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THE QUANTIZATION OF A FOURTH-ORDER EQUATION WITHOUT A SECOND-ORDER LAGRANGIAN

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We present an equation of the fourth-order which does not possess a second-order Lagrangian and demonstrate by means of the method of reduction of order that one can obtain a first-order Lagrangian for it. This opens the way to quantization through the construction of an Hamiltonian which is suitable to be quantized according to the procedure of Dirac with the correct physical attributes.

Keywords: Fourth-order ODE; Lagrangian; Lie symmetries; Hamiltonian; quantization.

1. Introduction

In the context of the quantization of higher-order field theories, it has been found that some unpleasant properties occur. A classic example is the model of Pais–Uhlenbeck [15] described by a second-order Lagrangian in the Action,

$$A = \frac{1}{2}\gamma \int \{ \ddot{z}^2 - (\Omega_1^2 + \Omega_2^2) \dot{z}^2 + \Omega_1^2 \Omega_2^2 z^2 \} dt, \quad (1.1)$$

so that the Euler–Lagrange equation is of the fourth-order, being

$$\ddot{\ddot{z}} + (\Omega_1^2 + \Omega_2^2) \ddot{z} + \Omega_1^2 \Omega_2^2 z = 0. \quad (1.2)$$

To bring this model within the context of Hamiltonian Theory, Davidson [6] and Mannheim [7] introduced a new variable y , to describe the model in terms of a system

with two degrees of freedom. The problem with the model of Pais–Uhlenbeck when it is quantized according to the method advanced by Dirac almost a century ago is that it possesses “ghost” states, i.e. the norm of the (quantum) state is negative. This is not acceptable mathematically as well as physically since the whole concept of a norm is rooted in the essence of being nonnegative. Consequently the model has been regarded as unphysical. A procedure which produced a satisfactory result in terms of the properties expected in the quantal description was provided by Bender and Mannheim [1] in the context of \mathcal{PT} -symmetric quantum mechanics. The construction of the Hamiltonian by the authors mentioned above was according to the procedure of Ostrogradski [14]. An alternate approach [12, 13] showed by means of the method of reduction of order [9] that it is possible to construct an Hamiltonian which gives rise to a satisfactory quantal description without having to depart from the standard methods of quantization. By resorting to the new methods of \mathcal{PT} -symmetric quantum mechanics, Bender and Mannheim were able to resolve the problem of the unfortunate properties of the Hamiltonian obtained from the fourth-order model of Pais–Uhlenbeck. Nucci *et al.* recognized that the problem of incorrect quantization following Dirac was caused by following the procedure of Ostrogradsky. Whilst it would be unfair to classify the former result as “two wrongs make a right”, the latter result showed that the avoidance of an unusual wrong made it unnecessary to introduce a new theory. One could remain within the purview of the theory of Dirac.

The fourth-order equation in the model of Pais–Uhlenbeck is the Euler–Lagrange equation obtained from the application of Hamilton’s Principle to the Action Integral (1.1). In this note we consider a fourth-order equation which does not have a second-order Lagrangian. Nevertheless we are able, by means of group theoretical methods, to obtain a Hamiltonian and consequently provide a route to quantization in the standard fashion given to us by Dirac [2] so that there is no ambiguity in the interpretation of the quantum mechanics.

The fourth-order equation which we consider is

$$\ddot{w} = 3w + 2\dot{w} + \frac{4(\dot{w} + \ddot{w})^2}{3(w + \ddot{w})}. \tag{1.3}$$

Fels [3] has demonstrated that a fourth-order equation of the form

$$u^{(iv)} = F(t, u, u', u'', u''') \tag{1.4}$$

admits a unique second-order Lagrangian iff the following conditions are satisfied:

$$\frac{\partial^3 F}{\partial (u''')^3} = 0, \tag{1.5}$$

$$\begin{aligned} \frac{\partial F}{\partial u'} + \frac{1}{2} \frac{d^2}{dt^2} \left(\frac{\partial F}{\partial u'''} \right) - \frac{d}{dt} \left(\frac{\partial F}{\partial u''} \right) - \frac{3}{4} \frac{\partial F}{\partial u'''} \frac{d}{dt} \left(\frac{\partial F}{\partial u'''} \right) \\ + \frac{1}{2} \frac{\partial F}{\partial u''} \frac{\partial F}{\partial u'''} + \frac{1}{8} \left(\frac{\partial F}{\partial u'''} \right)^3 = 0. \end{aligned} \tag{1.6}$$

A simple calculation shows that Eq. (1.3) does not satisfies condition (1.6) — although satisfies condition (1.5). Therefore Eq. (1.3) does not possess a second-order Lagrangian.

Despite the absence of a Lagrangian equation (1.3) does have six Lie point symmetries. They are

$$\begin{aligned} \Gamma_1 &= \sin t \partial_w & \Gamma_4 &= \partial_t \\ \Gamma_2 &= \cos t \partial_w & \Gamma_5 &= \sin 2t \partial_t + w \cos 2t \partial_w \\ \Gamma_3 &= w \partial_w & \Gamma_6 &= \cos 2t \partial_t - w \sin 2t \partial_w \end{aligned} \tag{1.7}$$

and the algebra is $sl(2, R) \oplus_s \{D \oplus_s 2A_1\}$, i.e. the semidirect sum of $sl(2, R)$ with the group of translations and dilations in the plane. This algebra is the same of that of the one-dimensional linear oscillator apart from the absence of the two non-Cartan symmetries of the latter.

2. Application of the Method of Reduction of Order

The essential feature of the method of reduction of order [9] is to rewrite the system under consideration as a set of first-order equations. We write (1.3) as

$$\dot{w}_1 = w_2 \tag{2.1}$$

$$\dot{w}_2 = w_3 \tag{2.2}$$

$$\dot{w}_3 = w_4 \tag{2.3}$$

$$\dot{w}_4 = 3w_1 + 2w_3 + \frac{4(w_2 + w_4)^2}{3(w_1 + w_3)} \tag{2.4}$$

in which it is quite obvious that the w_1 of (2.1) is the dependent variable in (1.3).

We calculate a Jacobi Last Multiplier from the system (2.1)–(2.4) using the formula [4, 16]

$$M = \exp \left[- \int \sum_{j=1}^4 \frac{\partial W_j}{\partial w_j} dt \right], \tag{2.5}$$

where W_j is the right-hand-side of the j th member of the system (2.1)–(2.4). Evidently the only contribution comes from (2.4) and we have

$$\begin{aligned} M &= \exp \left[- \int \frac{8(w_2 + w_4)}{3(w_1 + w_3)} dt \right] \\ &= (w_1 + w_3)^{-8/3}, \end{aligned} \tag{2.6}$$

in which we have made use of (2.1) and (2.3) to perform the quadrature. We can also use the symmetries of (1.7), expressed in the variables of (2.1)–(2.4), to calculate further multipliers [5, 10]. When we use $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 , we obtain the matrix

$$C_{1234} = \begin{pmatrix} 1 & W_1 & W_2 & W_3 & W_4 \\ 0 & \sin t & \cos t & -\sin t & -\cos t \\ 0 & \cos t & -\sin t & -\cos t & \sin t \\ 0 & w_1 & w_2 & w_3 & w_4 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.7}$$

The reciprocal of its determinant is another multiplier, i.e.

$$M_{1234} = \frac{3}{(w_2 + w_4)^2 + 9(w_1 + w_3)^2}. \tag{2.8}$$

These two multipliers are singular if $w_2 + w_4 = 0$ and $w_1 + w_3 = 0$. This suggests [11] the introduction of two new variables through

$$w_4 = r_4 - w_2 \quad \text{and} \quad w_3 = r_3 - w_1. \tag{2.9}$$

To raise the system to the second order we make the substitutions [9]

$$r_4 = \dot{r}_3 \quad \text{and} \quad w_1 = -\dot{w}_2 + r_3 \tag{2.10}$$

and obtain the pair of second-order equations,

$$\ddot{w}_2 = \dot{r}_3 - w_2, \tag{2.11}$$

$$\ddot{r}_3 = 3r_3 + \frac{4\dot{r}_3^2}{3r_3}. \tag{2.12}$$

The variable r_3 has its own equation in (2.12) and it is a simple matter [8] to determine that (2.12) possesses eight Lie point symmetries and so is linearisable. Under the transformation

$$r_3 = R_3^{-3} \tag{2.13}$$

the system (2.11), (2.12) becomes

$$\ddot{w}_2 = -w_2 - \frac{3\dot{R}_3}{R_3^4}, \tag{2.14}$$

$$\ddot{R}_3 = -R_3. \tag{2.15}$$

A further simplification is achieved by means of the transformation^a

$$w_2 = r_2 - \frac{\dot{R}_3^2 - i\dot{R}_3 R_3 - R_3^2}{2(\dot{R}_3^2 + R_3^2)(\dot{R}_3 - iR_3)R_3^2} \tag{2.16}$$

^aEquation (2.14) can be written as

$$\ddot{w}_2 = -w_2 - 3i \frac{c_1 \exp[it] - c_2 \exp[-it]}{(c_1 \exp[it] + c_2 \exp[-it])^4},$$

if one substitutes for the general solution of (2.15). Then its particular solution is

$$-i \frac{3c_1^2 \exp[4it] + c_2^2}{16c_1^2 c_2 \exp[it](c_1 \exp[2it] + c_2)^2},$$

namely

$$\frac{\dot{R}_3^2 - i\dot{R}_3 R_3 - R_3^2}{2(\dot{R}_3^2 + R_3^2)(\dot{R}_3 - iR_3)R_3^2}.$$

for then the system (2.14), (2.15) becomes

$$\ddot{r}_2 = -r_2, \quad (2.17)$$

$$\ddot{R}_3 = -R_3. \quad (2.18)$$

By means of the application of the method of reduction of order, the fourth-order nonlinear equation (1.3), has been reduced to an uncoupled pair of linear equations which represents an isotropic two-dimensional linear oscillator. The process of quantization is now elementary. It is obvious that system (2.17), (2.18) possesses the Lagrangian

$$L = \frac{1}{2}(\dot{r}_2^2 + \dot{R}_3^2 - r_2^2 - R_3^2), \quad (2.19)$$

so that the conjugate momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}_2} = \dot{r}_2 \quad \text{and} \quad P_R = \frac{\partial L}{\partial \dot{R}_3} = \dot{R}_3. \quad (2.20)$$

Under the Legendre transformation

$$H = p_r \dot{r}_2 + P_R \dot{R}_3 - L$$

we obtain the Hamiltonian

$$H = \frac{1}{2}(p_r^2 + P_R^2 + r_2^2 + R_3^2). \quad (2.21)$$

It is then straightforward to see that the corresponding Schrödinger equation is

$$2iu_t = -u_{r_2 r_2} - u_{R_3 R_3} + (r_2^2 + R_3^2)u, \quad (2.22)$$

i.e. once the original fourth-order nonlinear equation (1.3), is reduced by the method of reduction of order to the system of two second-order equations (2.17) and (2.18), the standard, i.e. Dirac's, method of quantization proceeds smoothly.

3. Discussion

A critical feature in the treatment of high-order field theories for the purposes of quantum mechanics is the manner in which the system is reduced to an Hamiltonian system. What we have done here is to demonstrate that even an equation of the form of (1.3), which does not have a Lagrangian and so is not amenable to traditional methods of treatment, can be written in a fashion which leads to the quantization of a physically acceptable system. The purpose of the reduction of the system to a set of first-order equations is to enable the selection of appropriate variables to render the system in terms of the usual second-order equations of classical mechanics from which the process of quantization is easily initiated.

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