G(t,x). \tag{2.18}$$

This Lagrangian is known [20]. Equation (1.4) does not possess any Lie point symmetry apart translation in t. Therefore Noether's theorem [23] applied to the autonomous

Lagrangian L in (2.18) yields the following known first integral [20]:

$$I = \frac{ax^2 + \dot{x}^2}{2(\lambda x^2 + 1)}. \tag{2.19}$$

Jacobi proved that in the case of a second-order differential equation, if one knows a first integral and a last multiplier, then the equation can be integrated by quadrature (a new Principle of Mechanics, indeed) [13, 14].

Equation (1.5) is obtained by the symmetry reduction transformation $x = u', t = u$ of the third-order equation:

$$u''' = -\frac{u'u''}{u}, \tag{2.20}$$

where $u(T)$ is a function of T. Equation (1.5) admits an eight-dimensional Lie point symmetry algebra and therefore is linearizable. In [33] it was shown that if one knows several (at least two) Lie symmetries of the second-order differential Eq. (2.8), i.e.

$$\Gamma_j = V_j(t,x)\partial_t + G_j(t,x)\partial_x, \quad j = 1, r, \tag{2.21}$$

then many Jacobi Last Multipliers could be derived by means of (2.7), i.e.

$$\frac{1}{M_{nm}} = \Delta_{nm} = \det \begin{bmatrix} 1 & \dot{x} & F(t,x,\dot{x}) \\ V_n & G_n & \dfrac{\mathrm{d}G_n}{\mathrm{d}t} - \dot{x}\dfrac{\mathrm{d}V_n}{\mathrm{d}t} \\ V_m & G_m & \dfrac{\mathrm{d}G_m}{\mathrm{d}t} - \dot{x}\dfrac{\mathrm{d}V_m}{\mathrm{d}t} \end{bmatrix}, \tag{2.22}$$

with $(n, m = 1, r)$, and therefore many Lagrangians can be obtained by means of (2.12). In particular fourteen different Lagrangians can be obtained if the equation admits an eight-dimensional Lie point symmetry algebra. We do not look for the fourteen Lagrangians of Eq. (1.5). Instead we use Eq. (2.14) to find a Jacobi Last Multiplier and consequently a Lagrangian. In fact from (2.14) and (2.15) we derive:

$$JLM = \frac{x^2}{t}, \tag{2.23}$$

and consequently

$$Lag = \frac{\dot{x}^2 x^2}{2t} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x). \tag{2.24}$$

If one applies Noether's theorem to Lag then the following five first integrals of Eq. (1.5) can be derived:

$$FI_1 = x^2(x - \dot{x}t)^2$$

$$FI_2 = -\frac{x^2\dot{x}(x - \dot{x}t)}{2t}$$

$$FI_3 = \frac{x^2\dot{x}^2}{2t^2}$$

$$FI_4 = x(x - \dot{x}t)$$
$$FI_5 = \frac{x\dot{x}}{t}. \tag{2.25}$$

We emphasize that the first integrals FI_1 and FI_4 could not be derived if the gauge function $G(t, x)$ was assumed to be equal to zero.

3. Equations with Space-dependent Coefficients

The class of Eq. (1.3) has an obvious Jacobi Last Multiplier and therefore Lagrangian if the following relationship holds between $f(x)$ and $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{f(x)}\right) = \alpha(1-\alpha)f(x), \tag{3.1}$$

where α is any constant $\neq 1$. In fact, if (3.1) holds, then Eq. (1.3) can be written as

$$\dot{u} + \alpha f(x)u = 0, \tag{3.2}$$

i.e.

$$f(x) = -\frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}(\log u) \quad \text{with } u = \dot{x} + \frac{g(x)}{\alpha f(x)} \tag{3.3}$$

and thus a Jacobi Last Multiplier for Eq. (1.3) is

$$M = \exp\left(\int f(x)\mathrm{d}t\right) = \exp\left(-\frac{1}{\alpha}\int \mathrm{d}(\log u)\right) = u^{-1/\alpha} \tag{3.4}$$

and the corresponding Lagrangian is

$$L = u^{2-1/\alpha} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x) = \left(\dot{x} + \frac{g(x)}{\alpha f(x)}\right)^{2-1/\alpha} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x), \tag{3.5}$$

with $G(t, x)$ an arbitrary gauge function.

We note that this Lagrangian is autonomous and therefore it admits at least the Noether point symmetry of translation in t and consequently the following first integral

$$In = \left(\dot{x} + \frac{g(x)}{\alpha f(x)}\right)^{1-1/\alpha} \frac{\alpha f(x)\dot{x} - f(x)\dot{x} - g(x)}{\alpha^2 f(x)^2}. \tag{3.6}$$

We would like to remark that because of property (d) of the Jacobi Last Multiplier we can obtain another Jacobi Last Multiplier $\overline{M} = In \times M$ and consequently another Lagrangian of Eq. (1.3). We do not pursue it here any further, but one can envision a deluge of Lagrangians obtained by simply taking any function of the first integral In in (3.6) and multiplying it by either M in (3.4) or $\overline{M}$ and so on ad libitum.

3.1. *Examples*

It is very easy to obtain a Jacobi Last Multiplier and therefore a Lagrangian for the following Liènard type nonlinear oscillator:

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda x = 0. \tag{3.7}$$

In fact we know that $M = \exp(\int kx\mathrm{d}t)$. If one can put the equation in the form

$$\dot{u}_1 + \alpha k x u_1 = 0, \tag{3.8}$$

with α a constant to be determined, i.e.

$$kx = -\frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}(\log u_1), \tag{3.9}$$

then

$$M = u_1^{-1/\alpha}. \tag{3.10}$$

In the case of Eq. (3.7) we have

$$\dot{u}_1 + \frac{1}{3}kxu_1 = 0 \quad \text{with } u_1 = \dot{x} + \frac{k}{3}x^2 + \frac{3}{k}\lambda, \quad \text{i.e.} \quad \alpha = \frac{1}{3}. \tag{3.11}$$

Therefore $M_1 = u_1^{-3}$ and consequently

$$L_1 = \frac{1}{u_1} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x) = \frac{1}{\dot{x} + \frac{k}{3}x^2 + \frac{3}{k}\lambda} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x). \tag{3.12}$$

Actually we can derive another Lagrangian because substituting $f(x) = kx,\ g(x) = \frac{k^2}{9}x^3 + \lambda x$ into Eq. (3.1) yields two different α, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g}{f}\right) - \alpha(1-\alpha)f = 0 \Longrightarrow 9\alpha^2 - 9\alpha + 2 = 0 \Longrightarrow \alpha_{1,2} = \frac{1}{3}, \frac{2}{3}. \tag{3.13}$$

The case $\alpha = \frac{1}{3}$ has been considered above. If we substitute $\alpha = \frac{2}{3}$ into Eq. (3.2), then we obtain $M_2 = u^{-3/2}$ and consequently

$$L_2 = \sqrt{u} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x) = \sqrt{\dot{x} + \frac{k}{6}x^2 + \frac{3}{2k}\lambda} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x). \tag{3.14}$$

Equation (3.7) admits an eight-dimensional Lie symmetry algebra and therefore is linearizable. Moreover one can determine at least twelve more Lagrangians [36]. We note that L_1 admits one Noether point symmetry while L_2 admits three Noether point symmetries.

A particular case of (3.7) is the second-order Riccati equation:

$$\ddot{x} + 3x\dot{x} + x^3 = 0, \tag{3.15}$$

a member of the Riccati-chain [1]. Equation (3.15) is linearizable to a third-order linear equation by the transformation $x = \dot{V}(t)/V(t)$, namely (3.15) transforms into $\dddot{V} = 0$. It also well-known that Eq. (3.15) is linearizable by means of a point transformation because it admits an eight-dimensional Lie symmetry algebra generated by the following operators:

$$\Gamma_1 = t^3(tx-2)\partial_t - t(xt-2)(x^2t^2+2-2xt)\partial_x$$

$$\Gamma_2 = xt^3\partial_t - (xt-1)(x^2t^2+4-2xt)\partial_x$$

$$\Gamma_3 = xt^2\partial_t - x(x^2t^2+2-2xt)\partial_x$$

$$\Gamma_4 = xt\partial_t - x^2(xt-1)\partial_x$$

$$\begin{aligned}
\Gamma_5 &= x\partial_t - x^3\partial_x \\
\Gamma_6 &= \partial_t \\
\Gamma_7 &= t\partial_t - x\partial_x \\
\Gamma_8 &= t^2\partial_t - 2(xt-1)\partial_x.
\end{aligned} \tag{3.16}$$

In order to find the linearizing transformation we have to look for a two-dimensional abelian intransitive subalgebra [19] and, following Lie's classification of two-dimensional algebras in the real plane [19], we have to transform it into the canonical form

$$\partial_{\tilde{x}}, \qquad \tilde{t}\partial_{\tilde{x}}$$

with $\tilde{t}$ and $\tilde{x}$ the new independent and dependent variables, respectively. We found that one such subalgebra is that generated by Γ_1 and $\Gamma_9 \equiv \Gamma_2 - \Gamma_8$. Then it is easy to derive that

$$\tilde{t} = \frac{tx-1}{t(tx-2)}, \quad \tilde{x} = -\frac{x}{2t(tx-2)}$$

and Eq. (3.15) becomes

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{t}^2} = 0. \tag{3.17}$$

We can derive fourteen different Lagrangians by using (2.22) and (2.12). Two of these Lagrangians admit five Noether symmetries, i.e.:

$$L_{56} = -\frac{1}{2(\dot{x}+x^2)} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x) \tag{3.18}$$

and

$$L_{19} = -\frac{1}{2t^4(x^2t^2+\dot{x}t^2-2xt+2)} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x) \tag{3.19}$$

which are derived from

$$JLM_{56} = -\frac{1}{(\dot{x}+x^2)^3}, \tag{3.20}$$

and

$$JLM_{19} = -\frac{1}{(t^2x^2+t^2\dot{x}-2tx+2)^3}, \tag{3.21}$$

respectively. We remark that JLM_{56} can be also obtained from (3.2). In fact Eq. (3.15) can be written as

$$\dot{u} + 3x\alpha u = 0, \quad u = \dot{x} + x^2, \quad \alpha = \frac{1}{3}. \tag{3.22}$$

If one applies Noether's theorem to L_{56} then the following five first integrals of Eq. (3.15) can be derived:

$$I_1 = \frac{(x^2t^2 - 2xt + \dot{x}t^2 + 2)^2}{4(x^2+\dot{x})^2}$$

$$
\begin{aligned}
I_2 &= \frac{(x^2t^2 - 2xt + \dot{x}t^2 + 2)(x^2t - x + \dot{x}t)}{2(x^2 + \dot{x})^2} \\
I_3 &= \frac{x^2 + 2\dot{x}}{2(x^2 + \dot{x})^2} \\
I_4 &= \frac{(x^2t - x + \dot{x}t)^2}{(x^2 + \dot{x})^2} \\
I_5 &= \frac{x^2t - x + \dot{x}t}{x^2 + \dot{x}}
\end{aligned}
\tag{3.23}
$$

while Noether's theorem applied to L_{19} yields:

$$
\begin{aligned}
In_1 &= \frac{x^2t - x + \dot{x}t}{x^2t^2 - 2xt + \dot{x}t^2 + 2} \\
In_2 &= \frac{(x^3t - 2x^2 - 2\dot{x})xt + (\dot{x}t^2 + 4)\dot{x} + (2\dot{x}t^2 + 3)x^2}{(x^2t^2 - 2xt + \dot{x}t^2 + 2)^2} \\
In_3 &= \frac{(x^2t - x + \dot{x}t)(x^2 + \dot{x})}{(x^2t^2 - 2xt + \dot{x}t^2 + 2)^2} \\
In_4 &= \frac{(x^2 + \dot{x})^2}{(x^2t^2 - 2xt + \dot{x}t^2 + 2)^2} \\
In_5 &= \frac{x^2 + 2\dot{x}}{2(x^2t^2 - 2xt + \dot{x}t^2 + 2)^2}.
\end{aligned}
\tag{3.24}
$$

We remark the importance of the gauge function $G(t, x)$. None of the first integrals above, apart I_3 and In_5, could be derived if the gauge function was assumed to be equal to zero.

The Lagrangian which admits the maximum number of Noether point symmetries seems to be that obtained by means of the Jacobi Last Multiplier that is derived from the determinant (2.22) with the two solution symmetries, namely the two-dimensional abelian intransitive subalgebra which yields the linearizing transformation. We could infer that this is the physical Lagrangian. Yet the Lagrangian obtained by using the Jacobi Last Multiplier (3.4) also possesses the maximum number of Noether point symmetries as shown in the case of Eq. (3.1). This may explain why Lagrangian (3.5) possesses nice physical properties as shown in [5].

4. Final Remarks

The many Lagrangians that Jacobi's method may produce could confuse a physicist seeing such a lot of mathematical formulas. Yet the correct application of Noether's theorem can actually discriminate among those Lagrangians. In fact the Lagrangians which possess the maximum number of Noether symmetries are those that possess the most useful physical properties, as it can be seen in the provided examples.

In the present paper, we do not claim to have been exhaustive in our presentation of the application of the Jacobi Last Multiplier for finding Lagrangians of any second-order differential equation. Indeed we would like to encourage other authors to apply Jacobi's method to their preferred equation. The key is to be able to solve Eq. (2.5). In this paper, we show that it is not an impossible task. Lie symmetries may also help. Noether symmetries should not be forgotten.

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