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Environmental Management through Coopetitive Urban Waste Recycling in Eco-Industrial Parks

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Abstract: In this article, we consider the coexistence of competing actors within a specific eco-industrial park. The competing firms dynamics evolves by means of an interplay agreement determined among the competitors themselves. In particular, we show a possible scenario in which the selected eco-industrial competitors could greatly benefit from a coopetitive interaction, within their common eco-park, while improving the general conditions of a near residential area. The associated dynamical coopetitive agreement, aims at the growth and improvement of the firms themselves and of their industrial network (within a virtuous environmental path). As an example, we assume the existence of two competitors selling the same good on the same market, so that, from a competitive point of view, we construct a classic Cournot duopoly model upon which we build up a multidimensional coopetitive agreement. Our eco-friendly deal allows to “enlarge the pie” of possible gains by diminishing sunk costs and other forms of costs, especially the environmental costs associated to the management of urban waste recycling. Consequently, we suggest production methods and production quantitative profiles in order to “share the gains fairly”. We show a complete mathematical analysis of our new economic game and show some of its possible and relevant solutions.

Keywords: sustainability; industrial symbiosis; game theory; coopetition; green technologies; industrial ecology

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1. Introduction

In this paper, we apply game theory and coopetition to the study and economic development of eco-industrial parks. In particular, we consider the possibility of coexistence of competing actors within eco-industrial parks, by means of agreements among the competitors themselves.

1.1. Industrial Symbiosis and Eco-Industrial Parks

Industrial ecology explains how to learn from natural ecosystems in order to reduce the environmental impacts of human activities to levels that natural systems could sustain. The use of energy and material is optimized and the generation of waste minimized. Wastes from one process can be used as raw materials in another [1]. Industrial ecology can have a few approaches: “the first approach uses tools like material flow analysis including substance flow analysis, life cycle analysis and design for environment. A local approach is taken in studying eco-industrial parks” (see [2]). An eco-industrial park is defined as “companies and institutes working together and building a production network to develop their environmental and economic performances”. According to this, an eco-industrial park is an “ecosystem where waste minimization is sought in the use of energy and raw materials, material and energy exchange is planned, and economic, ecologic, and social relations are established” [3].

1.2. Literature Review on Industrial Symbiosis and Game Theory

In the scientific literature, we find essentially three different types of game theory approach for the applications in the industrial symbiosis context.

The first type of game theory applications in that field is linked to the theory of cooperative games, games in which the values of individual components are evaluated with respect to possible coalitions of participants (see [4]). An example of game-theory application is presented in Fiestras-Janeiro et al. (2011), where the authors solved cost allocation problems through cooperative games applied to natural resources and power industry [5]. A more recent work of Jin et al. (2020), instead, establishes a game model of government value compensation for different types of industrial waste recycling in industrial parks and examines the standards for recycling firm waste based on maximizing government benefits [6].

A second type of applications concerns non-cooperative finite games, where essentially the scholars are looking for possible balances and optimal gains for the participants in the eco-industrial park (see [7]). This approach leads to articles of a more strict applicative nature but with developments that appear still quite elementary and not widely useful (used) in the general case. An existing model that could be easily extended to analyze competition and cooperation in an eco-industrial park is presented in Attanasi et al. (2012), where the authors analyze environmental agreements among two parties negotiating over environmental standards [8]. Similarly, the model of strategic preservation/developing choice of two parties holding a similar environmental resource presented in [9], could be easily applied to two companies in the same eco-industrial park. Only recently, Yazan et al. (2020) propose a non-cooperative game-theoretical model based on coopetition [10]: a "coopetition" problem where companies need to cooperate to reduce waste discharge costs and traditional input purchase costs and dive into competition to pay a minimum share of additional costs (i.e., waste treatment, waste transportation, and transaction costs) of operating industrial symbiosis.

A third line of research adopts the theory of evolutionary games for the analysis of growth dynamics within eco-industrial parks, in order to understand if an eco-industrial park could grow or shrink over time and which of its components can be successful or losers in the short and medium term. In this case, we also find software and graphics applications, typical of dynamic systems, with a probabilistic approach and characterized by the resolution of ordinary first order differential equations (see [11]). Examples are presented, for instance, in Luo et al. (2019), where the authors construct a dynamic evolutionary game model between three game players of E-commerce industrial park symbiosis system and analyze the replication dynamic equations [12] or in Tilman et al. (2020), where a new general framework for eco-evolutionary games is proposed [13].

1.3. Our Game Theory Approach

We should notice that our game theory approach could be thought as a dynamic game theory approach, in a wide sense. Usually, dynamic games are conceived as sequences of non-cooperative games (discrete case) or curves of non-cooperative games (continuous case), where the indexing parameters are essentially of time nature [14]. Our game model can be considered a generalized dynamic game model, in particular a differentiable curve of non-cooperative games (and in more complex cases it could be a differential parametric manifold in the space of non-cooperative games). Nevertheless, the parameter indexing the parametric curve is the cooperative strategy chosen by the players. This new kind of dynamic games, by its own nature, requires new kind of solutions. The two players can choose the cooperative strategy together and simultaneously optimizing the corresponding manifolds of classic solutions. For example, more specifically, each point-game of the coopetitive game shows a Nash equilibrium, therefore the players can choose inside a curve of infinitely many Nash equilibria. Clearly, they will find an optimization technique to choose the correspondingly optimal Nash equilibrium. Selection of the above type, a selection of an optimal Nash equilibrium, will be called a purely coopetitive solution.

1.4. Novelty of the Approach in an Industrial Symbiosis Context

Therefore, our approach could be considered significantly new because, on the one hand, we use infinite non-cooperative games for the determination of equilibria and Pareto boundaries in a possible realistic competitive interaction outside the eco-industrial park; on the other hand, we introduce a collaborative dynamics to find efficient policies for the expansion of the eco-industrial park and for the improvement of its facilities as well as the environmental protection and respect for public health. Other studies in which the authors have used this new approach applied to different environmental contexts are [15–19].

Our Coopetitive Game Approach

We observe that our approach would fall within the coopetitive game qualitative idea of Brandenburger and Nalebuff (see, e.g., [20,21]). However, we should notice that there is no general consensus in quantitatively modeling such empirical and epistemological idea. In our case, we choose explicitly the modeling suggested by Carfì in 2010 [22]: he essentially (and substantially) suggests that a possible model of Brandenburger and Nalebuff's idea is that of a "parametric manifold of non-cooperative games indexed (parametrized) by a shared cooperative strategy space". Even more complex, when introducing these games, is the problem of providing a definition of coopetitive solution (which cannot be an "equilibrium solution" because of the presence of its cooperative part). In the next subsection, we recall briefly the definition of coopetitive game and some related solutions.

1.5. Coopetitive Games and Dynamic Games

We desire to point out that, even if our game can be considered as a dynamic game, nevertheless, the third strategy, that is the continuous variable z—which represents a common investment in green technologies—is our dynamic parameter: what in usual dynamic games would be time, in our game G is the cooperative strategy z. In this sense, the trajectories we design and construct here are trajectories in a continuous cooperative strategy, not in time. From a mathematical point of view, this economic interpretation doesn't change much, but from an economic point of view the solutions that we need to find rely to a completely different decision problem. We are not searching here for optimal times or optimal noncooperative strategies within a time horizon, but we are searching for optimal cooperative strategies z^*, some Nash equilibria $N(z^*)$ corresponding to those optimal cooperative strategies z^* and finally fair sharings W' of the consequential Nash payoffs $N'(z^*)$. The approach considered in the article relies on graphical and analytical (not probabilistic) descriptions. Our trajectories are solutions of second-order differential equations, but we determine those trajectories not by solving differential equations, rather, by a method proper of game theory: we have analytically determined the curve of all Nash equilibria, depending on the continuous variable z, and simply obtain the objective trajectories as the continuous curves formed by those equilibria and by the payoffs of those equilibria, respectively. The novelty of this approach, for studying eco-park systems, lies not in the use of classic dynamic games, but in the use of new techniques that can be also interpreted in a dynamic fashion. That new approach contains a more specific economic and strategic meaning. That new coopetitive meaning requires the construction of a completely new mathematical toolbox of definitions and procedures, which is proper of the growing field of coopetitive games. Nevertheless, from an economic and intuitive point of view, that new game theoretical toolbox can be fully appreciated and understood without going too deeply into the mathematics.

1.6. Coopetitive Games and Their Solutions: Formal Definitions

A coopetitive game is a game in which two or more players (participants) can interact cooperatively and non-cooperatively at the same time. Even Brandenburger and Nalebuff [21], creators of coopetition, did not define, precisely, a *quantitative way to implement coopetition* in the Game Theory context. The problem to implement the notion of coopetition in Game Theory is summarized in the following question:

how do, in normal-form games, cooperative and non-cooperative interactions coexist simultaneously, in a Brandenburger-Nalebuff sense?

In order to answer the above question, consider a classic two-player normal-form gain game $G = (f, >)$—such a game is a pair in which f is a vector valued function defined on a Cartesian product $E \times F$ with values in the Euclidean plane $\mathbb{R}^2$ and $>$ is the natural strict sup-order of the Euclidean plane itself (the sup-order is indicating that the game, with payoff function f, is a gain game and not a loss game). Let E and F be the strategy sets of the two players in the game G. The two players can choose the respective strategies $x \in E$ and $y \in F$

- cooperatively (exchanging information and making binding agreements);
- non-cooperatively (not exchanging information or exchanging information but without possibility to make binding agreements).

The above two behavioral ways are mutually exclusive, at least in normal-form games:

- the two ways cannot be adopted simultaneously in the model of normal-form game (without using convex probability mixtures, but this is not the way suggested by Brandenburger and Nalebuff in their approach);
- there is no room, in the classic normal-form game model, for a simultaneous (non-probabilistic) employment of the two behavioral extremes *cooperation* and *non-cooperation*.

Carfì [22,23] has proposed a manner to overcome this *impasse*, according to the idea of coopetition in the sense of Brandenburger and Nalebuff. In a Carfì's coopetitive game model,

- the players of the game have their respective strategy-sets (in which they can choose cooperatively or not cooperatively);
- there is a common strategy set C containing other strategies (possibly of different type with respect to those in the respective classic strategy sets) that *must be chosen cooperatively*;
- the strategy set C can also be structured as a Cartesian product (similarly to the profile strategy space of normal-form games), but in any case the strategies belonging to this new set C must be chosen cooperatively.

1.6.1. Two-Player Coopetitive Games

Definition 1. *Definition (of coopetitive game).* Let E, F and C be three nonempty sets. We define as two-player coopetitive gain game carried by the strategic triple (E, F, C) any pair of the form $G = (f, >)$, where f is a function from the Cartesian product $E \times F \times C$ into the real Euclidean plane $\mathbb{R}^2$ and the binary relation $>$ is the usual sup-order of the Cartesian plane (defined component-wise, for every couple of points p and q, by $p > q$ iff $p_i > q_i$, for each index i).

Remark 1. *Remark (coopetitive games and normal-form games). The difference between a two-player normal-form (gain) game and a two player coopetitive (gain) game is the fundamental presence of the third strategy Cartesian-factor C. The presence of this third set C determines a total change of perspective with respect to the usual exam of two-player normal-form games, since we now have to consider a normal-form game $G(z)$, for every element z of the set C; we have, then, to study an entire ordered family of normal form games in its own totality, and we have to define a new manner to study this kinds of game families.*

1.6.2. Solutions of a Coopetitive Games

The two players of a coopetitive game G—according to the general economic principles of *monotonicity of preferences* and of non-satiation—should choose the cooperative strategy z in C in order that:

- the reasonable Nash equilibria of the game G_z are f-preferable than the reasonable Nash equilibria in each other game $G_{z'}$;

- the supremum of G_z is greater (in the sense of the usual order of the Cartesian plane) than the supremum of any other game $G_{z'}$;
- the Pareto maximal boundary of G_z is higher than that of any other game $G_{z'}$;
- the Nash bargaining solutions in G_z are f-preferable than those in $G_{z'}$;
- in general, fixed a common kind of solution for any game G_z, *say* $S(z)$ the set of these kind of solutions for the game G_z, we can consider the problem to find all the optimal solutions (in the sense of Pareto) of the set valued path S, defined on the cooperative strategy set C. Then, we should face the problem of selection of reasonable Pareto strategies in the set-valued path S via proper selection methods (Nash-bargaining, Kalai-Smorodinsky and so on).

Moreover, we shall consider the maximal Pareto boundary of the payoff space $\mathrm{im}(f)$ as an appropriate zone for the bargaining solutions. The payoff function of a two-player coopetitive game is (as in the case of normal-form game) a vector valued function with values belonging to the Cartesian plane $\mathbb{R}^2$. We note that in general the above criteria are multi-criteria and so they will generate multi-criteria optimization problems.

1.6.3. Nash (Proper) Solution of a Coopetitive Game

Let $N := \mathcal{N}(G)$ be the union of the Nash-zone family of a coopetitive game G, that is the union of the family $(\mathcal{N}(G_z))_{z\in C}$ of all Nash-zones of the game family $g = (g_z)_{z\in C}$ associated to the coopetitive game G. We call *Nash path of the game* G the multi-valued path

$$z \mapsto \mathcal{N}(G_z)$$

and Nash zone of G the trajectory N of the above multi-path. Let N^* be the Pareto maximal boundary of the Nash zone N. We can consider the bargaining problem

$$P_{\mathcal{N}} = (N^*, \inf(N^*), \sup(N^*)).$$

Definition 2. *If the above bargaining problem $P_{\mathcal{N}}$ has a Kalai-Smorodinsky solution k, we say that k is the properly coopetitive solution of the coopetitive game G.*

The term "properly coopetitive" means that:

- *this solution k is determined by cooperation on the common strategy set C and to be selfish (competitive in the Nash sense) on the bi-strategy space $E \times F$.*

For a specific study of n-players coopetitive games and of other possible solutions of coopetitive games (bargaining solutions, Pareto compromise solutions, purely coopetitive solution, transferable utility solutions, win-win solutions) we recommend [22,23].

1.7. Motivations and Objectives of the Paper

Why should two companies of the same type, belonging to the same eco-industrial park, establish a coopetitive relationship?

In reality, belonging to an industrial park is a clear collaboration, in the strict sense, that not only benefits the individual elements of the park but also determines positive repercussions on the environment and on the health of the community, within the park and its surroundings. However, the classic literature and theory on the subject does not consider one aspect of the matter: the economic coexistence dynamics of different factories of the same industrial park producing the same good. We focus precisely on what happens dynamically when two products of the same type, produced by two different competitors living in the park, end up for sale on the same market. Of course, two companies of the same type end up competing in the market; this could also generate negative repercussions on the ecological park.

1.7.1. Symmetric Competition and Asymmetric Cooperation

As explained, we model the interaction between two companies "of the same type". They are of the same type in the sense that they produce the same good and sell it in the same market with similar variable costs, according to classical duopoly theory; from an economical point of view, we have chosen to satisfy the general duopoly assumptions, in a classic microeconomics settings (which determines the non-cooperative core of the game). From a cooperative point of view, we shall add a quadratic perturbation acting on the common cooperative strategy space C. Generally speaking, in coopetitive games, the "type of a producer" is defined by the non-cooperative core payoff function (which is a classic Cournot payoff function). However, in our numerical example, we choose the cooperative part in such a way that the second player will show only a linear cooperative perturbation, so the two companies differ in their payoff function: comparing Equation (1) with Equation (2), company 1 has a proper quadratic cooperative-strategy function, while company 2 has a linear one (which is, nevertheless, a particular case of quadratic function). The cooperative part of the payoff function could be put in the form

$$v(x,y,z) = A(z,z^2),$$

for a convenient bi-dimensional square matrix A. We choose, for our numerical example, a more variegated asymmetric matrix A, just to show the versatility of the model: therefore, our model, is symmetric in the first two variables and asymmetric in the third one.

1.7.2. Improvement of the Park and Possible Additional Earnings

We want to demonstrate, in this article, that there exist scenarios in which, not only the competition can be sustained in equilibrium—without negative effects on the park—but we can dynamically conceive a way to improve the park itself. In this respect, the coopetitive approach and game theory help us to build up quantitative models that examines possible strategic scenarios.

We, in our model, will deal exclusively with the interaction between two companies of the same type and their possible additional earnings due to the proposed coopetitive interaction and we will not deal specifically with the positive effects that our approach provides for all the other elements of the park. When we talk about the "possible additional earnings" due to the proposed coopetitive interaction, usually we consider as the reference earnings, those that the two companies would get in the absence of the proposed coopetitive interaction: those coming from a classic non-cooperative solution, that is the equilibrium obtained for the shared cooperative strategy z equal to 0; in other terms, the Nash equilibrium of the "initial" game G_0. In our specific case, since we base our analysis on a classic duopoly, we have a unique Nash equilibrium for every value of the cooperative strategy z and, in particular, for z equal zero. More generally, since the possible coopetitive solutions are determined within convenient "coopetitive paths", starting from convenient "disagreement points" (initial equilibria or other classic choices of defense or menace in game theory), we could consider (and propose) various options in order to measure the "additional" earnings that our proposed coopetitive agreement would allow to the companies. Anyway, in our case, we choose as a starting point for the bargaining the Nash equilibrium of the game G_0.

1.7.3. From Cournot Duopoly to Coopetitive Clusters

Our approach, as we already underlined, is dynamic, since it foresees an evolution that develops within some temporal horizon. In the present model, we have chosen a medium-term time horizon; this modeling choice allows us to use a quadratic translation vector field in addition to the competitive payoff functions. In the present case, our competitive core game comes from consolidated economic models and analyzed in details (see [24–26]): the duopoly models. Therefore, we start from a classic duopoly interaction and then we improve it through cooperative binding agreements between competitors.

The collaborative agreements that we initially propose are positively affecting the entire park and, in the medium term, bring more revenues and/or benefits to the two companies of our game in terms of saving production costs, saving in the more efficient disposal of waste materials.

Moreover, we desire to emphasize how the presence of two (or more) competitors in the same industrial eco-park might dynamically produce beneficial effects on the entire industrial park. We do not analyze quantitatively the positive effects of the two companies' production choices on the whole industrial park, however, from a qualitative standpoint:

- the presence of two similar industries in the same eco-industrial park might determine the critical financial availability for a "technology leap" and, in particular, for a transition to a greener and more efficient production (which represents *per se* an improvement of the eco-park, beyond the possible financial positive effects that our model might forecast for all the other elements of the park);
- the win-win coexistence of more competitive producers in the park might represent an incentive for new potentially innovative enterprises to enter the park, allowing the park itself to grow and improve from technical and ecological standpoints;
- one could conceive of situations in which an industrial park arises precisely from the needs of two companies competing on the same market;
- the presence of more competitors within the same industrial park determines a sort of dynamic imbalance that may translate into economic growth both for the companies themselves and for the various members of the industrial park, because of the greater production capacity of the arising "coopetitive clusters";
- the empowerment of the park determines positive effects on the safeguarding of the local environment, improving the nearby natural resources and contributing to the safeguard of public health.

Our model can be generalized to several companies (coopetitive oligopoly clusters).

2. Study Case: Eco-Industrial Park of Styria

We consider the eco-industrial park of Styria, located in Austria. This is one of the most familiar eco-industrial parks in the literature. "Styria is an industrial zone where there are firms from different sectors with high innovation potential. Primary sectors are paper and wood products, machinery, metal and steel, and automotive. Clustering of firms in a region help them reduce their costs as well as make it possible to establish a network among firms where wastes are used. It is possible to recycle materials such as paper, gypsum, iron parts, used oil, and rubber in the established system. With these non-expensive by-products, costs of the firms are reduced and at the same time, environmental benefits are obtained. Hence, whole region benefits from the established recycling system" [3].

We consider a part of the Styria recycling network (Figure 1)—another excerpt of the real eco-industrial park could be found in [27]—and we focus our attention upon the two producers of the same type, the two paper factories. They collaborate because they belong to the eco-industrial park and in particular the waste paper of the factory 2 is used as input material for the paper factory 1, but they are also competitors because they sell on the same market. We propose two possible agreements among the two competitors, agreements that increase the total gain of the two competitors in the eco-industrial park (specifically, one maximizing the total payoff and at the same time equalizing the two competitors' payoffs and one maximizing the total payoff and at the same time differentiating the two competitors' payoffs proportionally to the financial participation in the initial investment.).

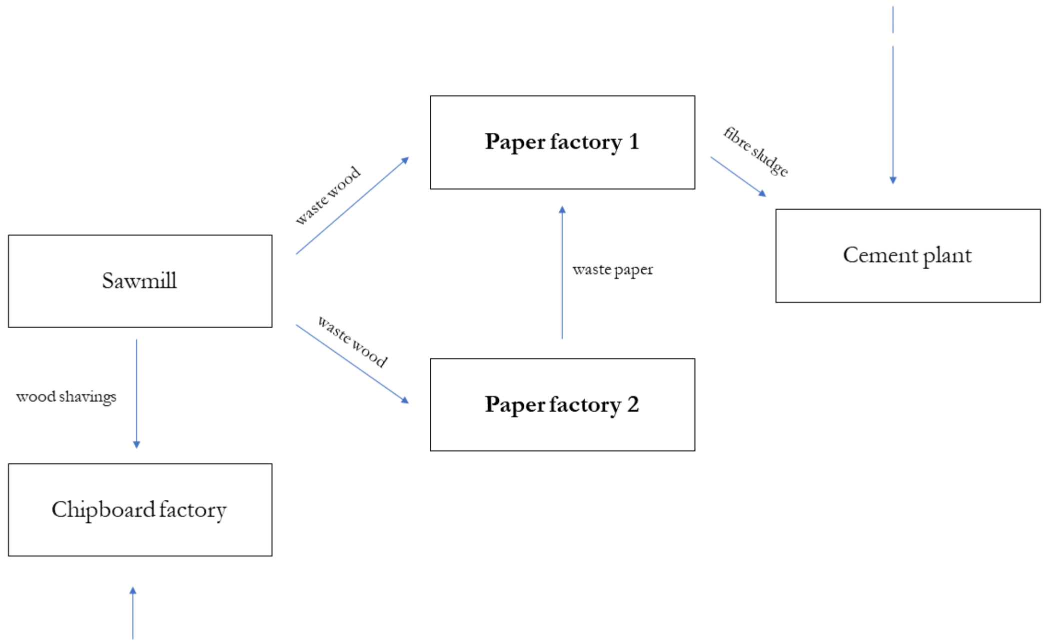

Figure 1. Part of the Styria recycling network.

3. Methods: The Economic Model

The two players of the game are the two paper factories, that invest on the development of the eco-industrial park, and in particular, their cooperative strategy is represented by the strategy z.

3.1. Sketch of the Model

Now we provide a verbal, intuitive sketch of our coopetitive game and of its related solution procedure. First of all, our game G can be defined and viewed as a continuous curve of classic non-cooperative Cournot games (a continuous family of Cournot games). To each noncooperative strategy z, we associate a classic duopoly game G_z. That duopoly game is a symmetric noncooperative game. The payoff functions of our game G reveal simply the payoff functions of the classic duopoly model translated by a quadratic vector field v, determined by a matrix A of coefficients containing information about the efficiency and costs of the new technology. In other terms, our payoff functions are the sum of a classic duopoly function plus a quadratic term.

Fixed a cooperative strategy z, we determine the Nash equilibrium, the classic Nash Equilibrium, of the game G_z, for every cooperative strategy z. So, we have constructed a curve N that we called the Nash path. Now, we need to find the optimal Nash equilibrium by using a reasonable criterium, we choose to maximize the collective gain. The optimal Nash solution with respect to the above reasonable criterium, the criterion of maximum collective gain, provides us the first part of our solution: according to the general idea of cooperation, we have enlarged the pie of possible gains with respect to the initial duopoly game G_0, with zero cooperation, by constructing an infinite set of new Cournot duopoly games which are at virtual disposal of the two industries. After that the two industries choose what duopoly G_z should be played (one that guarantees "more profits" with respect to G_0) and how to share the revenues. Infact, the second part of the solution consists in the sharing of the maximum collective gain, for which we use a classic linear selection.

3.2. *Strategies*

The two players' strategies are:

1. the strategy
$$x \in E := U = [0,1],$$
representing any paper quantity produced by paper factory 1;
2. the strategy
$$y \in F := U = [0,1],$$
representing any paper quantity produced by paper factory 2;
3. the shared strategy
$$z \in C := [0,3](\text{million dollars}),$$
representing the cooperative strategy. The set C is determined together by the two players.

3.2.1. Measure Units

Any shared strategy z in C is expressed in million dollars (it is a shared investment of the two enterprises). Differently, for what concerns the units of measure of strategies x in E and y in F, they are not expressed in million dollars, but they are expressed (as usual in Cournot games) by a normalized quantity unit, chosen once and for all in our game. This normalized unit is equal for both players and reduces the strategy sets E and F to the unit interval $[0,1]$. The maximum 1 is the so called critical quantity of the Cournot game: any player certainly loses by choosing a strategy $s > 1$, independently of the choice of the other player (this is true for the non-cooperative core of our game).

Now, we need to observe three fundamental aspects of the model:

1. the model is asymmetric in z, but it is absolutely symmetric in x and y, so the units of measure of E and F are the same; our asymmetric game G is a curve of symmetric games G_z;
2. the payoff
$$4x(1-x-y),$$
as well as
$$4y(1-x-y),$$
is monetary and should be measured in money units—previous multiplications times unit price have transformed quantities into money. Now, without losing generality, we have assumed that the unit of measure is a million dollar—any other possible measure unit determines only the change of the coefficient 4.
3. we prefer to distinguish, at least at a nominal level, the sets E and F, although they are exactly the same set $U = [0,1]$.

3.2.2. Cooperative Strategy

The strategy z in C is the aggregate investment for the environmental sustainability economic approach, specifically
$$z \in C = [0,3]$$
represents the investments to acquire advanced green technologies for recycling paper waste. This strategy could be implemented by acquiring an advanced paper production machinery that allows to reach sustainable development and create environmentally friendly products.

We remind that, on average, the need to produce a ton of paper pulp is 2.2–4.4 tons of wood, while, for obtaining the same quantity of paper pulp, we need to recycle 1.4 tons of waste paper. Moreover, recycling of 1 ton waste paper will save energy, material and natural resources ("17 trees, 380 gallons of oil, three cubic yards of landfill space, 4000 kilowatts of energy, and 7000 gallons of water, that represent a 64% energy savings, a 58% water savings, and 60 pounds less of air pollution" [28]).

3.2.3. Recap: Interpretation of the Strategies

At this point of the paper, we desire to underline that:

1. any z in C is a possible shared total investment by the two firms (namely decomposable into respective parts $\beta_1 z$ and $\beta_2 z$). We call it "cooperative strategy" because the strategy set C introduces and defines together with the translating vector field v the coopetitive agreement of the two enterprises in its cooperative part (the competitive one is defined by the classic duopoly model).
2. strategies x in E and y in F are the classic Cournot production strategies: they are quantities of production chosen separately and simultaneously by the two players, exactly as in the duopoly game.

We observe again that payoff functions of the duopoly game and the translating vector field v transform the strategy triple (x, y, z) into money. Strictly speaking, the vector field v should be considered as defined upon the entire tri-dimensional coopetitive strategy set P, not only upon the one-dimensional strategy set C, firstly because that is the general case and secondly because mathematics requires to add functions defined on the same domains; for instance, we consider even the non-cooperative part defined upon the entire strategy space P, even if that part is constant with respect to z.

In the case of z, both players control this cooperative strategy, in a shared and consensual fashion. They may contribute asymmetrically to z by different amounts $\beta_1 z$ and $\beta_2 z$, with $\beta_1 + \beta_2 = 1$. The cooperative strategy z is chosen by the two companies together, as their total investment to acquire advanced technologies for recycling paper waste, and this technology, once obtained, is available to both companies and they can use it according to the convex pair β (first player uses the machine for β_1 of the total production time and the second player for β_2 of the total production time).

For what concerns the timing in the game G, we observe that the two players:

1. firstly analyze the general situation,
2. secondly choose the cooperative strategy z,
3. thirdly, they play the fixed duopoly game G_z, determined by the strategy z,
4. finally they share the gains according to the coopetitive solution.

3.3. *Payoff Functions*

For what concerns the duopoly core G_0, we have simply used the payoff function in the reduced form proposed in [25,29,30].

In the payoff function of any player i, paper factory 1 controls x, paper factory 2 controls y, and the two paper factories choose z together and consensually (the two players choose to invest a total aggregate amount z, contributing with respective complementary parts $\beta_1 z$ and $\beta_2 z$). Furthermore, the two values of x and y are chosen independently and simultaneously (as usually happens in non-cooperative game theory), after both of them have chosen together and implemented the cooperative strategy z.

The payoff function of the paper factory 1 is the function f_1 of the parallelepiped $P = E \times F \times C$ into the real line, defined by

$$f_1(x, y, z) \quad = \quad 4x(1 - x - y) + m_1(z - az^2) \tag{1}$$

for every triple (x, y, z) in the parallelepiped P where m_1 is a real parameter representing the interest rate of the investments decided by the two players upon the economic performances of the paper factory 1 (see Section 3.5 for the specific evaluation) and a is a random erosion coefficient.

The payoff function of the paper factory 2 is the function f_2 of the parallelepiped P into the real line, defined by

$$f_2(x, y, z) \quad = \quad 4y(1 - x - y) + m_2 z \tag{2}$$

for every triple (x, y, z) in the parallelepiped P where m_2 is a real parameter representing the interest rate of the investments decided by the two players upon the economic performances of the paper factory 2 (see Section 3.5 for the specific evaluation).

The payoff function of the coopetitive game G is given by

$$\begin{aligned} f(x,y,z) &= (4x(1-x-y) + m_1(z - az^2), 4y(1-x-y) + m_2 z), \\ &= 4(x(1-x-y), y(1-x-y)) + v(x,y,z), \end{aligned} \tag{3}$$

for every triple (x, y, z) in the compact parallelepiped P, where

$$v(x,y,z) = (m_1(z - az^2), m_2 z)$$

defines our translation vector field perturbing the classic Cournot duopoly. We assume the erosion coefficient a equal to 1.5, but it could be considered a stochastic variable.

The Cooperative Translating Vector Field V

We write $v(x, y, z)$, because we need two functions defined over the same strategy space P in order to properly define their summation and also to obtain a function f defined (similarly) on the same space P. Our scalar and the vector fields are correctly and properly defined over the tri-dimensional strategy space P. Nevertheless, we observe that here the vector field v depends properly only on the strategy z. In fact, it is a constant function with respect to x and y (although it is defined on the entire three-dimensional space P). In more complex situations the coefficients m_1 and m_2 could also properly depend on x, y and z, but that is not the case in our model; our general case is well represented by constant coefficients.

3.4. The Game G as a Curve of Infinitely Many Duopoly Games

We desire to underline that the two players are not playing one duopoly game at the time, they should consider all the infinitely many duopoly games G_z, optimize with respect to z, for which we need a reasonable criterium for determining the optimal duopoly game to play, then they should select the Nash equilibrium strategies of that optimal game and furthermore they need to share fairly the collective gain at that Nash equilibrium $N(z^*)$, which can be done also by a Nash bargaining procedure, that is not the problem. The real technical problem is to find the set of all possible Nash equilibria and to find there the "best" Nash equilibrium with respect to a reasonable criterium. Then, we can classically determine a possible compromise. In other terms, the problem in solving a coopetitive game G consists of:

1. analytically determine the Pareto boundary H of all possible focal (incomparable) strategy profiles (for instance, all possible Nash equilibria of all possible classic non-cooperative games forming G);
2. then propose/use a first reasonable selection procedure of the "win-win" solution in H (for instance, the collective maximum criterion);
3. then determine the way to share the "win-win" solution (for example a linear selection, Kalai selection, a Nash bargaining selection... and so on...).

Dealing with infinitely many games at a time, at our best knowledge, we cannot use any simpler way to determine the final solution $(N(z^*), W')$.

3.5. Determination of the Interest Rate of the Investment M

From Figure 1, we can see that the paper factory 2 of the eco-industrial park produces paper starting only from the waste wood of the sawmill, while the paper factory 1 produces paper from the waste wood of the sawmill and from waste paper of the paper factory 2. So, let us assume that the input materials are free for the two paper factories. If the paper factories desire to produce a greater quantity of paper (according to the market demand), with the same available waste material as input, we propose the joint investment for the purchase of a more efficient, sustainable and modern machinery producing paper from recycled paper.

Moreover, we assume that the waste paper could come from cities in the nearby of the eco-industrial park that enter into an agreement with the paper factories and deposit their paper waste at a zero price for the factories, while the cities could avoid landfill (which as we know implies not only high management costs, but more and more heavy and unsustainable environmental issues).

3.5.1. Description of the New Recycling Cycle

Consequently, we assume that a quantity q_w of free waste paper comes from the near cities and from the eco-industrial park in a year. From the quantity q_w, one new machine produces another quantity q_e of recycled paper in a year. The quantity q_w never appears in the analysis. We assume here, that the quantity nq_e produced by n machines (bought by the enterprises by means of z) can be sold on the market at a price p_e (price proposed by the two enterprises themselves and obtained by minimizing the company's costs which depend on the input quantity q_w). In other terms, we assume the existence of an equilibrium pair (p_e, q_e) in the market, the equilibrium pair price-quantity coming from the intersection of a possible market demand multi-curve and an enterprise's supply multi-curve. Here, we desire to observe that, in our model, we assume the old production technology described by the classic duopoly and the new production described by the matrix A strictly separated from an economic point of view. In other terms, we assume that the two enterprises maintained the old production together with the new one.

With this, we have that q_e is the production of one machine, which is completely sold at the price p_e, the quantity q_e is aggregately sold by both players (player i sells $\beta_i q_e$ of the total quantity q_e), so the pair (p_e, q_e) does not depend on i because it is not an individual pair but an aggregate pair. In other terms, the aggregate supply of the two enterprises (coming from the new production technology) is q_e for any machine production at unit price p_e. On the other hand, m_i depends on i exactly by the factor β_i, because that factor specifies the quantity sold by player i in the market, the quantity $\beta_i q_e$ produced by one machine. The quantity q_e is the annual production of one new machine, in particular $n\beta_i q_e$ is the annual production of all n machines sold in the market by the i-th player, but, this does not mean that x is equal to $\beta_i q_e$, because not all the production of the first factory depends on the new machines (x is the old paper production of factory 1 that coexists together with the new production $n\beta_i q_e$). Similar remark holds for the second player.

3.5.2. Analytic Expression of the Coefficient M_i

We calculate now the annual interest rate m_i of the i-th player, which is independent of z; m_i depends on i by the factor β_i. The payoff asymmetry in z is not relevant here because m_i depends only on one machine, that belongs to both players in proportion β.

The annual interest rate of this investment is

$$\begin{aligned} m_i &= \beta_i(np_e q_e - z - c(p_e)nq_e)/z \\ &= \beta_i(p_e q_e z/p_m - z - c(p_e)q_e z/p_m)/z \\ &= \beta_i(p_e q_e/p_m - 1 - c(p_e)q_e/p_m), \end{aligned} \tag{4}$$

where:

- n is the number of the recycling machines bought by the investment z;
- p_m is the price of one machine, so that, $n = z/p_m$;
- p_e is the unit selling price of the paper in \$/tons (in the examined year);
- $p_e q_e$ is the revenue of the paper factories from selling a quantity q_e of paper at price p_e;
- nq_e is the paper produced by n machines in a year;
- z is the total investments of the enterprises in acquiring the recycling machines, which is our cooperative strategy;

- $c(p_e)$ is the marginal cost of production, depending on the selling price p_e. We assume that the value $c(p_e)$ belongs to the interval

$$[0,1]p_e = [0, p_e].$$

The marginal cost $c(p_e)$ represents an erosion coefficient containing information about taxes, general costs of the factory, cost of using electric power and fresh water;
- β_i is the percentage of the common investment paid by player i = 1, 2;
- in other terms, we assume that a choice of a shared investment z determines a state of the market and, consequently, an equilibrium pair (p_e, q_e).

3.5.3. Possible Dependence of M_i upon Z

We note that the dependence of m_i upon z is only apparent in the above Equation (4), which is the definition of those coefficients. The two functions m_i reveal constant functions, so that we identify those constant functions with their respective values. We shall clarify better later.

As we tried to explain before, all the part of the payoff space determined by the cooperative strategy shows a general quadratic behavior for both players, but, for sake of simplicity and in order to take into account the asymmetric economic situation of the two industries—for which the first enterprise will possibly receive less material from the second, because the second one will use part of the material previously devoted to the first enterprise for himself—we decided an asymmetric matrix A. We desire here to underline that our model clearly, in general, can easily handle any possible choice of A, in particular those choices in which both enterprises possess a proper quadratic form of the cooperative payoff. The asymmetry, in our case, derives exactly from the different role played by the two enterprises initially: without coopetitive strategy, the first one receives also from the second one; on the contrary, the presence of the coopetitive strategy could diminish the free row material coming from the second one, but we underline that the choice of A is just for the sake of example.

3.5.4. The Subdivision Strategy β

We hypothesize that the investment z is subdivided into two additive terms according to a convex pair of percentages (β_1, β_2), with

$$\beta_1 + \beta_2 = 1.$$

We don't assume that the two players should contribute symmetrically to the investment z, we find a symmetric assumption too restrictive. Moreover, we shall see that the choice of the erosion coefficient a will determine uniquely the optimal subdivision strategy β, solving the entire decision problem.

3.6. Numerical Example

Table 1 shows production details of a machinery that produces A4 paper from paper waste [31].

Moreover, we assume a total production quantity per machine q_e equal to 4000 tons (in a year) at equilibrium price p_e of 150$ per tons (by "equilibrium price" we means simply that at unit price p_e the market is capable of acquiring the entire production q_e).

We assume that the marginal cost for producing p_e is equal to

$$c(p_e) = 0.3p_e.$$

Table 1. Detailsof production of the machinery. Source: [31].

Name	A4 Paper Production Unit
Price	60,059 USD –150,000 USD
Product type	A4 paper, photocopier paper
Power (W)	100–300 KW
Warranty	1 years
Applicable industries	Production plant, Paper industry
Place of origin	Henan, China (Mainland)
Brand	Leizhan
Voltage	380 V
Certification	CE, BV, ISO
Production capacity	20–270 t/d
Raw material	Paper waste, virgin pulp
Paper output	A4 paper, copy paper, office paper

So, assuming the maximum possible price for one machine (p_m = 150,000 \$), Equation (4) becomes

$$\begin{aligned} m_i &= \beta_i(p_e q_e/p_m - 1 - c(p_e)q_e/p_m) \\ &= \beta_i(0.7 p_e q_e/p_m - 1) \\ &= \beta_i(0.7 \times 150[\$/t] \times 4000[t]/150,000[\$] - 1) \\ &= \beta_i(2.8 - 1) \\ &= 1.8\beta_i. \end{aligned}$$

We assume, for the sake of example, that the two paper factories contribute to the investment z according to a percentage pair

$$(\beta_1, \beta_2) = (1/3, 2/3).$$

Remark 2. *Our model can analyze completely the situation for any choice of the convex pair*

$$\beta = (\beta_1, \beta_2).$$

To exemplify, we have chosen the pair $(1/3, 2/3)$*. One possible economic justification of the chosen scenario would be that the payoff function of the first player is properly quadratic in* z *with a negative coefficient of* z^2*, but we desire to underline that here the choice of the convex pair* β *is exogenous (it is another decision parameter of the model) and the investment of the first player reveals, in any case, repaid by the convenient interest rate* m_1*. We will see later that, fixed the erosion coefficient* a*, the optimal choice of the strategy* β *will reveal uniquely determined.*

3.7. Recap of the Coopetitive Game G

- The game proposed is a particular case of a coopetitive game which is symmetric in the first two variables, indeed, fixing any strategy z, the game is a classic symmetric duopoly game.
- The dependence on z is, in the general case, quadratic for both players, even if in some cases some quadratic dependence may reveal indeed linear, since the coefficient of the z^2 may vanish.
- The Nash equilibrium path N, path of all Nash equilibria, is a constant path when projected in $E \times F$: the projection contains only the bi-strategy $(1/3, 1/3)$, because we do not consider the third strategy z. In the strategy space P, the Nash equilibrium path N is the parametric straight-line associating to any z the triple

$$N(z) = (1/3, 1/3, z).$$

- On the other hand the Nash payoff path is represented in a two-dimensional payoff space and it is the parametric curve N' associating to any z the pair

$$N'(z) = (4/9, 4/9) + (m_1(z - az^2), m_2 z).$$

- When we fix a strategy z, we obtain indeed a symmetric Nash equilibrium; we obtain the classic symmetric Nash equilibrium and we could have expected that because, when we fix the cooperative strategy z, we are dealing simply with a symmetric duopoly. On the other hand, the asymmetry of the Nash payoff N path depends only on the strategy z.

4. Results

4.1. Study of the Game

Fixed a cooperative strategy z in the interval $C = [0, 3]$, the game

$$G(z) = (g_z, \geq),$$

with payoff function g_z, defined on the square U^2 by

$$g_z(x, y) = f(x, y, z),$$

is the translation of the game G_0 by the vector field

$$v(x, y, z) = (m_1(z - az^2), m_2 z),$$

so that we can study the game G_0 and then we can translate the various information of the game G_0 by the vector $v(x, y, z)$.

We show, in Figure 2, the payoff space of the game G_0. Then, we translate the payoff space of the game G_0 by the vector $v(x, y, z)$, and obtain the coopetitive dynamical path of the initial payoff (see Figure 3).

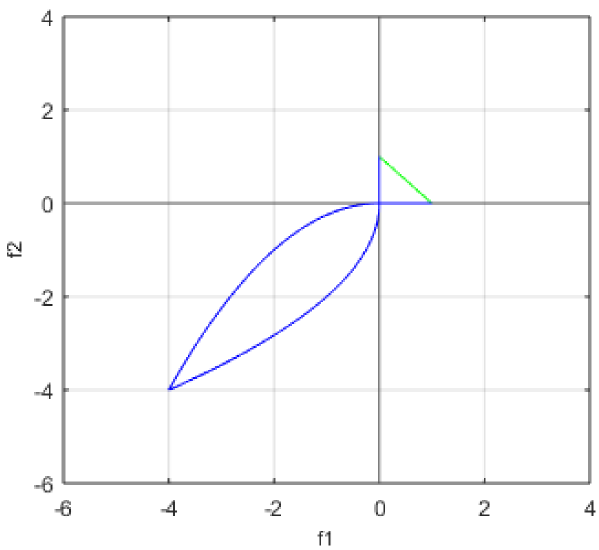

Figure 2. Payoff space of the game G_0.

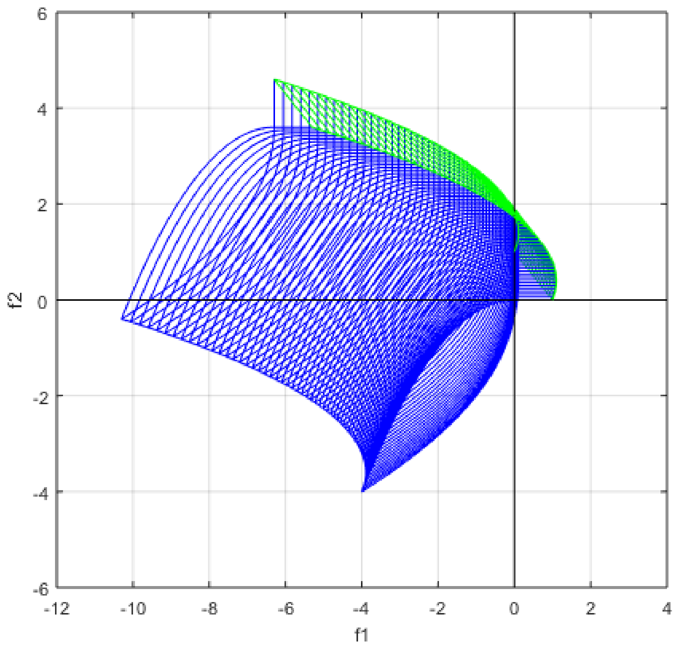

Figure 3. Payoff space of the game G.

4.2. Payoff Universe, Payoff Spaces and Their Interpretations

We just remind that the space represented in Figure 2 is the payoff universe of the game G_0, it is a two dimensional Euclidean space $\mathbb{R}^2$ in which the first coordinate represents the payoffs of the first player and the second coordinate represents the payoffs of the second player. The blue and green lines in the payoff universe constitute the boundary of the payoff space of the game G_0: all possible payoffs in the Cournot duopoly G_0 lie inside the above boundary. It is clear that the zone in which both players gain is the triangle contained into the first quadrant, while the other part of the payoff space condemns the players to certain losses.

4.3. Possible Coopetitive Solutions

We propose two possible solutions of the game: in a purely coopetitive fashion, we provide the strategy-sharing solutions

$$(N(1), T')$$

and

$$(N(1), W'),$$

although we think the second is more adequate to solve our problem.

They are the pairs in which:

- the first member $N(1)$ is a Nash equilibrium ($N(1)$ is the Nash equilibrium

$$(1/3, 1/3, 1),$$

according to the notation of the Nash equilibrium path N);
- the second member is the sharing between the two players of the collective gain

$$f_1(N(1)) + f_2(N(1));$$

- the Nash equilibrium $N(1)$ is obtained by maximizing the collective payoff function upon the parametric curve (or trajectory) N determined by all the Nash equilibria of the coopetitive game G (this maximum point is attained precisely at the value 1 of the parameter z, as we can easily prove);

- the second member T' (or W') of our solution is determined by a Kalai-Smorodinsky procedure, specifically, we solve a compromise decision problem on the Pareto boundary, which is nothing but the straight-line of the maximum collective gain passing through the point
$$N'(1) = f(N(1))$$
and by the threat point $N'(0)$.

4.3.1. Geometric Determination of the Purely Coopetitive Solution $N(1)$

The purely coopetitive solution $N(1)$ [23] can be obtained also by maximizing the intercept k of the straight-line
$$X + Y = k,$$
(which represents the collective gain of the payoff pair (X, Y)) upon the curve N' of Nash points (the yellow curve in Figure 4), in other terms we consider the highest intercept k for which the above straight-line touches the yellow curve. In our context, we consider the parametric curve
$$N' : C \to \mathbb{R}^2 : z \mapsto f(N(z)),$$
which is the image of the parametric curve N determined by all the Nash equilibria of the parametric game G. We observe, in fact, that every member $G(z)$ of our parametric game G offers one unique Nash Cournot equilibrium
$$N(z) = (1/3, 1/3, z)$$
whose payoff $N'(z)$ is $f(N(z))$. In particular, we will see that
$$N(1) = \left(\frac{1}{3}, \frac{1}{3}, 1\right),$$
while
$$N'(1) = f(N(1)) = \left(\frac{4}{9}, \frac{4}{9}\right) + \left(-\frac{3}{10}, \frac{6}{5}\right) = \left(\frac{13}{90}, \frac{74}{45}\right).$$

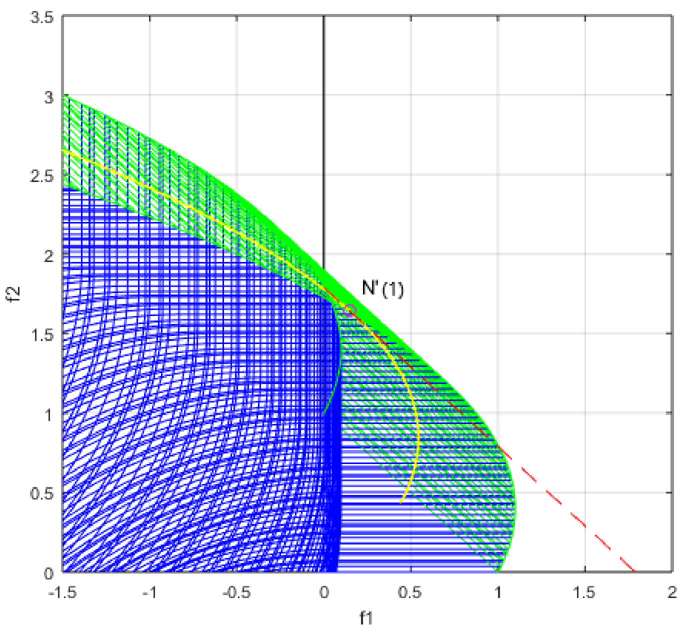

Figure 4. Maximum collective gain Nash payoff.

4.3.2. Remark: Curve of Cournot Duopolies and Its Nash Path

Throughout the paper, we define our game as Cournot-like in the sense that it is a parametric family, continuous family, of classic Cournot duopoly models. In this regards, by better highlighting our notations, we have constructed a corresponding continuous family of Nash equilibria

$$N(z) = (1/3, 1/3, z)$$

and associated payoffs

$$N'(z) = f(N(z)).$$

It is extremely natural to examine and analyze a coopetitive game as a family of non-cooperative games and, correspondingly, to analyze the family of the Nash equilibria of games forming the coopetitive game, viewing that Nash family as a locus of possible candidate-solutions of the game G itself. For example, here, our purely coopetititve solution is chosen among the locus of all Nash coopetitive equilibria by a procedure of optimization based on the cooperative strategy z.

4.3.3. Symmetries of the Purely Coopetitive Solution

Here we underline that the 1 in $N(1)$ refers to the optimal cooperative strategy

$$z^* = 1.$$

We, moreover, underline that N is a function of z, it is a parametric curve containing all the Nash equilibria of the game G in the form

$$N(z) = (1/3, 1/3, z).$$

Then, for $z = 1$ we have

$$N(1) = (1/3, 1/3, 1),$$

where $(1/3, 1/3)$ is the Nash equilibrium of the game $G(1)$. We observe that $(1/3, 1/3, 1)$ represents a symmetric Nash equilibrium of the game G, and this is possible because the asymmetry between company 1 and company 2's payoff functions (respectively, Equations (1) and (2)) holds only with respect to the cooperative strategy z while the two payoff functions are perfectly symmetric with respect to non-cooperative strategies x and y. Indeed, the asymmetry of the two payoff functions, at the level of the solution, is completely contained in the optimal value $z^* = 1$ and in the sharing T' or W'. On the other hand, there is no other natural and straightforward candidate as equilibrium solution of our game G.

4.3.4. Economic Interpretation of the Purely Coopetitive Solution

The payoff vector

$$N'(1) = (13/90, 74/45)$$

means that, after deciding together the cooperative strategy $z = 1$ (because it maximizes the collective gain upon all possible Nash equilibrium payoffs), the first player gains

$$X^* = f_1(N(1)) = 0,1\bar{4} \text{ (million dollars)},$$

while the second player gains

$$Y^* = f_2(N(1)) = 1,6\bar{4} \text{ (million dollars)}.$$

We underline that the point $N'(1)$ is the maximum collective gain Nash payoff. The maximum collective gain upon the Nash path is indeed

$$X^* + Y^* = 1,7\bar{8} \text{ (million dollars)}.$$

We should notice that $N'(1)$ maximizes the payoff sum $X+Y$ on the path N', but it does not provide (by no means) a fair subdivision of the collective gain: we need to find a partition of $X^* + Y^*$.

4.3.5. Transferable Utility Solutions

The relation between $N'(1)$ and T' (or W') is simply that they have the same collective gain, but T' nor W' are Nash (vector) payoffs: they provide a fair division of the collective gain in $N(1)$. We follow the usual procedures in bargaining problems: we determine a point belonging to a Pareto boundary which condensates within itself the substance of the deal. In our case, the Pareto boundary is the straight line of maximum collective gain, the procedure we used to select a possible point belonging to that straight line is a linear selection procedure (see Appendix A for the direct reachability of the payoff T').

4.3.6. Purely Coopetitive Solutions and Super-Cooperative Solutions

We should moreover underline that $N'(1)$ is what the two companies obtain by cooperating on strategy z and (just) competing on strategies x,y, while T' and W' represent further cooperation to find a fair subdivision of the gains. The ultimate aim of coopetition is to enlarge the pie of possible gains and share it fairly.

We finally note that if the players would cooperate both on z and x, y, that would provide another even better solution, which, however, would be used in order to construct another linear selection of fair division.

4.3.7. Threat Points and Fair Sharing

The second member of our solution, that comes from the threat point

$$M' = N'(0)$$

and which is the intersection of the maximum collective gain straight-line

$$X + Y = \frac{161}{90}$$

with Kalai-Smorodinsky straight-line

$$X = Y$$

(in Figure 5) and

$$M' + \mathbb{R}(1/3, 2/3)$$

(in Figure 6), are the two payoff profiles

$$T' = \left(\frac{161}{180}, \frac{161}{180}\right) \approx (0.89, 0.89)$$

and

$$W' = \left(\frac{201}{270}, \frac{282}{270}\right) \approx (0.74, 1.04).$$

We desire to stress that the maximum collective gain Nash payoff $N'(1)$ is the maximum of the Nash payoff curve N' with respect to the total preorder induced by the collective payoff function $f_1 + f_2$, when we identify the parametric curve N' with its graph.

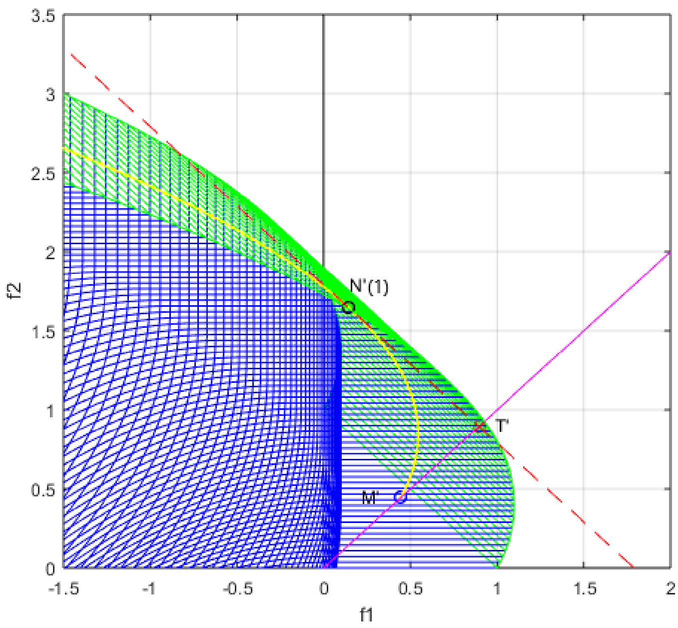

Figure 5. Sharing solution T'.

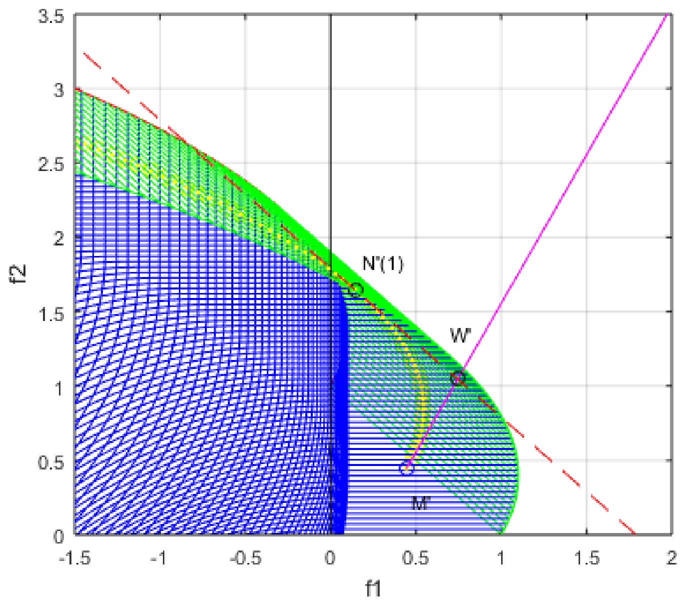

Figure 6. Sharingsolution W'.

5. Discussion

5.1. Results Discussion

The results of the mathematical study prove that, fixed an investment sharing β, we can find two possible coopetitive solutions: the pairs

$$(N(1), T')$$

and

$$(N(1), W').$$

They are the pairs in which the first member $N(1)$ is one Nash equilibrium and the second member is the sharing between the two players of the collective gain

$$f_1(N(1)) + f_2(N(1)).$$

Those solutions reveal advantageous both for the firms involved, for the eco-industrial park, for the environment nearby the park, for the cities cooperating with the park.

Definition of a Transferable Utility Coopetitive Solution

We desire to underline here that a transferable utility coopetitive solution, in general, can be configured as a hybrid-pair in which the first component belongs to the strategy space of the game and the second component belongs to the payoff space of the game. The first component indicates what kind of actions the two players must adopt, individually and collectively, in order to obtain the payoff indicated by the second component of the solution itself. Clearly, we cannot obtain the second component T' (or W') by a mere transformation of the strategy $N(1)$ by the payoff function f, because a coopetitive solution requires a two stage process: firstly we optimize a collective gain upon a reasonable enlarged pie, secondly we teach to the players how to fairly share the maximum collective gain just obtained.

5.2. Model Discussion

Our approach can be considered also dynamic in time, therefore it foresees an evolution that develops within some temporal horizon. In the present model, we have chosen a medium term time horizon; this choice allows us to use a quadratic translation vector field in addition to the competitive payoff function, because the cooperative strategy z can affect the payoffs of the two enterprises acting on the level of revenues and costs.

Remark 3. *We desire to observe that*

- *firstly, in the long term very plausibly we should use more powers of z and greater dimensions for the coefficient matrix A;*
- *secondly, in the medium term, we use a translating vector field of the form*

$$v(x, y, z) = A(z, z^2),$$

for any possible choice of the (2, 2) matrix A, which, as it could be readily understood, in the majority instance provides vector fields with both components properly quadratic, sometimes gives only one properly quadratic component and in a negligible remaining cases provides a linear translation, depending on the coefficients in A: the point is that our model can describe faithfully any possible choice of A and our methods can analyze any possible consequence in details.

5.2.1. Econometric Determination of the Technical Matrix A

We desire however to specify that we have used, as an example, a case in which one out of the two companies of our game is properly quadratic and the other is linear, because, in the real case of the considered industrial park, the two industries are slightly different in the way they receive waste paper. It remains, however, intended that the coefficients in the matrix A should be evaluated in the actual and real state of the industrial area by a careful examination of the real costs and economic frictions of the two enterprises. So, the present values of A must be intended only for the sake of example.

5.2.2. Indirect Positive Effect on the Entire Park

The coopetitive agreement, that we propose, positively affects the entire park because:

- it shows how to compensate the negative effects of the competition inside the park;
- it allows possible technology leaps towards a more efficient model of waste disposal, by new innovative machines;
- in the medium term, it brings more revenues and/or benefits to the two companies (or possible competitive clusters) of our game, in terms of production costs and more efficient disposal of waste materials (because all the coopetitive agreement is founded

upon the acquisition and use of new technologies capable to mitigate the production costs and bringing a more efficient and profitable disposal of waste materials).

Although we had quantitatively and particularly focused on the two companies' coopetitive agreement, and related optimal strategies, together with the fair sharing of the acquired earnings, we desire to stress that:

- all the production costs and market information of the competitive base game G_0 (that is, "status quo" before the additional cooperative agreement) is contained in the g_0 payoff vector function, which is the classic payoff vector function of the duopoly;
- all the production costs, economic values and market information of the coopetitive agreement interaction G (continuous curve of non-cooperative symmetrical Cournot games) is contained in the g_0 payoff function and in the matrix A, which should be determined by econometric and statistical analysis upon the real status of the two industries, times by times.

5.2.3. Efficient Disposal of Waste Materials and Technology Leap

For what concerns the efficient disposal of waste materials, any information about the new more efficient and more green use of the paper wastes produced inside the industrial park and in the vicinity of it (i.e., towns near the industrial park or other nearby facilities) is completely contained in the features of the machines that constitute the technological core of the agreement itself.

The choice of the machine is a decision problem which we consider solved and we can reduce that choice to the amount z of money which serves to buy n machines.

The quantitative information about the technical efficiency of the agreement changes with the machine and ultimately it is contained again in the matrix A.

Although we quantitatively do not compute explicitly all the positive effects on the entire park (if we exclude the more efficient and more sustainable use of the wastes inside the park by the new technology), we observe that, from a qualitative point of view, we have constructed a mathematical model that allows a rational and controllable way to enlarge the industrial park in a semi-free market regime and improve the technological level of the park itself towards a greener and more sustainable status.

5.3. Sensitivity Analysis: Optimal Strategy β^* and Z^*_β and General Optimal Solutions

5.3.1. The General Solution in Analytic Form

We recall that the Nash payoff path is the parametric curve N' associating to any z the pair

$$N'(z) = (4/9, 4/9) + (m_1(z - az^2), m_2 z).$$

Let $\gamma \in \mathbb{R}_+$ be such that

$$m_i = \gamma\beta_i.$$

The collective gain at the Nash equilibrium $N(z)$ is

$$\mu(z) := f_1(N(z)) + f_2(N(z)),$$

that is the function μ defined by

$$\mu(z) = 8/9 + m_1 z - m_1 a z^2 + m_2 z = 8/9 + \gamma z - m_1 a z^2,$$

where

$$\gamma = m_1 + m_2.$$

The first derivative of the function μ is defined by

$$\mu'(z) = \gamma - 2m_1 a z = \gamma - 2(\gamma\beta_1) a z,$$

for every $z \in C$, which is greater than 0 if and only if

$$z < \frac{1}{2\beta_1 a},$$

since $\gamma > 0$. So, the maximum point of μ is the strategy

$$z^*_\beta = \frac{1}{2\beta_1 a},$$

if

$$\frac{1}{2\beta_1 a} \leq 3,$$

that is if

$$\beta_1 \geq \frac{1}{6a}.$$

Otherwise, the maximum z^* is 3. Therefore, the optimal Nash equilibrium, after fixing the vector β, is

$$N(z^*_\beta) = \left(1/3, 1/3, \frac{1}{2\beta_1 a}\right)$$

for all $\beta_1 \in [1/(6a), 1]$ and

$$N(z^*_\beta) = (1/3, 1/3, 3)$$

for $\beta_1 < 1/(6a)$. The collective gain, in the first case, is

$$\begin{aligned} \mu(z^*_\beta) &= \frac{8}{9} + \gamma z^*_\beta - \gamma \beta_1 a (z^*_\beta)^2 = \\ &= \frac{8}{9} + \gamma \frac{1}{2\beta_1 a} - \gamma \beta_1 a (\frac{1}{4\beta_1^2 a^2}) = \\ &= \frac{8}{9} + \frac{\gamma}{4\beta_1 a}. \end{aligned}$$

In this first case, the term

$$\gamma/(4\beta_1 a)$$

is greater than 0 and that the maximum value of $\mu(z^*_\beta)$ is attained at the minimum possible value of β_1, which is

$$\beta_1^* = \frac{1}{6a}.$$

The maximum value of the function associating with β the above value $\mu(z^*_\beta)$, is the value

$$\begin{aligned} \mu(z^*_{\beta^*}) &= \frac{8}{9} + \frac{\gamma}{4\beta_1^* a} = \\ &= \frac{8}{9} + \gamma \frac{6a}{4a} = \\ &= \frac{8}{9} + \frac{3}{2}\gamma. \end{aligned}$$

5.3.2. Example

Provided that:

$$\gamma = 9/5 \text{ and } a = 3/2,$$

we have

$$\beta^* = (\beta_1^*, \beta_2^*) = \left(\frac{1}{6a}, \frac{6a-1}{6a}\right) = \left(\frac{1}{9}, \frac{8}{9}\right)$$

and

$$z^* = \frac{1}{2\beta_1^* a} = 3.$$

The collective gain $\mu(z^*_{\beta^*})$ is

$$\mu(z^*_{\beta^*}) = \frac{8}{9} + \frac{3\gamma}{2} = \frac{323}{90} = 3,5\bar{8} \text{ (million dollars)}.$$

If, however, the two players cannot invest the optimal $z^* = 3$ and they can invest only $z = 1$ (million dollars), as in our game, we obtain a non-optimal β_1, that is, for fixed $a = 3/2$,

$$\beta_1 = \frac{1}{2z^* a} = \frac{1}{3},$$

and so the pair β is equal to

$$\beta = \left(\frac{1}{3}, \frac{2}{3}\right).$$

5.3.3. Graphical Representation for Every Value of β_1

Here we represent, in Figure 7, the payoff space of the game G for $\beta_1 = 0, 1/2, 1$.

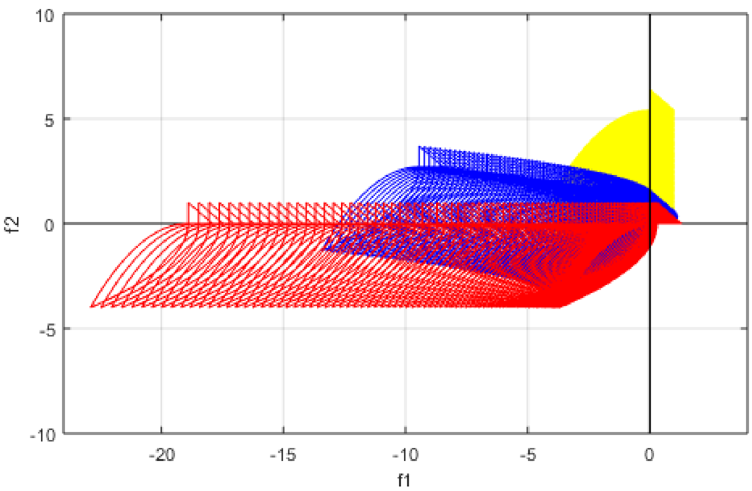

Figure 7. Payoff spaces for three values of β_1 and $z \in [0, 3]$.

In Figure 8 we represent the payoff space of the game G for $\beta_1 \in [0, 1]$ and the Nash payoff trajectories.

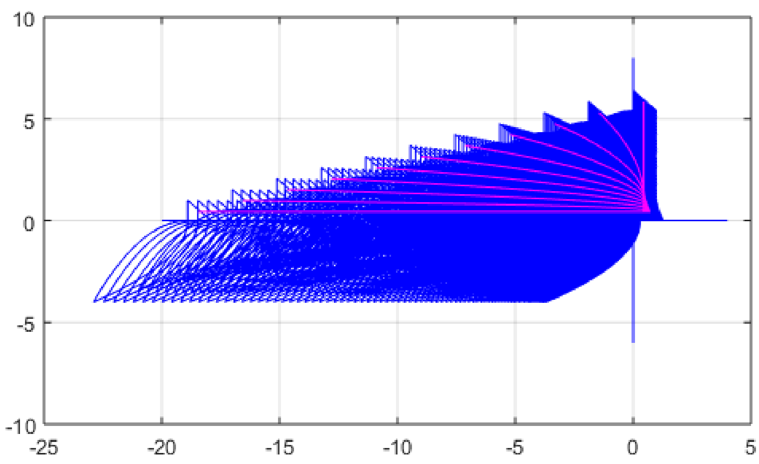

Figure 8. Payoff spaces and Nash trajectories for different values of β_1 and $z \in [0, 3]$.

In Figure 9 we represent the Nash trajectories of the game G for $\beta_1 \in [0, 1]$ together with the optimal point $N(z^*_{\beta^*})$. Matlab Code showed in Appendix B.

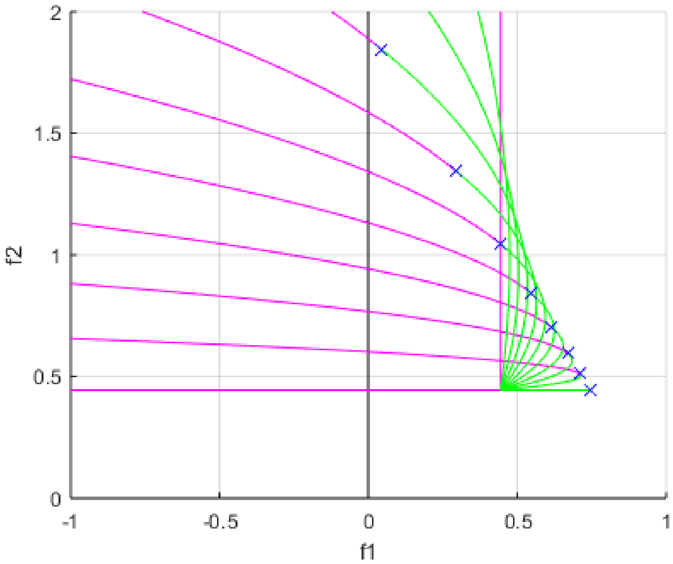

Figure 9. Payoff Nash trajectories and related optimal point $N'(z^*_{\beta^*})$.

5.4. Limitations and Future Research Directions

A limitation, as we previously pointed out, regards the coefficients of the matrix A. They should be evaluated in the actual and real state of the industrial area by a careful examination of the real costs and economic frictions of the two enterprises. So, the present values of A must be intended only for the sake of example.

Our game theory approach could also consider quantitative information that we have not yet included in the initial network constituting the park, in terms of costs, revenues, savings on raw materials and waste disposal.

Moreover, we would, in future research, propose additional agreements that encourage the participation of other firms and the development of the entire eco-industrial park in order to increase the total gain of any enterprise belonging there. In fact, our model can be generalized to several companies. We have proposed possible agreements among the two paper factories that increase the total gain of the two competitors with respect to the earnings they would obtain without any coopetitive agreement. The final net bi-gain of the agreement in one year is the vector

$$W' - M' \text{ (or } T' - M')$$

which is a vector strictly greater than the null vector.

We desire to stress that, before the agreement, we have a duopoly G_0, while after the coopetitive agreement we have a greatly enlarged game-curve G. Our future research proposal foresees a proof of the increase in the total gain of any (new) competing company (entering and) belonging to the park, thanks to the capacity of the coopetition to enlarge the economic pie.

Oligopoly Theory and Coopetition

The increase in total gain of any competing company in oligopoly conditions is not inconsistent with the predictions of oligopoly theory, which clearly indicates that under Cournot competition new firms entering the market will reduce the individual payoff of each firm. Indeed, letting

$$W' - M'$$

be the net bi-gain from passing from duopoly to our previous coopetitive agreement, we put a third competitor on the industrial park, so we pass from M' to a Nash equilibrium M'' of the 3-poly, with

$$M'' < M';$$

we now could easily imagine the possibility to acquire more machines with more efficient technology represented by an interest rate vector m (or to develop such new more efficient technology) that can generate a payoff triple

$$(X^*_m, Y^*_m, Z^*_m),$$

whose sum can be greater than the sum of the pair

$$W' - M'$$

plus the theoretical 3-poly payoff Z^{**} of the new enterprise. It should exist a theoretical efficiency m of the machine such that:

$$(X^*_m + Y^*_m + Z^*_m) - 3Z^{**} > \sum(W' - M') + Z^{**}.$$

Anyway, the above inequality represents the condition of feasibility for the entering of a third enterprise in the industrial park starting from an already successful coopetitive agreement of two players. If, *ab initio*, we have no previous agreement between two industries, we have only to satisfy the below inequality:

$$(X^*_m + Y^*_m + Z^*_m) > 3Z^{**},$$

which is more easily possible.

As other possible future research directions, to stress that our approach is a dynamic one, our model can be generalized in order to study the evolution over time of the payoff of each factory by using a stochastic process such as a Markov process (as in, e.g., D'Amico et al., 2021) [32] or more complex processes.

6. Conclusions

Our coopetitive model has considered the possibility of coexistence of competing actors within a specific eco-industrial park, by means of possible agreements among the competitors themselves.

In particular, we have showed a possible scenario in which the selected eco-industrial competitors could greatly benefit from a coopetitive interaction, within their common eco-park, while improving the general conditions of a near residential area.

The associated dynamical coopetitive agreement has aimed at the growth and improvement of the firms themselves and of their industrial network (within a virtuous environmental path).

We have assumed the existence of two competitors selling the same good on the same market, so that, from a competitive point of view, we have constructed a Cournot duopoly as a base upon which we have build up a multidimensional coopetitive agreement. This policy concerns investments to acquire advanced green technologies for recycling paper waste.

Specifically, we model the interaction between two companies "of the same type" (in the sense that they produce the same good and sell it in the same market with similar variable costs and so on); from an economic point of view, we have chosen to satisfy the general duopoly assumptions, in a classic microeconomics settings (which determines the non-cooperative core of the game). From a cooperative point of view, we shall add a quadratic perturbation acting on the common cooperative strategy space C. We have chosen the cooperative part in such a way that the second player will show only a linear cooperative perturbation, so the two companies differ in their payoff function: comparing

Equation (1) with Equation (2), company 1 has a proper quadratic cooperative-strategy function, while company 2 has a linear one (which is a particular case of quadratic function). The asymmetry, in our case, derives exactly from the different role played by the two enterprises initially: without coopetitive strategy, the first one receives material also from the second one; on the contrary, the presence of the coopetitive strategy diminishes the free row material coming from the second one.

Our eco-friendly deal allowed to "enlarge the pie" of possible gains—by diminishing sunk costs and other forms of costs, especially the environmental costs associated to the management of urban waste recycling. Consequently, we suggested production methods and production quantitative profiles in order to "share the gains fairly".

We have shown the complete analysis of our proposed game and we suggested some its possible solutions.

We have demonstrated how the presence of two competitors in the same industrial eco-park can dynamically produce beneficial effects on the entire industrial park; indeed, one could conceive of situations in which an industrial park arises precisely from the need of two companies competing on the same market. In fact, the presence of these competitors within the same industrial park determines a sort of dynamic imbalance that our approach can translate into economic growth. Last but not least, our proposal helps the safeguarding of the environment and environmental resources and aims to improve the public health and the quality of life.

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Appendix A. Direct Reachability of the Payoff T'

We could ask if we can obtain the payoff T' (or W') by using strategies belonging to the strategy space by means of a mere transformation. In other words, we ask if we can obtain directly the payoff T' (or W') by choosing a convenient strategy profile. Indeed, as we see immediately for instance for T', we could find infinite many strategy profiles T such that

$$f(T) = T'.$$

Here we need to explicitly observe the existence of strategy point T in P, possible triple (x, y, z) such that

$$f(T) = T'$$

and of τ, the infinite set (continuous curve, as the reader shall see soon) of all such triples (x, y, z)—in technical terms, τ is the anti-image of T' by f. What is the point in considering such curve τ? Well, any point of that curve specifies what actions the players should implement in order to obtain immediately the compromise solution T'. We should notice that the knowledge of such actions and consequently the knowledge of the existence of such direct actions is interesting *per se* (from a classic game theory perspective) but it is not necessary in our generalized coopetitive context, because, from a coopetitive point of view, we have already explained how to obtain indirectly the solutions $(N(1), T')$ and $(N(1), W')$. Indeed, in a more practical and realistic way (by a compromise procedure after collective optimization) we have already explained how to obtain such coopetitive solutions by a linear Pareto selection.

In particular, we can obtain a curve τ of infinite possible profile strategies T such that

$$\forall T \in \tau : f(T) = T'.$$

The curve τ is the intersection of two surfaces (see Figures A1–A3). Specifically, we have the intersection

$$f(x,y,z) = T' = (161/180, 161/180)$$

that is

$$\begin{cases} 4x(1-x-y) + \frac{3}{5}z - \frac{9}{10}z^2 = \frac{161}{180} \\ 4y(1-x-y) + \frac{6}{5}z = \frac{161}{180} \end{cases}$$

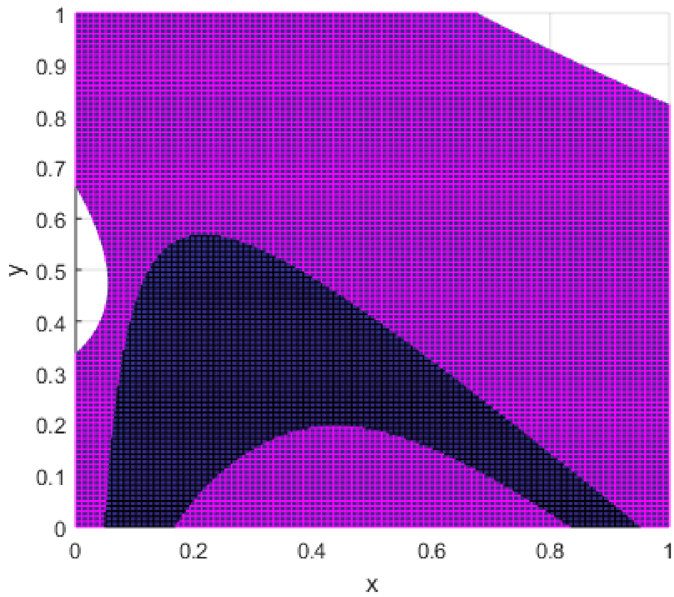

Figure A1. Projection of the curve τ upon the (x,y) plane (curve starting approximately from 0.2 to 0.8).

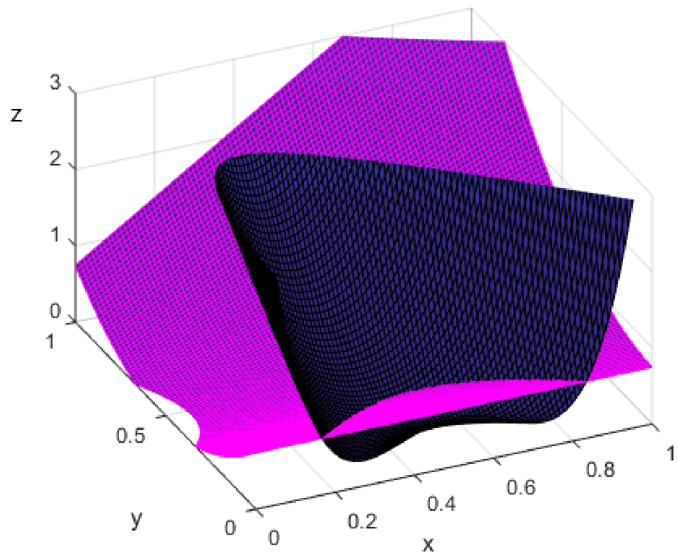

Figure A2. Curve τ as intersection of the two surfaces in the strategy space.

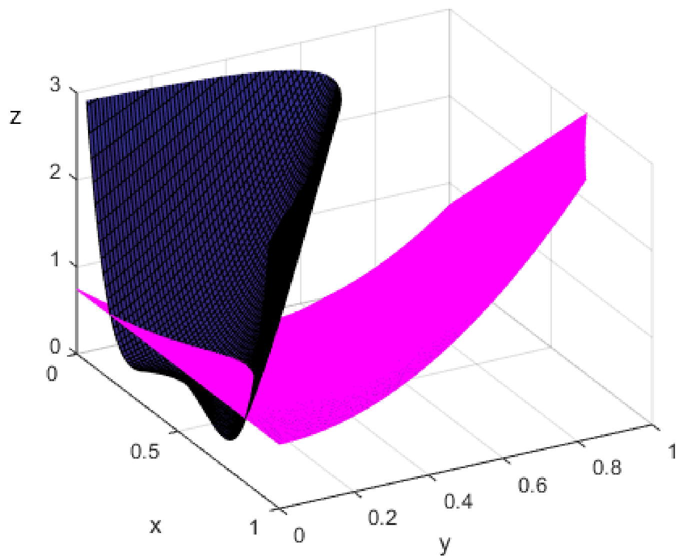

Figure A3. Curve τ as intersection of the two surfaces in the strategy space.

Appendix B. Matlab Code for the Graphical Representation of the Nash Optimal Point $N'(Z^*_{\beta^*})$

The code is

```
for β1 = [0:0.1:1];
β2 = 1-β1;
a = 1.5;
p = 4;
z = linspace(0,1/(2.*β1.*a));
x = 1/3;
y = 1/3;
N1 = 4*x.*(1-x-y) + ((β1)*(0.7*p*z-z)).*(1-a*z);
N2 = 4*y.*(1-x-y) + ((β2)*(0.7*p*z-z))
plot(N1,N2,'g')
end
```

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