On minimal-α-spaces

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Abstract

An α-space is a topological space in which the topology is generated by the family of all α-sets (see [N]). In this paper the minimal-$\alpha\mathcal{P}$-space (where $\mathcal{P}$ denotes several separation axioms) are investigated. Some new characterization for α-spaces are also obtained.

1 Introduction

The family of all topologies on a set X is a complete atomic lattice. There has been a considerable amount of interest in topologies which are minimal or maximal in this lattice with respect to certain topological properties.

Given a topological property $\mathcal{P}$, we say that a topology on a set X is $\mathcal{P}$*-minimal* if every weaker topology on X does not possess property $\mathcal{P}$

Throughout this paper, the word "space" will mean topological space, the topology on a space X is denoted by $\tau(X)$, int_τ and cl_τ (or int_X and cl_X when no confusion is possible about the topology on X) will denote respectively the interior and the closure operators respect to $\tau(X)$ and if σ is a topology on the underlying set of X, then σ is called an *expansion* (respectively a *compression*) of $\tau(X)$ if $\tau(X) \subseteq \sigma$ (resp. $\sigma \subseteq \tau(X)$).

A subset R of a space X is said *regular open* if $int_\tau(cl_\tau(R)) = R$. The family of all regular open sets of X is denoted by $RO(X)$ and forms a base

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for a topology $\tau_s(X)$ on X which is a compression of $\tau(X)$ and it is called the *semiregularization* of X. We say that X is semiregular if $\tau_s(X) = \tau(X)$.

The notion of α-set was introduced in 1965 by Njåstad [N]. Given a space X, we say that $A \subseteq X$ is a *α-set* if $A \subseteq int_\tau(cl_\tau(int_\tau(A)))$. It is easy to prove that the family $\alpha\tau(X)$ of all α-set of X is a topology on X which will be called the *α-topology* induced by $\tau(X)$ and it consists of all the subsets A of X such that there exists some open set $U \in \tau(X)$ such that $U \subseteq A \subseteq int_\tau(cl_\tau(U))\}$. The members of $\alpha\tau(X)$ will be called the *α-open* set of X while their complementary will be called the *α-closed* set of X. Evidently, every open set is an α-set and hence the α-topology is an expansion of $\tau(X)$. We say that X is an *α-space* if $\tau(X) = \tau_\alpha(X)$.

Obviously, the partial ordered set of all α-topologies contain both a maximum (the discrete topology) and a minimum (the trivial topology) element.

If $\mathcal{P}$ is a topological property, we denote by $\alpha\mathcal{P}$ the class of α-spaces which satisfies property $\mathcal{P}$. In this paper, we investigate the minimal-$\alpha\mathcal{P}$ property with particular regard to the properties $\mathcal{P} = T_0, T_1, T_2$ and $T_{2\frac{1}{2}}$.

2 Basic facts on α-space

The α-topology $\alpha\tau(X)$ of a space $(X, \tau(X))$ has some interesting similarities with the notion of *semiregularization* $\tau_s(X)$ (see, for example, [PW]).

Lemma 1 ([N]). *For any α-open set A and any α-closed set C of a space X, we have:*

(1) $cl_{\alpha\tau}(A) = cl_\tau(A)$;

(2) $int_{\alpha\tau}(C) = int_\tau(C)$;

(3) $int_{\alpha\tau}(cl_{\alpha\tau}(A)) = int_\tau(cl_\tau(A))$.

Proposition 2 ([N]). *For any space X, we have $\alpha\,(\alpha\tau(X)) = \alpha\tau(X)$.*

Proof. Evidently $\alpha\tau(X) \subseteq \alpha\,(\alpha\tau(X))$. Let $B \in \alpha\,(\alpha\tau(X))$. Then, there is some $A \in \alpha\tau(X)$ such that $A \subseteq B \subseteq int_{\alpha\tau}(cl_{\alpha\tau}(A))$. So, there exists some $U \in \tau(X)$ such that $U \subseteq A \subseteq int_\tau(cl_\tau(U))$. Hence, $int_\tau(cl_\tau(U)) = int_\tau(cl_\tau(A))$ and, by 1(2), we have

$$U \subseteq A \subseteq B \subseteq int_{\alpha\tau}(cl_{\alpha\tau}(A)) = int_\tau(cl_\tau(A)) = int_\tau(cl_\tau(U)).$$

This proves that $B \in \alpha\tau(X)$. $\square$

Definition 1 ([N]). *A space X is called an α-space if $\tau(X) = \alpha\tau(X)$ or, equivalently, if $\tau(X) = \alpha(\sigma(X))$ for some topology $\sigma(X)$ on X.*

Proposition 3. *Let $(X, \tau(X))$ be a space. The following are equivalent:*

(1) X is an α-space;

(2) $\tau(X) = \{U \cup \{p\} : U \in \tau(X), p \in int(cl(U))\}$;

(3) $\{U \cup \{p\} : U \in \tau(X)$ is dense in X and $p \in X\} \subseteq \tau(X)$.

Proof. (1) $\Rightarrow$ (2) Evidently, $\tau(X) \subseteq \{U \cup \{p\} : U \in \tau(X), p \in int_\tau(cl_\tau(U))\}$ (it suffices to take $p \in U$). Conversely, let $U \in \tau(X)$ and $p \in int_\tau(cl_\tau(U))$. Then $U \subseteq U \cup \{p\} \subseteq U \cup int_\tau(cl_\tau(U)) = int_\tau(cl_\tau(U))$. Since X is an α-space, $U \cup \{p\} \in \tau(X)$.
(2) $\Rightarrow$ (3) Obvious because, when $cl_\tau(U) = X$, every $p \in int_\tau(cl_\tau(U)) = X$.
(3) $\Rightarrow$ (1) Suppose that $\{U \cup \{p\} : U \in \tau(X)$ is dense in X and $p \in X\} \subseteq \tau(X)$ and let $A \in \alpha\tau(X)$, i.e. $A \subseteq int_\tau(cl_\tau(int_\tau(A)))$. The set $D = int_\tau(A) \cup int_\tau(X \setminus int_\tau(A))$ is open and dense because

$$\begin{aligned}
cl(D) &= cl_\tau\big(int_\tau(A)\big) \cup cl_\tau\big(int_\tau(X \setminus int_\tau(A))\big) \\
&= cl_\tau\Big(cl_\tau(int_\tau(A))\Big) \cup cl_\tau\Big(X \setminus cl_\tau(int_\tau(A))\Big) \\
&= cl_\tau\Big(cl_\tau(int_\tau(A)) \cup \big(X \setminus cl_\tau(int_\tau(A))\big)\Big) \\
&= cl_\tau(X) \\
&= X\,.
\end{aligned}$$

So, for any $p \in A$, by hypothesis we have that $D \cup \{p\} \in \tau(X)$. Now, we consider the open set:

$$\begin{aligned} W_p &= \big(D \cup \{p\}\big) \cap int_\tau(cl_\tau(int_\tau(A))) \\ &= \Big(int_\tau(A) \cup int_\tau(X \setminus int_\tau(A)) \cup \{p\}\Big) \cap int_\tau(cl_\tau(int_\tau(A))) \\ &= \Big(int_\tau(A) \cap int_\tau\big(cl_\tau(int_\tau(A))\big)\Big) \\ &\quad \cup \Big(int_\tau\big(X \setminus int_\tau(A)\big) \cap int_\tau\big(cl_\tau(int_\tau(A))\big)\Big) \\ &\quad \cup \Big(\{p\} \cap int_\tau\big(cl_\tau(int_\tau(A))\big)\Big) \\ &= int_\tau(A) \cup \{p\} \end{aligned}$$

as

$$\begin{aligned} &\big(int_\tau(X \setminus int_\tau(A))\big) \cap int_\tau\big(cl_\tau(int_\tau(A))\big) \\ &\quad = \Big(X \setminus cl_\tau\big(int_\tau(A)\big)\Big) \cap int_\tau\big(cl_\tau(int_\tau(A))\big) \\ &\quad \subseteq \Big(X \setminus int_\tau\big(cl_\tau\big(int_\tau(A)\big)\big)\Big) \cap int_\tau\big(cl_\tau(int_\tau(A))\big) = \emptyset \end{aligned}$$

Thus, for every $p \in A$, $W_p = int_\tau(A) \cup \{p\} \in \tau(X)$ and so, also the union $\bigcup_{p \in A} W_p = int_\tau(A) \cup A = A$ is an open set of X. This proves that $\alpha\tau(X) \subseteq \tau(X)$ and hence that X is an α-space. □

In [N] it is proved the following:

Proposition 4. *A space X is an α-space if and only if all the nowhere dense sets are closed sets.*

The following proposition improves the previous one.

Proposition 5. *A space (X, τ) is an α-space if and only if every nowhere dense set is discrete.*

Proof. Suppose that X is an α-space and that N is a nowhere dense subset of X. By Proposition 4, N is a closed set and so $X \setminus N$ is an open dense set. For any $p \in N$, by 3(3), the set $U = (X \setminus N) \cup \{p\}$ is open in X. Since $U \cap N = \{p\}$, it follows that N is discrete.

Conversely, let U be an open dense set of X and $p \in X$. Then $X \setminus U$ is a nowhere dense set and by hypothesis, it is discrete. So, if $p \in X \setminus U$, there exists some $V \in \tau$ such that $V \cap (X \setminus U) = \{p\}$. Hence $U \cup \{p\} = U \cup V \in$

$\tau(X)$. Since the case when $p \in U$ is trivial, by 3(3), it is proved that X is an α-space. $\square$

It is shown in [Lo] that the operator α is not monotonic, i.e. that, in general, for two topologies $\tau(X)$ and $\sigma(X)$ on a set X, $\tau(X) \subseteq \sigma(X)$ does not imply $\alpha\tau(X) \subseteq \alpha\sigma(X)$. However, we have the following:

Lemma 6. *Let $\tau(X)$ and $\sigma(X)$ be topologies on a set X such that $\tau(X) \subseteq \sigma(X)$ and $\tau_s(X) = \sigma_s(X)$. Then $\alpha\tau(X) \subseteq \alpha\sigma(X)$.*

Proof. Let $A \in \alpha\tau(X)$. Then there exists some $U \in \tau(X)$ such that $U \subseteq A \subseteq int_\tau(cl_\tau(U))$. So, being

$$int_\sigma(cl_\sigma(U)) = int_{\sigma_s}(cl_{\sigma_s}(U)) = int_{\tau_s}(cl_{\tau_s}(U)) = int_\tau(cl_\tau(U)),$$

we have $U \subseteq A \subseteq int_\sigma(cl_\sigma(U))$ with $U \in \tau \subseteq \sigma$, that is $A \in \alpha\sigma(X)$.

$\square$

Let us recall that a space $(X, \tau(X))$ is called:

- *T_2-closed* (resp. $T_{2\frac{1}{2}}$-closed) if it is closed in every Hausdorff (resp. $T_{2\frac{1}{2}}$) space containing X as a subspace
- *minimal-T_2* (resp. minimal-$T_{2\frac{1}{2}}$) if it is a T_2- (resp. $T_{2\frac{1}{2}}$-) space and there is no strictly coarser T_2 (resp. $T_{2\frac{1}{2}}$) topology on the same set X.

The following properties are well-known and will be used later.

Proposition 7. *A Hausdorff space is T_2-closed if and only if every open ultrafilter on X is fixed.*

Proposition 8. *A space $(X, \tau(X))$ is Hausdorff if and only if its semiregularization $(X, \tau_s(X))$ is Hausdorff.*

Corollary 9. *A space $(X, \tau(X))$ is T_2-closed if and only if its semiregularization $(X, \tau_s(X))$ is minimal-T_2.*

Proposition 10. *Every regular closed subspace of a T_2-closed space is T_2-closed.*

Proposition 11. *A Hausdorff space X is minimal-T_2 if and only if it is semiregular and T_2-closed.*

Proposition 12. *If $\sigma(X)$ and $\tau(X)$ are two topologies on a set X such that $\tau_s(X) \subseteq \sigma(X) \subseteq \tau(X)$ then $\tau_s(X) = \sigma_s(X)$.*

Proof. Let R be a regular open set of $(X, \tau(X))$. Then there exists some $U \in \tau(X)$ such that $R = int_\tau\big(cl_\tau(U)\big)$. So, $R \in \tau_s(X) \subseteq \sigma(X)$. Since $\tau_s(X) \subseteq \sigma(X)$ and using a well-known property of the closure of the semiregularization (see [PW]), we have that

$$cl_\sigma(R) \subseteq cl_{\tau_s}(R) = cl_\tau(R)$$

while, being $\sigma(X) \subseteq \tau(X)$, it follows

$$int_\sigma\big(cl_\sigma(R)\big) \subseteq int_\tau\big(cl_\tau(R)\big) = R.$$

Obviously, being $R \in \sigma(X)$, we also have that $R \subseteq int_\sigma\big(cl_\sigma(R)\big)$ and hence that $R = int_\sigma\big(cl_\sigma(R)\big)$. Since $R \in \sigma(X)$, this means that R is a regular open set of $(X, \sigma(X))$.

On the other hand, let S be a regular open set of $(X, \sigma(X))$. Then there exists some $V \in \sigma(X)$ such that $S = int_\sigma\big(cl_\sigma(V)\big)$.

Since $S \in \sigma(X) \subseteq \tau(X)$, we have that

$$cl_\tau(S) \subseteq cl_\sigma(S)$$

Hence, being $\tau_s(X) \subseteq \sigma(X)$ and by some well-known properties of the interior of the semiregularization it follows that

$$int_\tau\big(cl_\tau(S)\big) = int_{\tau_s}\big(cl_\tau(S)\big) \subseteq int_\sigma\big(cl_\sigma(S)\big) = S.$$

Obviously, being $S \in \tau(X)$, we also have that $S \subseteq int_\tau\big(cl_\tau(S)\big)$ and hence that $S = int_\tau\big(cl_\tau(S)\big)$, that is S is a regular open set of $(X, \tau(X))$.

Thus, the topologies generated by these families of regular open sets, i.e. the semiregularization of $(X, \tau(X))$ and $(X, \sigma(X))$ coincide and we can conclude that $\tau_s(X) = \sigma_s(X)$. □

Proposition 13. *Let $\mathcal{U}$ be a free open ultrafilter on a Hausdorff space X and p be a fixed point in X. Then, there exists a Hausdorff topology $\tau_{\mathcal{U}}$ on X such that $\alpha_{\mathcal{U}}(X) \subsetneqq \alpha\tau(X)$.*

Proof. Let us consider the family $\tau_{\mathcal{U}}(X) = \{U \in \tau(X) : p \in U \Rightarrow U \in \mathcal{U}\}$. It is a simple routine to verify that $\tau_{\mathcal{U}}(X)$ forms a topology on X such that $\tau_{\mathcal{U}}(X) \subseteq \tau(X)$. The space $(X, \tau_{\mathcal{U}}(X))$ is T_2. In fact, for every $x \neq y$ in X, since $(X, \tau(X))$ is Hausdorff, there are $U, V \in \tau(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. If $p \notin U \cup V$, $U, V \in \tau_{\mathcal{U}}(X)$ and we are done. Otherwise, if, for example, $p \in U$, we have $p \notin V$ and $V \in \tau(X)$. Furthermore, since $\mathcal{U}$ is free respect to $(X, \tau(X))$, there exist some $N \in \tau(X)$ and some $W \in \mathcal{U}$ such that $y \in N$ and $W \cap N = \emptyset$. Thus $W \subseteq U \cup W$ implies $U \cup W \in \mathcal{U}$ and hence $U \cup W \in \tau_{\mathcal{U}}(X)$, while being $p \notin V \cap N \in \tau(X)$ it follows that $V \cap N \in \tau_{\mathcal{U}}(X)$. So, $U \cup W$ and $V \cap N$ are two open neighborhoods of x and y respectively in $\tau_{\mathcal{U}}(X)$ such that

$$\begin{aligned}\big(U \cup W\big) \cap \big(V \cap N\big) &= \big(U \cap (V \cap N)\big) \cup \big(W \cap (V \cap N)\big) \\ &\subseteq (U \cap V) \cup (W \cap N) \\ &= \emptyset \cup \emptyset = \emptyset\end{aligned}$$

and this proves that the space $(X, \tau_{\mathcal{U}}(X))$ is Hausdorff.

Since it is immediate to see that every neighborhood of p in $\tau_{\mathcal{U}}(X)$ belongs to $\mathcal{U}$ and $(X, \tau_{\mathcal{U}}(X))$ is T_2, it follows that p is the unique convergence point of $\mathcal{U}$ respect to $\tau_{\mathcal{U}}(X)$.

In order to show that $\alpha\tau_{\mathcal{U}}(X) \subseteq \alpha\tau(X)$, we observe first that, for every $U \in \tau_{\mathcal{U}}(X)$, it results:

$$cl_{\tau_{\mathcal{U}}}(U) = \begin{cases} cl_{\tau}(U) & \text{if } U \notin \mathcal{U} \\ cl_{\tau}(U) \cup \{p\} & \text{if } U \in \mathcal{U} \end{cases} \tag{1}$$

In fact, $\tau(X) \subseteq \tau_{\mathcal{U}}(X)$ implies, in any case, $cl_{\tau_{\mathcal{U}}}(U) \subseteq cl_{\tau}(U)$. Now, consider the case $U \notin \mathcal{U}$ and suppose, by contradiction that there is some $x \in cl_{\tau_{\mathcal{U}}}(U)$ such that $x \notin cl_{\tau}(U)$. So, there exists some neighborhood N of x in $\tau(X)$ such that $N \cap U = \emptyset$. Since $\mathcal{U}$ is an open ultrafilter on $(X, \tau(X))$, $U \notin \mathcal{U}$

implies that $X \setminus cl_\tau(U) \in \mathcal{U}$. Now, $W = \big(X \setminus cl_\tau(U)\big) \cup N$ is an open neighborhood of x respect to $\tau_\mathcal{U}(X)$ (because $W \in \tau(X)$, $x \in N \subseteq W$, and $X \setminus cl_\tau(U) \subseteq W$ implies $W \in \mathcal{U}$) but it results

$$W \cap U = \Big(\big(X \setminus cl_\tau(U)\big) \cup N\Big) \cap U = \Big(\big(X \setminus cl_\tau(U)\big) \cap U\Big) \cup \big(N \cap U\big) = \emptyset \cup \emptyset = \emptyset$$

which is a contradiction to $x \in cl_{\tau_\mathcal{U}}(U)$.

Let us consider the case $U \in \mathcal{U}$. Evidently $p \in cl_{\tau_\mathcal{U}}(U)$ as for every open neighborhood N of p in $\tau_\mathcal{U}(X)$, it follows that $N \in \mathcal{U}$ and hence $N \cap U \in \mathcal{U}$ implies $N \cap V \neq \emptyset$. Conversely, suppose, by contradiction, that there is some $x \in cl_{\tau_\mathcal{U}}(U)$ such that $x \neq p$ and $x \notin cl_\tau(U)$. Then, there exists some open neighborhood N of p in $\tau(X)$ such that $N \cap U = \emptyset$ and, it must be $N \notin \tau_\mathcal{U}(X)$, that is $p \in N \notin \mathcal{U}$. Since $(X, \tau(X))$ is T_2 and $x \neq p$, there is some open neighborhood G of $x \in \tau(X)$ such that $p \notin G$. Thus, $p \notin N \cap G \in \tau(X)$ implies that $N \cap G \in \tau_\mathcal{U}(X)$. So, $N \cap G$ is an open neighborhood of x in $\tau_\mathcal{U}(X)$ such that $\big(N \cap G\big) \cap U \subseteq N \cap U = \emptyset$. A contradiction to $x \in cl_{\tau_\mathcal{U}}(U)$.

Applying the usual duality rules to formula (1), we also obtain that, for every $U \in \tau_\mathcal{U}(X)$, it results:

$$int_{\tau_\mathcal{U}}\big(cl_{\tau_c U}(U)\big) = \begin{cases} int_\tau\big(cl_\tau(U)\big) \setminus \{p\} & \text{if } U \notin \mathcal{U} \\ int_\tau\big(cl_\tau(U)\big) & \text{if } U \in \mathcal{U} \end{cases} \tag{2}$$

In fact, if $U \notin \mathcal{U}$, as $\mathcal{U}$ is an open ultrafilter on $(X, \tau(X))$, $X \setminus cl_\tau(U) \in \mathcal{U}$ and so we have that:

$$\begin{aligned} int_{\tau_\mathcal{U}}\big(cl_{\tau_c U}(U)\big) &= int_{\tau_\mathcal{U}}\big(cl_\tau(U)\big) \\ &= X \setminus cl_{\tau_\mathcal{U}}\Big(X \setminus \big(cl_\tau(U)\big)\Big) \\ &= X \setminus cl_\tau\Big(X \setminus \big(cl_\tau(U) \cup \{p\}\big)\Big) \\ &= \Big(X \setminus cl_\tau\big(X \setminus cl_\tau(U)\big)\Big) \cap \big(X \setminus \{p\}\big) \\ &= int_\tau\big(cl_\tau(U)\big) \setminus \{p\} \end{aligned}$$

If $U \in \mathcal{U}$, since $\mathcal{U}$ is an open ultrafilter on $(X, \tau(X))$, it necessarily results

$X \setminus cl_\tau(U) \notin \mathcal{U}$ and we have:

$$\begin{aligned} int_{\tau_\mathcal{U}}\big(cl_{\tau_\mathcal{U}}(U)\big) &= X \setminus cl_{\tau_\mathcal{U}}\big(X \setminus cl_{\tau_\mathcal{U}}(U)\big) \\ &= X \setminus cl_\tau\big(X \setminus cl_{\tau_\mathcal{U}}(U)\big) \\ &= int_\tau\big(cl_{\tau_\mathcal{U}}(U)\big) \\ &= int_\tau\big(cl_\tau(U) \cup \{p\}\big) \\ &= int_\tau\big(cl_\tau(U)\big) \end{aligned}$$

where the last equality is due to the fact that $p \in cl_\tau(U)$ (because for every neighborhood N of p in $\tau(X)$, we have $N \in \mathcal{U}$ and so, being $U \in \mathcal{U}$, it follows that $N \cap U \in \mathcal{U}$ and $N \cap U \neq \emptyset$.

Now, for every $A \in \alpha\tau_\mathcal{U}(X)$, there exists some $U \in \tau_\mathcal{U}(X)$ such that $U \subseteq A \subseteq int_{\tau_\mathcal{U}}\big(cl_{\tau_\mathcal{U}}(U)\big)$ and by formula (2), it immediately follows, in both cases, that $U \subseteq A \subseteq int_\tau\big(cl_\tau(U)\big)$ with $U \in \alpha\tau(X) \subseteq \tau(X)$ and so that $A \in \alpha\tau(X)$. Thus $\alpha\tau_\mathcal{U}(X) \subseteq \alpha\tau(X)$.

To finish the proof, we will show that $\alpha\tau_\mathcal{U}(X) \neq \alpha\tau(X)$. Since $\mathcal{U}$ is free, p is not an adherent point for $\mathcal{U}$ and so there exist some neighborhood V of p in $\tau(X)$ and some $U \in \mathcal{U}$ such that $U \cap V = \emptyset$. Thus $V \notin \mathcal{U}$ and, being $p \in V$, by definition of $\tau_\mathcal{U}(X)$, it follows that $V \notin \tau_\mathcal{U}(X)$. Evidently $V \in \alpha\tau(X)$ (as $V \in \tau(X)$) but $V \notin \alpha\tau_\mathcal{U}(X)$. Suppose, by contradiction, that there exists some $W \in \tau_\mathcal{U}(X)$ such that $W \subseteq V \subseteq int_{\tau_\mathcal{U}}\big(cl_{\tau_\mathcal{U}}(W)\big)$. Since $V \notin \mathcal{U}$, it follows a fortiori that $W \notin \mathcal{U}$ and so, by formula (2), we have that $W \subseteq V \subseteq int_\tau\big(cl_\tau(W)\big) \setminus \{p\}$ and thus that $p \notin V$. A contradiction. This proves that $\alpha\tau_\mathcal{U}(X) \subsetneq \alpha\tau(X)$ and concludes our proof. □

3 Minimal-αT_0-space

Definition 2. *Let $\mathcal{P}$ be a topological property. A space X is said to be a $\alpha\mathcal{P}$-space if it is a α-space and it holds property $\mathcal{P}$.*

Definition 3. *Let $\mathcal{P}$ be a topological property. A space X is called minimal-$\alpha\mathcal{P}$ if it is a $\alpha\mathcal{P}$-space and there is no strictly coarser $\alpha\mathcal{P}$-topology on the same set.*

Since the unique T_1-minimal-space is the cofinite topology which is an α-space (by Proposition 3), it is evident that the class of minimal-αT_1-spaces coincides with the well-known class of minimal-T_1.
One would suspect that if X is minimal-T_0, then $(X, \alpha\tau(X))$ is minimal-αT_0. Here is a counterexample.

Example 14 . Consider $\mathbb{R}$ with $\tau(\mathbb{R}) = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. Then $\mathbb{R}$ is minimal-T_0. If $\varnothing \neq U \in \tau(\mathbb{R})$, $cl_{\mathbb{R}}(U) = \mathbb{R}$, then $int_{\mathbb{R}}\big(cl_{\mathbb{R}}(U)\big) = \mathbb{R}$. So, if $a < b$ and $c = a - 1$, then $(-\infty, c) \cup \{a\}$, $(-\infty, c) \cup \{b\} \in \alpha\tau(\mathbb{R})$. In particular, it follows that $(\mathbb{R}, \alpha\tau(\mathbb{R}))$ is T_1. If σ is the cofinite topology on $\mathbb{R}$, then $\sigma \subset \tau(\mathbb{R})$.

Lemma 15. *Let X be a set, $p \in X$, and $\tau(X) = \{U : p \in U$ and $X \setminus U$ is finite$\} \cup \{\varnothing\}$. Then X is minimal-αT_0.*

Proof. Clearly, X is T_0. If $\varnothing \neq U \in \tau(X)$, then $p \in U$, $cl_X U = X$, and $int_X cl_X U = X$. Now $U \cup \{q\}$ is open for all $q \in X$, by Proposition 3, since X is αT_0. Let $\sigma \subseteq \tau(X)$ and (X, σ) be αT_0. If $\varnothing \neq U \in \tau(X)$, then $p \in U$, and $X \setminus U = \{q_1, \cdots, q_n\}$ is finite. Since $\varnothing \neq V \in \sigma \subseteq \tau(X)$ implies $p \in V$, then for each $1 \leq i \leq n$, there is a $V_i \in \sigma$ such that $p \in V_i \subseteq X \setminus \{q_i\}$. For $T = \bigcap\{V_i : 1 \leq i \leq n\}$, $p \in T \in \sigma$ and $T \subseteq U$. As (X, σ) is αT_0 and $cl_\sigma T = X$, then $U = \bigcup\{T \cup \{q\} : q \in U \setminus T\} \in \sigma$. □

In the topology just considered, every point different from p is closed. Now, let us consider the case in which some point of X is not closed.

Lemma 16. *Let X be an αT_0-space, and $p \in X$ such that $cl_X\{p\} \neq \{p\}$. Then $cl_X(\{p\})$ is a regular-closed set and $p \in int_X\big(cl_X(\{p\})\big)$.*

Proof. Let $U = X \setminus cl_X(\{p\})$. It suffices to show that U is regular-open. If $p \in cl_X(U)$, then $cl_X(U) = X$ and $int_X\big(cl_X(U)\big) = X$. So, for $q \in cl_X(\{p\}) \setminus \{p\}$, $U \cup \{q\}$ is open and $(U \cup \{q\}) \cap \{p\} = \varnothing$. That is, $q \notin cl_X(\{p\})$, a contradiction. So, $p \notin cl_X(U)$. For $V = X \setminus cl_X(U)$, we have that $p \in V \subseteq cl_X(\{p\})$. Thus, $cl_X(V) = cl_X(\{p\})$. □

In order to obtain a characterization of minimal-T_0-space, we need some other lemmas.

Lemma 17. *Let X be a αT_0-space, and $p, q \in X$ such that $cl_X(\{p\}) \neq \{p\}$, $cl_X(\{q\}) \neq \{q\}$, and $p \neq q$. Then $p \notin cl_X(\{q\})$ and $q \notin cl_X(\{p\})$.*

Proof. Since X is T_0, then $cl_X(\{p\}) \neq cl_X(\{q\})$. Assume that $q \in cl_X(\{p\})$. Then $p \notin cl_X(\{q\})$ as $cl_X(\{p\}) \neq cl_X(\{q\})$. So, $cl_X(\{q\}) \subseteq cl_X(\{p\})$. By Lemma 16, $q \in int_X\big(cl_X(\{q\})\big) \subseteq int_X\big(cl_X(\{p\})\big) \subseteq cl_X(\{p\})$. As $p \notin cl_X(\{q\})$, it follows that $p \notin int_X\big(cl_X(\{q\})\big)$. Therefore, $q \notin cl_X(\{p\})$, a contradiction. □

Lemma 18. *Let X be a minimal-αT_0-space, and $p, q \in X$ such that $cl_X(\{p\}) \neq \{p\}$, $cl_X(\{q\}) \neq \{q\}$, and $p \neq q$. Then $int_X\big(cl_X(\{p\})\big) = \{p\}$ and $int_X\big(cl_X(\{q\})\big) = \{q\}$.*

Proof. Let $\tau = \tau(X)$ be the topology on X, $r \in int_X\big(cl_X(\{p\})\big) \setminus \{p\}$ and consider the topology $\sigma = \{U \in \tau : r \in U$ implies $q \in U\}$ on X. Clearly, $\sigma \subseteq \tau$. There are a number of cases to verify that (X, σ) is T_0 but each case is straightforward. Next we show that (X, σ) is an α-space. Let U be an open and dense in (X, σ) and $t \in X \setminus U$. We want to show that $U \cup \{t\} \in \sigma$. If $r \in U$, then U is dense and open in (X, τ) and hence, $U \cup \{t\} \in \sigma$. Suppose that $r \notin U$. Our first goal is to show that U is dense in τ. If $cl_\tau(U) \neq cl_\sigma(U)$, then $r \in cl_\sigma(U) \setminus cl_\tau(U)$. Then $p \notin U$ and $cl_\tau(\{p\}) \cap U = \varnothing$. In particular, $int_\tau\big(cl_\tau(\{p\})\big) \cap U = \varnothing$. There is an $V \in \tau$ such that $p \in V$ and $r \notin V$. Now $p \in int_\tau\big(cl_\tau(\{p\})\big) \cap V \in \sigma$ and $int_\tau\big(cl_\tau(\{p\})\big) \cap V \cap U = \varnothing$. So, U is not dense in (X, σ), a contradiction. Thus, U is dense in τ. Also the above proof shows that $p \in U$. If $q \notin U$, then $cl_\tau(\{q\}) \cap U = \varnothing$. In particular, $int_\tau\big(cl_\tau(\{q\})\big) \cap U = \varnothing$. But $q \in int_\tau\big(cl_\tau(\{q\})\big) \in \sigma$, a contradiction. Thus, $q \in U$. With both $p, q \in U$, we have that $U \cup \{r\} \in \sigma$. For $t \neq r$, then $U \cup \{t\} \in \sigma$ as $U \cup \{t\} \in \tau$ and $r \notin U \cup \{t\}$. □

Proposition 19. *Let X be a minimal-αT_0-space, and $P = \big\{p \in X : cl_X(\{p\}) \neq \{p\}\big\}$ such that $|P| \geq 2$. Then P is dense in X and if $V \in \tau(X)$ and $V \setminus P \neq \varnothing$, then $P \subseteq V$.*

Proof. Let $\tau = \tau(X)$ be the topology of the space X. First we show that $Q = \{q \in X : \{q\} \in \tau\}$ is dense. Clearly, $Q \supseteq P$. Fix $r \in X \setminus cl_X(Q)$. Note that the topology $\sigma = \{U \in \tau : r \in U \text{ implies } Q \subseteq U\} \subseteq \tau$. If we show that (X, σ) is αT_0, it will follow that Q is dense. Since $\{q\} \in \sigma$ for all $q \in Q$, q can be T_0-separated from all $p \in X \setminus \{q\}$ in σ. As $(X \setminus cl_X(Q)) \cup Q \in \sigma$ and $(X \setminus cl_X(Q)) \cup Q \cup \{t\} \in \sigma$ for $t \in cl_X(Q) \setminus Q$, a point $t \in cl_X(Q) \setminus Q$ can be T_0-separated from all $p \in X \setminus \{t\}$ in σ. Let $s, t \in X \setminus cl_X(Q)$. As τ is T_0, there is some $V \in \tau$ such that $s \in V$ and $t \notin V$ or vice versa. Now, $s \in V \cup Q \in \sigma$ and $t \notin V \cup Q$. this completes the proof that (X, σ) is T_0. Next we show that (X, σ) is an α-space. Let U be an open and dense subset of (X, σ). Note that $Q \subseteq U$. If $cl_\tau(U) \neq cl_\sigma(U)$, then $r \in cl_\sigma(U) \setminus cl_\tau(U)$. There is an $V \in \tau$ such that $r \in V$ and $V \cap Q = \varnothing$. As $r \notin P, cl_\tau(\{r\}) = \{r\}$ and $\varnothing \neq V \setminus \{r\} \in \sigma$ and $V \setminus \{r\} \cap U = \varnothing$, a contradiction as U is dense in (X, σ). This shows that $cl_\tau(U) = X$. As τ is an α topology, for $t \in X, U \cup \{t\} \in \tau$. As $Q \subseteq U$, $U \cup \{t\} \in \sigma$. This completes the proof that (X, σ) is an αT_0-space.

Finally, we show that $Q = P$. Assume that $q \in Q \setminus P$. The topology $\sigma = \{U \in \tau : q \in U \text{ implies } P \subseteq U\} \subseteq \tau$. Similar to the above, it is straightforward to show that (X, σ) is T_0 and an α-space, a contradiction as τ is minimal-αT_0. Thus, $P = Q$. □

Proposition 20. *Let X be a minimal-αT_0-space, and $P = \{p \in X : cl_X(\{p\}) \neq \{p\}\}$ such that $|P| \geq 2$. Then $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \bigcup \{\{p\} : p \in P\}$.*

Proof. This is an obvious consequence of Proposition 19. □

Proposition 21. *Let X be a set and $P \subsetneq X$ such that $|P| \geq 2$. Let $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \bigcup \{\{p\} : p \in P\}$. Then X is a minimal-αT_0-space.*

Proof. Let $\tau = \tau(X)$ be the topology on X and $\sigma \subseteq \tau$ be an αT_0 topology. For $p \in P, (X \setminus P) \cup \{p\} = cl_\tau(\{p\}) \subseteq cl_\sigma(\{p\})$. Thus, for all $p \in P$, $cl_\sigma(\{p\}) \neq \{p\}$. By Lemma 17, for $q \in P \setminus \{p\}$, $q \notin cl_\sigma(\{p\})$. Hence $cl_\sigma(\{p\}) = (X \setminus P) \cup \{p\} = cl_\tau(\{p\})$. Thus, $P \setminus \{p\} \in \sigma$ for all $p \in P$.

By Lemma 16, $p \in int_\sigma\big(cl_\sigma(\{p\})\big)$. As $|P| \geq 2$, let $q \in P \setminus \{p\}$. Then $int_\sigma\big(cl_\sigma\{q\}\big) \cap P \setminus \{p\} = \{q\} \in \sigma$ and $P \in \sigma$. As $X = cl_\tau(P) \subseteq cl_\sigma(P)$, $X = int_\sigma\big(cl_\sigma(P)\big)$. Since (X, σ) is an αT_0-space, it follows that for $q \in X \setminus P, P \cup \{q\} \in \sigma$. This shows that $\sigma = \tau$. □

Finally, we obtain the characterization of αT_0-space.

Theorem 22. *Let X be an αT_0-space and $P = \{p \in X : cl_X(\{p\}) \neq \{p\}\}$. Then X is minimal-αT_0 iff $P \neq \varnothing$ and*

(i) if $P = \{p\}$, then $\tau(X) = \{U : p \in U$ and $X \setminus U$ is finite $\} \cup \{\varnothing\}$, or

(ii) if $|P| \geq 2$, then $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \bigcup \{\{p\} : p \in P\}$.

Proof. ($\Longleftarrow$) It follows from Lemma 18 and Proposition 20.

($\Longrightarrow$) It follows from Lemma 15 and Proposition 21. □

4 Minimal-αT_2-space

Proposition 23. *Let X be an αT_2-space. Then X is a minimal-αT_2-space if and only if $(X, \tau_s(X))$ is minimal-T_2 and $\tau(X) = \alpha\tau_s(X)$.*

Proof. ($\Longrightarrow$) Suppose that X be a minimal-αT_2-space. Thus $\alpha\tau(X) = \tau(X)$. Then the space $(X, \tau(X))$ is T_2-closed. In fact, if, by contradiction, it is not, by Proposition 7, there exists some free open ultrafilter $\mathcal{U}$ on $(X, \tau(X))$ and by Proposition 13, there exists a strictly coarser αT_2 topology $\tau_{\mathcal{U}}$. A contradiction to the α-minimality of $(X, \tau(X))$. Hence, by Corollary 9, $(X, \tau_s(X))$ is minimal-T_2.

Furthermore, being $\tau_s(X) \subseteq \tau(X)$ and, obviously, $\big(\tau_s(X)\big)_s = \tau_s(X)$, by Lemma 6, we have that $\alpha\tau_s(X) \subseteq \alpha\tau(X)$, i.e. $\alpha\tau_s(X) \subseteq \tau(X)$ where $\alpha\tau_s(X)$ is Hausdorff by Proposition 11 as $\tau(X)$ is Hausdorff and the T_2 axioms is expansive. Since $(X, \tau(X))$ is minimal-αT_2, we conclude that $\alpha\tau_s(X) = \tau(X)$.

($\Longleftarrow$) Let us suppose that $(X, \tau_s(X))$ is minimal-T_2 and $\tau(X) = \alpha\tau_s(X)$. Then $(X, \tau(X))$ is H-closed by Corollary 9. Now, let $\sigma(X)$ be a αT_2 topology on X such that

$$\sigma(X) \subseteq \tau(X)$$

For every R regular open set of $(X, \tau(X))$, $X \setminus R$ is a regular closed set and hence, by Proposition 10, it is a T_2-closed subspace of the Hausdorff space $(X, \sigma(X))$. Thus $X \setminus R$ is a closed set of $(X, \sigma(X))$ and $R \in \sigma(X)$. This proves that

$$\tau_s(X) \subseteq \sigma(X)$$

Hence, by Proposition 12, $\tau_s(X) = \sigma_s(X)$ and so, applying Lemma 6 to $\tau_s(X) \subseteq \sigma(X)$, we have that $\alpha\tau_s(X) \subseteq \alpha\sigma(X)$. Since $\sigma(X)$ is an α topology and, by hypothesis $\alpha\tau_s(X) = \tau(X)$, it follows that $\tau(X) \subseteq \sigma(X)$ and so that $\tau(X) = \sigma(X)$. This shows that $(X, \tau(X))$ is a minimal-αT_2-space and concludes our proof. □

Corollary 24. *Let $(X, \tau(X))$ be a space with a dense set D of isolated points. Then:*

(1) The α-topology $\alpha\tau(X)$ coincides with the topology generated by $\tau(X) \cup \{D \cup \{x\} : x \in X\}$, i.e. is a simple extensions of the subspace D.

(2) X is an α-space if and only if $X \setminus D$ is discrete.

(3) If X is a semiregular α-space, it results:

$$\big\{\sigma : \sigma \text{ is a topology on } X \text{ such that } \sigma_s = \tau(X)\big\} = \{\tau(X)\}.$$

Proof. Straightforward applications of propositions 3, 5 and 6. □

Example 25 . Let us consider the set $X = \mathbb{R} \times [0, +\infty[$. It is easy to verify that

$$\tau_s(X) = \big\{U \subseteq X : (x, 0) \in U \Rightarrow \exists\epsilon > 0 \text{ such that }]x - \epsilon, x + \epsilon[\times[0, \epsilon[\subseteq U\big\}$$

defines a Tychonoff topology on X and that $D = \mathbb{R}\times]0,+\infty[$ is a dense set of isolated points. Now, if we consider another topology on X,

$$\begin{aligned}\tau(X) = \ & \big\{U \subseteq X : (x,0) \in U \setminus \mathbb{Q}\times\{0\} \Rightarrow \exists\epsilon > 0 \\ & \quad \text{such that }]x-\epsilon, x+\epsilon[\times[0,\epsilon[\subseteq U \\ & \quad \text{and } (x,0) \in U \cap \mathbb{Q}\times\{0\} \Rightarrow \exists\epsilon > 0 \text{ such that} \\ & \qquad]x-\epsilon, x+\epsilon[\times]0,\epsilon[\cup\big(]x-\epsilon, x+\epsilon[\cap\mathbb{Q}\times\{0\}\big) \subseteq U\big\}\end{aligned}$$

it is a simple routine to check that $\tau_s(X)$ is the semiregularization of $\tau(X)$ and that $\tau(X) \neq \tau_s(X)$. Furthermore, since $C = X \setminus (\mathbb{R}\times]0,+\infty[)$ is a closed nowhere dense subset with respect to both $\tau(X)$ and $\tau_s(X)$, by Corollary 24 follows that $\alpha\tau(X) = \alpha\tau_s(X)$ coincides with the topology generated by $\tau(X)\cup\big\{(\mathbb{R}\times]0,+\infty[)\cup\{(x,0)\} : x \in \mathbb{R}\big\}$ and so that $\tau(X) \neq \alpha\tau(X)$. Thus, we have that

$$\tau_s(X() \subsetneqq \tau(X) \subsetneqq \alpha\tau(X) = \alpha\tau_s(X).$$

Example 26 (A Tychonoff α-space which is not minimal-T_2). Let us recall that two sets are said *almost disjoint* if their intersection is a finite set. It is a simple routine to show (by using Zorn's Lemma) that there exists a maximal almost disjoint family $\mathcal{M}$ of subsets of $\mathbb{N}$. The space generated on the set $\psi = \mathbb{N} \cup \mathcal{M}$ by the base

$$\mathcal{B} = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \big\{\{M\} \cup S : M \in \mathcal{M} \text{ and } S \text{ is a cofinite subset of } M\big\}$$

is a 0-dimensional (and hence Tychonoff) but not normal (and hence not minimal-T_2). This space is known in literature as an ψ-space (see 1N, [PW]). Since, every closed nowhere dense set of ψ is discrete, by Proposition 5, it is evident that ψ is an α-space.

Example 27 (A minimal-T_2, α-space). Let us consider the set

$$Z = \left\{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\right\} \cup \left\{\left(\frac{1}{n}, \frac{1}{m}\right) : n \in \mathbb{N},\, m \in \mathbb{Z}\right\}$$

with the topology $\tau(Z)$ induced by the usual topology on $\mathbb{R}^2$. Let $X = Z \cup \{a, b\}$ and define a topology $\tau(X)$ on X by saying that a subset $U \subset Z$

is open if $U \cap Z \in \tau(Z)$ and if $a \in U$ (respectively, $b \in U$) there exists some $k \in \mathbb{N}$ such that $\big\{\big(\frac{1}{n}, \frac{1}{m}\big) : n \in \mathbb{N}, m \geq k\big\} \subseteq U$ (respectively, $\big\{\big(\frac{1}{n}, -\frac{1}{m}\big) : n \in \mathbb{N}, m \geq k\big\} \subseteq U$). It is well-known (see 4.8(d), [PW]) that the space X is Urysohn, not compact and minimal-T_2. Furthermore, since every its nowhere dense (namely, $\big\{\big(\frac{1}{n}, 0\big) : n \in \mathbb{N}\big\} \cup \{a, b\}$) is discrete, it follows from Proposition 5 that X is an α-space.

Let us note that, by 24(3), in both of the above spaces, we have that $\big\{\sigma : \sigma \text{ is a topology on X such that } \sigma_s = \tau(X)\big\} = \{\tau(X)\}$. Thus, the question remains: *is there some semiregular α-space $\big(X, \tau(X)\big)$ such that $\{\tau(X)\} \subsetneqq \big\{\sigma : \sigma$ is a topology on X such that $\sigma_s = \tau(X)\big\}$?*
We now provide an example of a Tychonoff α-space X such that $\{\tau(X)\} \subsetneq \{\sigma : \sigma \text{ is a topology on } X \text{ such that } \sigma_s = \tau(X)\}$.

Example 28 . Recall that a measurable subset A of $\mathbb{R}$ has density 1 if

$$A = \{x \in \mathbb{R} : \lim_{h \to 0} \frac{m(A \cap [x-h, x+h])}{2h} = 1\}.$$

The set of $\{A \subseteq \mathbb{R} : A \text{ measurable with density } 1\}$ forms a topology $\delta(\mathbb{R})$, called the density topology, on the set $\mathbb{R}$. The space $\big(\mathbb{R}, \delta(\mathbb{R})\big)$ is a Tychonoff space without isolated points, strictly finer than the usual topology $\tau(\mathbb{R})$, and has the property that every nowhere dense subset is closed and discrete (see 2.7 in [T]). In particular, $\big(\mathbb{R}, \delta(\mathbb{R})\big)$ is an α-space by Proposition 4. We need one additional fact about $\big(\mathbb{R}, \delta(\mathbb{R})\big)$. Note that for $x \in U \in \delta(\mathbb{R})$, $U \cap (x, \infty) \neq \varnothing$ and $U \cap (-\infty, x) \neq \varnothing$. In particular, the $\delta(\mathbb{R})$ open neighborhood filter is contained in two distinct $\delta(\mathbb{R})$ open ultrafilters on $\mathbb{R}$. Before continuing with this example, we need a result about absolutes of spaces.
For a Hausdorff space X and let $EX = \{\mathcal{U} : \mathcal{U}$ is a convergent, open ultrafilter on $X\}$. For $U \in \tau(X)$, let $O(U) = \{\mathcal{U} \in EX : U \in \mathcal{U}\}$. For $U, V \in \tau(X)$, it is easy to verify (see [PW]) that $O(\varnothing) = \varnothing$, $O(X) = EX, O(U \cap V) = O(U) \cap O(V), O(U \cup V) = O(U) \cup O(V), EX \backslash O(U) = O(X \backslash cl_X(U))$, and $O(U) = O(int_X cl_X(U))$. EX with the topology generated by $\{O(U) : U \in \tau(X)\}$ is an extremally disconnected Tychonoff space, called the **absolute** of

X. The function $k : EX \to X$ defined by $k(\mathcal{U})$ is the unique convergent point of $\mathcal{U}$ is called a covering function and has the properties that k is irreducible, $\theta-$continuous, perfect and onto. If X is regular, then k is also continuous. If $D \subseteq EX$ such that $k[D] = X$, then D is dense in EX.

Proposition 29. *Let X be a regular α-space and Y a subspace of EX such that for each $x \in X$, $|k^{\leftarrow}((x))| = 2$. Then Y is an α-space.*

Proof. Let $\overline{k} = k|Y$. The function $\overline{k} = k|Y : Y \to X$ is continuous and onto. So, Y is dense inEX and is extremally disconnected. Let N be a nowhere dense subset of Y. Suppose $U = int_{EX}\big(cl_{EX}(N)\big) \neq \varnothing$. Then, as Y is dense, $\varnothing \neq U \cap Y \subseteq cl_{EX}(N) \cap Y = cl_Y(N)$, contradicting that N is nowhere dense in Y. Thus, N is nowhere dense in EX. By 6.5d(2) in [PW], $k[N] = \overline{k}[N]$ is nowhere dense in X and hence discrete in X. Let $p \in N$. There is an open set V in X such that $V \cap \overline{k}[N] = \{k(p)\}$. So, $N \cap \overline{k}^{\leftarrow}[V] = \overline{k}^{\leftarrow}(\{p\})$. But $\overline{k}^{\leftarrow}(\{p\})$ has only two points and there is an open set W in Y such that $N \cap \overline{k}^{\leftarrow}[V] \cap W = \{p\}$. This shows that N is discrete in Y and Y is an α-space. □

We are ready to apply Proposition 29 to the regular α-space, $\big(\mathbb{R}, \delta(\mathbb{R})\big)$. First note that the covering function $k : E\mathbb{R} \to (\mathbb{R}, \delta(\mathbb{R}))$ has the property that $|k^{\leftarrow}(r)| \geq 2$ for each $r \in \mathbb{R}$ since each open neighborhood filter is contained in two distinct $\delta(\mathbb{R})$ open ultrafilters on $\mathbb{R}$. Let $Y \subseteq E\mathbb{R}$ with the property that $|k^{\leftarrow}(r) \cap Y| = 2$ for each $r \in \mathbb{R}$, and let $Z \subset Y$ such that $|k^{\leftarrow}(r) \cap Z| = 1$ for each $r \in \mathbb{R}$. Now, both Y and Z are dense in $E\mathbb{R}$. By Proposition 29, Y is a Tychonoff α-space. Let σ be the topology on Y generated by $\tau(Y) \cup \{Z\}$. Clearly, $\tau(Y) \subsetneq \sigma$, and it is straightforward to show that $\sigma_s = \tau(Y)$. The space Y is the desired space.

5 A problem

Since it is false that the property to be $T_{2\frac{1}{2}}$-closed passes to the regular open subspaces, we can not use no result like 10 in order to prove some proposition similar to 23. So, it remains the following:

Conjecture. Let X be an $\alpha T_{2\frac{1}{2}}$-space. Then X is a minimal-$\alpha T_{2\frac{1}{2}}$- space if and only if $(X, \tau_s(X))$ is semiregular, $T_{2\frac{1}{2}}$-closed and $\tau(X) = \alpha\tau_s(X)$.

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