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A RADNER EQUILIBRIUM PROBLEM: A VARIATIONAL APPROACH WITH PREFERENCE RELATIONS

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ABSTRACT. In this paper, we study an economic equilibrium problem under uncertainty using a variational approach. In particular, the equilibrium conditions involve the maximization of agents preferences, which are supposed to be not complete. Hence, we will reformulate the equilibrium problem by means of a quasi-variational inequality without representing the preferences by a utility function.

1. Introduction

This paper is focused on the analysis of an economic equilibrium model under time and uncertainty by using a variational approach to maximize the preferences of individuals, without completeness or transitivity assumptions. The variational inequality theory was introduced by Kinderlehrer and Stampacchia (1980) and in the sequel a large class of equilibrium problems was studied; see for instance, Donato *et al.* (2014), Milasi (2014), Daniele and Giuffrè (2015), Allevi *et al.* (2018), Donato *et al.* (2018a,b), and Scrimali and Mirabella (2018).

Debreu (1959) introduced an economic equilibrium model which evolves in a sequence of markets under uncertainty on the future conditions. Subsequently, Radner (1972) generalized such equilibrium model by introducing the possibility of agents to transfer wealth among all possible future time. Throughout two different market structures, forward and spot markets, consumers' and firms' choices will depict not only their taste concerning the goods but also their beliefs regarding the event chosen by Nature.

Milasi *et al.* (2019) considered an economic equilibrium problem in which the consumers preferences are described by a binary relation which is complete, transitive, continuous, non-satiated, and semistrictly convex. In the latter assumptions, they studied the equilibrium problem by using a variational approach without representing the consumer's preferences by the utility function. The assumption of completeness means that an agent should be

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able to compare any pair of possible alternatives. One can imagine real-life situations, for instance, as under uncertainty, in which the completeness does not hold, that is the consumer is not able to rank his preferences between two or more choices. This led us to consider an economic problem by dropping the assumptions of completeness and transitivity. However, the considered assumptions are not sufficient to guarantee the existence of a utility function representing the preference relation. Hence, we cannot consider the known results to characterize the maximum problem by means of a variational inequality. However, we are able to adapt to this more general assumptions the results obtained in Milasi *et al.* (2019). Hence, we reformulate the problem of maximizing preferences by means of a variational inequality problem without representation by utility.
The paper is organized as follows: in Section 2 we introduce the economic equilibrium model. In Section 3 the equilibrium problem of plans, prices, and price expectations has been characterized as a solution to a generalized quasi-variational inequality. We will use the variational formulation to obtain the equilibrium existence.

2. The Radner Equilibrium with preference relations

Let us consider a market which stars at time $t = 0$ and evolves in a finite sequence of T future dates; the uncertainty is expressed through a finite set of all possible situations which can occurs at each time. We set $\mathscr{T} := \{1,\dots,T\}$ and $\mathscr{T}_0 := \{0\} \cup \mathscr{T}$; $\{\xi_0\}$ is the initial situation of the market and the set $\Xi_t := \{\xi_t^1,\dots,\xi_t^{k_t}\}$ represents all possible situations that can occurs at time t; the element ξ_t^j is called *contingency*. We pose $\Xi := \Xi_1 \cup \ldots \cup \Xi_T$ and $\Xi_0 = \{\xi_0\} \cup \Xi$, with $|\Xi_0| = N$. Hence the evolution of the market can be represented by means of the oriented graph $\mathscr{G}$ with vertices Ξ. In this structure, we set a market economy in which a finite number of agents, with the same information, trade and consume a finite number of different commodities. We denote by $\mathscr{I} := \{1,\dots,i,\dots,I\}$ and $\mathscr{H} := \{1\dots,h\dots,H\}$, respectively, the set of agents and commodities, and, for sake of notations, we pose $G_t = Hk_t$, $G = H(N-1)$ and $G_0 = HN$. Each agent is characterized by a preference relation $\succ_i$ over the consumption set $\mathbb{R}_+^{G_0}$ and by the vector $e_i(\xi_t^j) \in \mathbb{R}_{++}^H$ which represents the endowment of commodity at the contingency ξ_t^j. Hence, the Economy is identified by the vector $\mathscr{E} := \Big(\mathscr{G}, (\succ_i, e_i)_{i\in\mathscr{I}}\Big)$.
The preference relations The preference relation $\succ_i$ is a binary relation which describes the preferences of the agent i over the set of alternatives $\mathbb{R}_+^{G_0}$. We write $x \succ_i y$ and we read *the agent i strictly prefers x to y*; for all $x \in \mathbb{R}_+^{G_0}$, we denote by $U_i(x)$ the *strict upper contour set*, that is the set of all elements of $\mathbb{R}_+^{G_0}$ strictly preferred to x, hence $U_i(x) := \{y \in X :\ y \succ_i x\}$. We recall some basic properties of preference relations. We say that $\succ_i$ is

- *complete:* for all $x, y \in \mathbb{R}_+^{G_0}$, $x \succ_i y$ or $y \succ_i x$;
- *lower semicontinuous:* if $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^{G_0}$ converges to x with $x \succ_i y$, then there exists $\nu \in \mathbb{N}$ such that $x_n \succ_i y$ for all $n > \nu$;
- *upper semicontinuous:* if $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^{G_0}$ converges to x with $y \succ_i x$, then there exists $\nu \in \mathbb{N}$ such that $y \succ_i x_n$ for all $n > \nu$;
- *continuous:* if it is lower and upper semicontinuous;
- *non-satiated:* if for all $x \in \mathbb{R}_+^{G_0}$ there exists $y \in \mathbb{R}_+^{G_0}$ s.t. $y \succ_i x$;

- *semistrictly convex:* if $x \succ_i y$ then $\lambda x + (1-\lambda)y \succ y$, for all $\lambda \in (0,1)$;
- *strictly increasing respect to component 1:* for all $x, y \in \mathbb{R}^{G_0}_+$ such that $x_k \geq y_k$ for all $k = 1, \ldots, G_0$ and $x_1 > y_1$ one has $x \succ_i y$.

The commodities The elements in the market are distinguishable not only by their physical characteristics, but also for the state of the word. This leads to introduce the notion of a vector which describes all the characteristics. A *state-contingent vector* is the vector

$$y := (y_0, \ldots, y_t, \ldots, y_T) \in \mathbb{R}^{G_0},$$

with $y_t := (y(\xi_t^1), \ldots, y(\xi_t^j), \ldots, y(\xi_t^{k_t})) = (y(\xi))_{\xi \in \Xi_t} \in \mathbb{R}^{G_t}$; $y(\xi_t^j) := (y^h(\xi_t^j))_{h \in \mathscr{H}} \in \mathbb{R}^H$. If the component at contingency ξ_0 is not included one has that $y \in \mathbb{R}^G$. A state-contingent vector y is understood as an entitlement to get $y(\xi_t^j)$ if state ξ_t^j occurs.

The markets The economy is characterized by two market structures: spot and forward markets. At each contingency $\xi_t^j \in \Xi$, markets open and each agent i consume or trade the amount of commodities $x_i(\xi_t^j) \in \mathbb{R}^H_+$ at prices $p(\xi_t^j)$. Hence, we set the state-contingency vectors

$$x_i := (x_{i0}, x_{i1}, \ldots, x_{it}, \ldots, x_{iT}) \in \mathbb{R}^{G_0}_+, \quad e_i := (e_{i0}, e_{i1}, \ldots, e_{it}, \ldots, e_{iT}) \in \mathbb{R}^{G_0}_{++}$$

$$p := (p_0, p_1, \ldots, p_t, \ldots, p_T) \in \mathbb{R}^{G_0}_+.$$

At time $t = 0$ agents does not know which state she will be in the next periods, but they make plans for each of them. So they can transfer wealth, in terms of commodity-1, among all future contingencies to have cash for spot market or for future contracts of commodity-1 in next contingencies. We introduce the forward contracts of each agent and the relative prices by the following state-contingency vectors

$$z_i := (z_{i1}, \ldots, z_{it}, \ldots, z_{iT}) \in \mathbb{R}^{N-1}, \qquad q := (q_1, \ldots, q_t, \ldots, q_T) \in \mathbb{R}^{N-1}_+;$$

$z_i(\xi_t^j)$ is the commodity-1 amount at ξ_t^j paid $q(\xi_t^j)$ at time 0. We observe that the components of z_i can be negative: if $z_i(\xi_t^j) < 0$, it is the amount to be delivered by the agent at ξ_t^j and $q(\xi_t^j) z_i(\xi_t^j)$ represents an *income* at ξ_0; while, if $z_i(\xi_t^j) > 0$, it is an amount to be received by the agent at ξ_t^j and $q(\xi_t^j) z_i(\xi_t^j)$ represents an *outcome* at ξ_0. Hence, the budget constraints of agent i at the current price system (p, q) is given by the following set:

$$\begin{aligned} B_i(p,q) := \{ &(x_i, z_i) \in \mathbb{R}^{HN}_+ \times \mathbb{R}^{N-1} : \langle p(\xi_0), x_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle q, z_i \rangle_{N-1} \leq 0 \\ &\langle p(\xi_t^j), x_i(\xi_t^j) - e_i(\xi_t^j) \rangle_H \leq p^1(\xi_t^j) z_i(\xi_t^j) \quad \forall \xi_t^j \in \Xi_t, t \in \mathscr{T} \} \end{aligned}$$

The first inequality represents the budget constraint at time 0 while, the second inequality represents the expected budget constraints at each contingency ξ_t^j, where $p^1(\xi_t^j)$ is the commodity-1 spot price at ξ_t^j.

In the market the aim of each agent i is to choose the maximal element in $B_i(p,q)$ respect to the preference relation $\succ_i$; moreover, the market clearing conditions must be satisfied. We have the following mathematical formulation of equilibrium.

Definition 1. *The vector* $\left((\bar{x}_i, \bar{z}_i)_{i \in \mathscr{I}}, \bar{p}, \bar{q}\right)$ *is an* equilibrium *of plans, prices and price expectations for the Economy* $\mathscr{E}$ *if and only if*

(i) *for any* $i \in \mathscr{I}$, $(\bar{x}_i, \bar{z}_i) \in B_i(\overline{p}, \overline{q})$ *and if* $x_i \succ_i \bar{x}_i$, *then* $\forall z_i \in \mathbb{R}^{N-1}$, $(x_i, z_i) \notin B_i(\bar{p}, \bar{q})$;

(ii) *for all* $t \in \mathscr{T}_0$: $$\sum_{i\in\mathscr{I}} \bar{x}_i(\xi_t^j) \le \sum_{i\in\mathscr{I}} e_i(\xi_t^j) \qquad \forall \xi_t^j \in \Xi_t;$$

(iii) *for all* $t \in \mathscr{T}$: $$\sum_{i\in\mathscr{I}} \bar{z}_i(\xi_t^j) = 0 \qquad \forall \xi_t^j \in \Xi_t.$$

When $\bar{x}_i$ satisfies condition (i), we say that it is a maximal element of $B_i(\overline{p},\overline{q})$ respect to the preference relation $\succ_i$.

Proposition 1. *Let $\succ_i$ be strictly increasing in commodity-1 and $(\bar{x}_i,\bar{z}_i)$ a maximal element of $B_i(\bar{p},\bar{q})$. Then $\bar{p}^1(\xi_t^j) > 0$ for all $\xi_t^j \in \Xi_0$ and $\bar{q}(\xi_t^j) > 0$ for all $\xi_t^j \in \Xi$.*

Proof. We suppose that there exists $\xi_{t^*}^{j^*} \in \Xi_0$ such that $\bar{p}^1(\xi_{t^*}^{j^*}) = 0$. We pose $\widetilde{x}_i$ such that $\widetilde{x}_i(\xi_t^j) := \bar{x}_i(\xi_t^j)$ for all $\xi_t^j \neq \xi_{t^*}^{j^*}$ and

$$\widetilde{x}_i(\xi_{t^*}^{j^*}) := \begin{cases} \bar{x}_i^1(\xi_{t^*}^{j^*}) + K, & \\ \bar{x}_i^h(\xi_{t^*}^{j^*}), & \forall h \neq 1 \end{cases}$$

with $K > 0$. It results $(\widetilde{x}_i,\bar{z}_i) \in B_i(\bar{p},\bar{q})$ and, since $\succ_i$ is strictly increasing in commodity-1, one has $\widetilde{x}_i \succ_i \bar{x}_i$ which contradicts the assumption.
The proof of $\bar{q}(\xi_t^j) > 0$, $\xi_t^j \in \Xi$, is closed to the previous one. $\square$

Thanks to Proposition 1, without loss of generality, we can consider the prices in the simplex-set $\Delta := \Delta_{\xi_0} \times \prod_{\xi_t^j \in \Xi} \Delta_{\xi_t^j}$, where

$$\Delta_{\xi_0} = \Big\{ (p(\xi_0), q(\xi_0)) \in \mathbb{R}_+^H \times \mathbb{R}_+^{N-1} : \sum_{h\in\mathscr{H}} p^h(\xi_0) + \sum_{\xi_t^j \in \Xi} q(\xi_t^j) = 1 \Big\};$$

$$\Delta_{\xi_t^j} = \Big\{ p(\xi_t^j) \in \mathbb{R}_+^H : \sum_{h\in\mathscr{H}} p^h(\xi_t^j) = 1 \Big\} \qquad \forall \xi_t^j \in \Xi.$$

3. Variational Approach

In this section we study the economy $\mathscr{E}$ introduced in the Section **2**. Firstly, we recall the definition of variational problem. Let C be a nonempty, convex, and compact subset of $\mathbb{R}^n$. Let $\Phi : C \rightrightarrows \mathbb{R}^n$ and $S : C \rightrightarrows C$ be two set-valued maps; a *generalized quasi-variational inequality* associated with C, S, Φ consists in the following:

$$Find\ \bar{x} \in S(\bar{x})\ such\ that\ \exists \varphi \in \Phi(\bar{x})\ with\ \langle \varphi, x - \bar{x} \rangle_n \ge 0 \quad \forall x \in S(\bar{x}). \tag{1}$$

Theorem 1 (See Tan (1985))**.** *Let Φ and S be two set-valued maps satisfying the following properties:*

(i) *Φ is upper semicontinuous with nonempty, convex, and compact values;*
(ii) *S is closed, lower semicontinuous, and with nonempty, convex, and compact values.*

Then, the generalized quasi-variational inequality (1) admits at least a solution.

Our main result is to characterize the equilibrium by means of a variational problem, and then by using the variational theory we will give an existence result. To this aim we suppose that the preference relations of agents are semistrictly convex, non-satiated, lower semicontinuous, and strictly increasing in commodity-1; hence, we are not considering completeness or transitivity assumptions. In this context we can adapt the map introduced by Aussel and Hadjisavvas (2005). To our aim, we pose, for all $i \in \mathscr{I}$, the map $N_i : \mathbb{R}^{G_0} \rightrightarrows \mathbb{R}^{G_0}$ such that for all $x_i \in \mathbb{R}^{G_0}_+$

$$N_i(x_i) := \{h_i \in \mathbb{R}^{G_0} : \langle h_i, y_i - x_i \rangle \leq 0 \quad \forall y_i \in U_i(x_i)\}$$

and $N_i(x_i) := \emptyset$ for all $x_i \notin \mathbb{R}^{G_0}_+$. We observe that, since $\succ_i$ is non-satiated and semistrictly convex, for all $x \in \mathbb{R}^{G_0}_+$ the sets $U_i(x)$ are convex and nonempty; hence $N_i(x)$ represents the normal cone to the set $U_i(x)$; moreover $N_i(x_i) \setminus \{0\} \neq \emptyset$. In order to obtain good properties on the operator, we consider the map $G_i : \mathbb{R}^{G_0}_+ \rightrightarrows \mathbb{R}^{G_0}$ such that for all $x_i \in \mathbb{R}^{G_0}_+$

$$G_i(x_i) := conv\,(N_i(x_i) \cap D)$$

where D is the boundary of the unit ball. We observe that for all $h_i \in N_i(x_i) \setminus \{0\}$, one has $h_i' = \dfrac{h_i}{\|h_i\|} \in G_i(x_i)$; hence the map G_i is nonempty, convex, and compact values. We pose $G(x) := \prod_{i \in \mathscr{I}} G_i(x_i)$, for all $x = (x_i)_{i \in \mathscr{I}} \in \mathbb{R}^{IG_0}_+$, and $\widetilde{B}(p,q) := B(p,q) \cap C$, where

$$C := \left[0, \sum_{h \in \mathscr{H}} \sum_{i \in \mathscr{I}} e_i^h(\xi_t^j)\right]^N \times \left[-\sum_{i \in \mathscr{I}} e_i^1(\xi_0), \sum_{i \in \mathscr{I}} e_i^1(\xi_0)\right]^{N-1}.$$

Finally, we introduce the generalized quasi-variational inequality problem:

$$\begin{gathered} \textit{Find } (\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in \widetilde{B}(\bar{p}, \bar{q}) \times \Delta \textit{ such that there exists } \ h := (h_i)_{i \in I} \in G(\bar{x}) \textit{ and} \\ \sum_{i \in \mathscr{I}} \langle h_i, x_i - \bar{x}_i \rangle_{G_0} - \langle (\sum_{i \in \mathscr{I}} (\bar{x}_i - e_i), \sum_{i \in \mathscr{I}} \bar{z}_i), (p,q) - (\bar{p}, \bar{q}) \rangle_{G_0+N-1} \geq 0 \\ \forall\,(x,z,p,q) \in \widetilde{B}(\bar{p}, \bar{q}) \times \Delta. \end{gathered} \tag{2}$$

Remark 1. *The vector $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a solution of the GQVI (2) if and only if the following inequalities simultaneously hold:*

(i) *for each $i \in \mathscr{I}$, $(\bar{x}_i, \bar{z}_i)$ is a solution to*

$$\langle h_i, x_i - \bar{x}_i \rangle_{G_0} \geq 0 \qquad \forall\,(x_i, z_i) \in \widetilde{B}_i(\bar{p}, \bar{q}), \tag{3}$$

(ii) *$(\bar{p}(\xi_0), \bar{q})$ is a solution to*

$$-\langle (\sum_{i \in \mathscr{I}} (\bar{x}_i(\xi_0) - e_i(\xi_0)), \sum_{i \in \mathscr{I}} \bar{z}_i), (p(\xi_0), q) - (\bar{p}(\xi_0), \bar{q}) \rangle_{H+N-1} \geq 0 \qquad \forall\,(p(\xi_0), q) \in \Delta_{\xi_0}, \tag{4}$$

(iii) *for all $\xi_t^j \in \Xi$, $\bar{p}(\xi_t^j)$ is a solution to*

$$-\langle \sum_{i \in \mathscr{I}} (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)), p(\xi_t^j) - \bar{p}(\xi_t^j) \rangle_H \geq 0 \qquad \forall p(\xi_t^j) \in \Delta_{\xi_t^j}. \tag{5}$$

In fact, let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be a solution to (2). Fixed $i^* \in \mathscr{I}$, we consider $(x,z,p,q) \in \widetilde{B}(\bar{p}, \bar{q}) \times \Delta$ such that $(p,q) = (\bar{p}, \bar{q})$, $(x_i, z_i) = (\bar{x}_i, \bar{z}_i)$ for all $i \neq i^*$ and (x_{i^*}, z_{i^*}) an element in $\widetilde{B}_{i^*}(\bar{p}, \bar{q})$. We can replace (x,z,p,q) in (2) and we obtain condition (3). Now we consider $(x,z,p,q) \in \widetilde{B}(\bar{p}, \bar{q}) \times \Delta$ such that $(x,z) = (\bar{x}, \bar{z})$, $p(\xi_t^j) = \overline{p}(\xi_t^j)$ for all ξ_t^j, with $t \neq 0$ and $(p(\xi_0), q) \in \Delta_{\xi_0}$. By replacing (x,z,p,q) in (2) we obtain (4). Finally, fixed $\xi_{t^*}^j$, we consider $(x,z,p,q) \in \widetilde{B}(\bar{p}, \bar{q}) \times \Delta$ such that $(x,z) = (\bar{x}, \bar{z})$, $(p(\xi_0), q) = (\bar{p}(\xi_0), \bar{q})$, $p(\xi_t^j) = \overline{p}(\xi_t^j)$ for all $\xi_t^j \neq \xi_{t^*}^j$,

and $p(\xi_{t^*}^j) \in \Delta_{\xi_{t^*}^j}$. *By replacing* (x,z,p,q) *in (2) we obtain (5).*
Viceversa, let $(\bar{x}_i, \bar{z}_i)$, $(\bar{p}(\xi_0), \bar{q})$ *and* $(\bar{p}(\xi_t^j))_{\xi_t^j \in \Xi}$ *satisfy (3), (4) and (5), then (2) is verified.*

Next result characterizes the equilibrium by means of the variational problem (2).

Theorem 2 (Characterization)**.** *Let* $\mathscr{E} := \left(\mathscr{G}, (\succ_i, e_i)_{i\in\mathscr{I}}\right)$ *be an economy such that for all* $i \in \mathscr{I}$ *the preference relation* $\succ_i$ *is semistrictly convex, lower semicontinuous, non-satiated, and strictly increasing in commodity-1. If* $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ *is a solution to the generalized quasi-variational problem (2), then it is an equilibrium vector for the economy* $\mathscr{E}$.

Proof.
Claim 1: For all $i \in \mathscr{I}$, $(\bar{x}_i, \bar{z}_i)$ is a maximal element of $\widetilde{B}_i(\bar{p}, \bar{q})$.
We suppose that there exists $(\tilde{x}_i, \tilde{z}_i) \in \widetilde{B}_i(\bar{p}, \bar{q})$ such that $\tilde{x}_i \succ_i \bar{x}_i$. From Remark 1 one has that $(\bar{x}_i, \bar{z}_i)$ is a solution of the variational problem (3), then $\langle h_i, \tilde{x}_i - \bar{x}_i \rangle_{G_0} \geq 0$. Moreover, being $h_i \in G_i(\bar{x}_i) = conv(N_i(\bar{x}_i) \cap D) \subseteq N_i(\bar{x}_i)$, one has $h_i \in N_i(\bar{x}_i)$, then since $\tilde{x}_i \succ_i \bar{x}_i$, it follows that $\langle h_i, \tilde{x}_i - \bar{x}_i \rangle_{G_0} \leq 0$. Hence $\langle h_i, \tilde{x}_i - \bar{x}_i \rangle_{G_0} = 0$.
Now, for all $n \in \mathbb{N}$, we pose $x_{i,n} := \tilde{x}_i + \dfrac{1}{n} h_i$; since $x_{i,n} \to \tilde{x}_i$ from lower semicontinuity of $\succ_i$ there exists $\nu \in \mathbb{N}$ such that $x_{i,n} \succ_i \bar{x}_i$ for all $n \geq \nu$. Hence $x_{i,n} \in U_i(\bar{x}_i)$ and $\langle h_i, x_{i,n} - \bar{x}_i \rangle \leq 0$. Then

$$0 \geq \langle h_i, x_{i,n} - \bar{x}_i \rangle_{G_0} = \langle h_i, \tilde{x}_i - \bar{x}_i \rangle_{G_0} + \frac{1}{n}\|h_i\|^2 = \frac{1}{n}\|h_i\|^2 \geq 0$$

this contradicts the fact that $h_i \neq 0$, hence, $h_i \notin G_i(\bar{x}_i)$.

Claim 2: $\sum_{i\in\mathscr{I}} \bar{z}_i(\xi_t^j) \leq 0$ for all $\xi_t^j \in \Xi$ and $\sum_{i\in\mathscr{I}} (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)) \leq 0$ for all $\xi_t^j \in \Xi_0$.
Since for all $i \in \mathscr{I}$, $(\bar{x}_i, \bar{z}_i) \in \widetilde{B}_i(\bar{p}, \bar{q})$, one has

$$\langle \sum_{i\in\mathscr{I}} (\bar{x}_i(\xi_0) - e_i(\xi_0)), \bar{p}(\xi_0) \rangle_H + \langle \sum_{i\in\mathscr{I}} \bar{z}_i, \bar{q} \rangle_{N-1} \leq 0. \tag{6}$$

Hence, from (6) and (4), one has:

$$\langle \sum_{i\in\mathscr{I}} (\bar{x}_i(\xi_0) - e_i(\xi_0)), p_0(\xi_0) \rangle_H + \langle \sum_{i\in\mathscr{I}} \bar{z}_i, q \rangle_{N-1} \leq 0 \qquad \forall (p_0, q) \in \Delta_{\xi_0}. \tag{7}$$

Now, fixed $h^* \in \mathscr{H}$ we pose $(\tilde{p}_0, \tilde{q})$ such that:

$$\tilde{q} = 0_{N-1} \quad and \quad \tilde{p}_0 = \begin{cases} \tilde{p}_0^{h^*} = 1 & \\ \tilde{p}_0^h = 0 & \forall h \neq h^* \end{cases}$$

Being $(\tilde{p}_0, \tilde{q}) \in \Delta_{\xi_0}$, by replacing it in (7) we obtain $\sum_{i\in\mathscr{I}} (\bar{x}_i^{h^*}(\xi_0) - e_i^{h^*}(\xi_0)) \leq 0$.
Fixed $t^* \in \mathscr{T}$ and $j^* = 1, \ldots, k_t$, we pose $(\tilde{p}_0, \tilde{q})$ such that:

$$\tilde{p}_0 = 0_H \quad and \quad \tilde{q} := \begin{cases} \tilde{q}(\xi_{t^*}^{j^*}) = 1 & t^* = t \quad j^* = j \\ \tilde{q}(\xi_t^j) = 0 & otherwise \end{cases} .$$

Being $(\tilde{p}_0, \tilde{q}) \in \Delta_{\xi_0}$, by replacing it in (7) we obtain $\sum_{i \in \mathscr{I}} \bar{z}_i(\xi_{t^*}^{j^*}) \leq 0$.

Moreover, from condition (5), from the second constraint of $\widetilde{B}_i(\bar{p}, \bar{q})$ and from the above inequality, for all $\xi_t^j \in \Xi$ one has:

$$\begin{aligned} \langle \sum_{i \in \mathscr{I}} (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)), p(\xi_t^j) \rangle_H &\leq \langle \sum_{i \in \mathscr{I}} (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)), \bar{p}(\xi_t^j) \rangle_H \\ &\leq \bar{p}^1(\xi_t^j) \Big(\sum_{i \in \mathscr{I}} \bar{z}_i(\xi_t^j) \Big) \leq 0 \quad \forall p(\xi_t^j) \in \Delta_{\xi_t^j}. \end{aligned} \tag{8}$$

Fixed $\xi_t^j \in \Xi$ and $h^* \in \mathscr{H}$, we pose $\widetilde{p}(\xi_t^j) \in \Delta_{\xi_t^j}$ such that $\widetilde{p}^{h^*}(\xi_t^j) = 1$ and $\widetilde{p}^h(\xi_t^j) = 0$ for all $h \neq h^*$; by replacing $\widetilde{p}(\xi_t^j)$ in (8) we get

$$\sum_{i \in \mathscr{I}} (\bar{x}_i^{h^*}(\xi_t^j) - e_i^{h^*}(\xi_t^j)) \leq 0 .$$

Claim 3: For all $i \in \mathscr{I}$ one has

$$\langle \bar{p}(\xi_0), \bar{x}_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \bar{q}, \bar{z}_i \rangle_{N-1} = 0 \tag{9}$$

$$\Big\langle \bar{p}(\xi_t^j), (\bar{x}_i(\xi_t^j) - e_i(\xi_t^j)) \Big\rangle_H = \bar{p}^1(\xi_t^j) \bar{z}_i(\xi_t^j) \qquad \forall \xi_t^j \in \Xi_t, t \in \mathscr{T} \tag{10}$$

Indeed, if there exists $i \in \mathscr{I}$ such that $\langle \bar{p}(\xi_0), \bar{x}_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \bar{q}, \bar{z}_i \rangle_{N-1} < 0$, we pose $\tilde{x}_i \in \mathbb{R}_+^{G_0}$ such that $\tilde{x}_i(\xi_t^j) = \bar{x}(\xi_t^j)$ for all $\xi_t^j \in \Xi$ and

$$\tilde{x}_i^h(\xi_0) := \begin{cases} \bar{x}_i^1(\xi_0) + K & \\ \bar{x}_i^h(\xi_0) & \forall h \neq 1 \end{cases}$$

with

$$0 < K \leq \min \Big\{ \Big(\sum_{h \in \mathscr{H}} \sum_{i \in \mathscr{I}} e_{i0}^1 \Big) - \bar{x}_{i0}^1, - \frac{\langle \bar{p}(\xi_0), \bar{x}_i(\xi_0) - e_i(\xi_0) \rangle_H + \langle \bar{q}, \bar{z}_i \rangle_{N-1}}{\bar{p}^1(\xi_0)} \Big\} .$$

We observe that, from Claim 2 and being $\bar{x}_i \in \mathbb{R}_+^{G_0}$ and $e_i \in \mathbb{R}_{++}^{G_0}$, one has

$$\bar{x}_i^1(\xi_0) \leq \sum_{i \in \mathscr{I}} \bar{x}_i^1(\xi_0) \leq \sum_{i \in \mathscr{I}} e_i^1(\xi_0) < \sum_{h \in \mathscr{H}} \sum_{i \in \mathscr{I}} e_i^h(\xi_0) . \tag{11}$$

Hence $(\tilde{x}_i, \bar{z}_i) \in \widetilde{B}_i(\bar{p}, \bar{q})$ and, since $\succ_i$ is strictly increasing in commodity-1, $\tilde{x}_i \succ_i \bar{x}_i$ which contradicts Claim 1.

Claim 4: For all $i \in \mathscr{I}$ if $x_i \succ_i \bar{x}_i$ then $x_i \notin B_i(\overline{p}, \overline{q})$.

We suppose that there exists $x_i' \in B_i(\overline{p}, \overline{q})$ such that $x_i' \succ_i \bar{x}_i$. Since $\succ_i$ is semistrictly convex, for all $\lambda \in (0,1)$ one has $m = \lambda x_i' + (1-\lambda)\bar{x}_i \succ_i \bar{x}_i$. Since $B_i(\overline{p}, \overline{q})$ is a convex set $m \in B_i(\overline{p}, \overline{q})$. Moreover, from (11) there exists $\varepsilon > 0$ such that $B(\bar{x}, \varepsilon) \cap \mathbb{R}_+^{HN} \subseteq C$, hence, for all $\lambda \in \Big(0, \dfrac{\varepsilon}{\|x_i' - \bar{x}_i\|}\Big)$ one has $m \in C$. Hence, one has $m \in \widetilde{B}_i(\overline{p}, \overline{q})$ and $m \succ_i \bar{x}_i$ which contradicts Claim 1.

Claim 5: $\sum_{i \in \mathscr{I}} \bar{z}_i(\xi_t^j) = 0$ for all $\xi_t^j \in \Xi$.

We suppose that there exists $\xi_t^j \in \Xi$ such that $\sum_{i\in\mathscr{I}} \bar{z}_i(\xi_t^j) < 0$; since, from Proposition 1, $\overline{q} \in \mathbb{R}_{++}^{N-1}$, from Claims 2 and 3, it follows that

$$\langle \bar{p}_0, \sum_{i\in\mathscr{I}} \bar{x}_{i0} - e_{i0}\rangle_H = -\langle \overline{q}, \sum_{i\in\mathscr{I}} \bar{z}_i\rangle > 0,$$

which contradicts Claim 2 since $\overline{p}_0 \in \Delta_0$.

From Claims 1-5 we can conclude that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ satisfies all equilibrium conditions for the economy $\mathscr{E}$. □

Theorem 3 (Existence). *Let $\mathscr{E} := \left(\mathscr{G}, (\succ_i, e_i)_{i\in\mathscr{I}}\right)$ be an economy such that for all $i \in \mathscr{I}$ the preference relation $\succ_i$ is semistrictly convex, continuous, non-satiated and strictly increasing in commodity-1. Then there exists a equilibrium of plans, price and price expectations for $\mathscr{E}$.*

Proof. Thanks to Theorem 2 it is sufficient to prove the existence of solution to variational problem (2). We can set the generalized quasi-variational inequality (2) in problem (1), where $C := \mathbb{R}_+^{IG_0} \times \mathbb{R}^{I(N-1)} \times \Delta$, and $\Phi : C \rightrightarrows \mathbb{R}^{G_0(I+1)+(N-1)}$ and $S : C \rightrightarrows C$ are the set-valued maps:

$$\Phi(x,z,p,q) := (G(x), \sum_{i\in\mathscr{I}} (x_i - e_i), \sum_{i\in\mathscr{I}} z_i) \qquad \forall (x,z,p,q) \in C\,,$$

$$S(x,z,p,q) := \widetilde{B}(p,q) \times \Delta \qquad \forall (x,z,p,q) \in C\,.$$

Claim 1: Φ is upper semicontinuous with nonempty, convex, and compact values.
From definition of Φ and from properties of G_i, one has that Φ is a map with nonempty, compact and convex values; hence, it is sufficient to prove that G is upper semicontinuous. From upper semicontinuity of $\succ_i$, it follows that the N_i is a closed map, and then with similar argument used in Theorem 3.2 of Milasi *et al.* 2019, we can prove that G_i is a closed map. Hence, being G_i compact and closed, it is upper semicontinuous, and we can conclude that G is upper semicontinuous.
Claim 2: S is closed, lower semicontinuous and with nonempty, convex and compact values. Firstly, we observe that for all $i \in \mathscr{I}$ and $(p,q) \in \Delta$, since $(e_i, 0_{N-1}) \in \widetilde{B}_i(p,q)$ one has $\widetilde{B}_i(p,q)$ is nonempty. It is sufficient to prove that $\widetilde{B}_i$ is a lower semicontinuous map. Let $\{(p_n,q_n)\}_{n\in\mathbb{N}} \subseteq \Delta$ be a sequence converging to (p,q); for all $(x_i,z_i) \in \widetilde{B}_i(p,q)$, we have to prove that there exists a sequence $\{(x_{i,n}, z_{i,n})\}_{n\in\mathbb{N}}$ converging to (x_i,z_i) and such that $(x_{i,n}, z_{i,n}) \in \widetilde{B}_i(p_n,q_n)$ for all $n \in \mathbb{N}$. If (x_i,z_i) is such that

$$\begin{aligned} &\langle p(\xi_0), x_i(\xi_0) - e_i(\xi_0)\rangle_H + \langle q, z_i\rangle_{N-1} < 0, \\ &\langle p(\xi_t^j), x_i(\xi_t^j) - e_i(\xi_t^j)\rangle_H < p^1(\xi_t^j) z_i(\xi_t^j) \qquad \forall \xi_t^j \in \Xi_t, t \in \mathscr{T}\,, \end{aligned} \tag{12}$$

we can choose $(x_{i,n}, z_{i,n}) = (x_i, z_i)$ for all n, and, form the Theorem of sign permanence $(x_i,z_i) \in \widetilde{B}_i(p_n,q_n)$. We suppose that at least one inequality of (12) is not satisfied. Let

$$Li\,\widetilde{B}_i(p_n,q_n) := \{(x_i,z_i) :\ (x_i,z_i) = \lim (x_{ik}, z_{ik}),\ (x_{ik},z_{ik}) \in \widetilde{B}_i(p_k,q_k)\ \text{ eventually}\}\,.$$

From Proposition 8.2.1 by Lucchetti (2006), one has that $Li\widetilde{B}_i(p_n,q_n)$ is a closed set; moreover, being $e_i(\xi_t^j) \in \mathbb{R}^H_{++}$ for all ξ_t^j, for all $(p,q) \in \Delta$, there exists x_i such that $(x_i, 0_{N-1}) \in int\,\widetilde{B}_i(p,q)$, then $\widetilde{B}_i(p,q) = cl\ int\ \widetilde{B}_i(p,q)$. Hence, one has:

$$\widetilde{B}_i(p,q) = cl\,int\,\widetilde{B}_i(p,q) \subset cl\ Li\widetilde{B}_i(p_n,q_n) = Li\widetilde{B}_i(p_n,q_n).$$

Then, we can conclude that $\widetilde{B}_i$ is lower semicontinuous.
Finally, thanks to Claims 1 and 2, from Theorem 1, there exists $(\bar{x},\bar{z},\bar{p},\bar{q})$, solution to variational problem which is an equilibrium of plans, price and price expectations for the economy $\mathscr{E}$. □

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