Spectral realization of the Riemann zeros by quantizing $H = w(x)(p + \ell^2_p/p)$: the Lie-Noether symmetry approach

To cite this article: M C Nucci 2014 *J. Phys.: Conf. Ser.* **482** 012032

View the article online for updates and enhancements.

Spectral realization of the Riemann zeros by quantizing $H = w(x)(p + \ell_p^2/p)$: the Lie-Noether symmetry approach

M C Nucci

Dipartimento di Matematica e Informatica, Università degli Studi di Perugia & INFN Sezione Perugia, 06123 Perugia, Italy

E-mail: `mariaclara.nucci@unipg.it`

Abstract. If t_n are the heights of the Riemann zeros $1/2 + it_n$, an old idea, attributed to Hilbert and Polya [6], stated that the Riemann hypothesis would be proved if the t_n could be shown to be eigenvalues of a self-adjoint operator. In 1986 Berry [1] conjectured that t_n could instead be the eigenvalues of a deterministic quantum system with a chaotic classical counterpart and in 1999 Berry and Keating [3] proposed the Hamiltonian $H = xp$, with x and p the position and momentum of a one-dimensional particle, respectively. This was proven not to be the correct Hamiltonian since it yields a continuum spectrum [23] and therefore a more general Hamiltonian $H = w(x)(p + \ell_p^2/p)$ was proposed [25], [4], [24] and different expressions of the function $w(x)$ were considered [25], [24], [16] although none of them yielding exactly t_n. We show that the quantization by means of Lie and Noether symmetries [18], [19], [20], [7] of the Lagrangian equation corresponding to the Hamiltonian H yields straightforwardly the Schrödinger equation and clearly explains why either the continuum or the discrete spectrum is obtained. Therefore we infer that suitable Lie and Noether symmetries of the classical Lagrangian corresponding to H should be searched in order to alleviate one of Berry's quantum obsessions [2].

1. Introduction

In [2] Michael Berry wrote:

Why should a physicist be concerned with the zeros of the Riemann zeta function $\zeta(s)$, and in particular the Riemann hypothesis, according to which all complex zeros of $\zeta(s)$ have $Res = \frac{1}{2}$?

Not for the reasons that motivate mathematicians, e.g. fluctuations in the distribution of primes. Rather, my interest grew from the study of quantum systems whose classical counterparts possess chaotic trajectories.

One hundred years ago Polya and Hilbert suggested that in order to prove the Riemann hypothesis one has to find a self-adjoint operator whose spectrum contains the imaginary

part of the nontrivial Riemann zeros [6]. Michael Berry suggested the existence of a classical Hamiltonian whose quantum version would realize the Polya-Hilbert conjecture. This conjectured Hamiltonian must satisfy the following conditions [25]:

(i) be chaotic, with isolated periodic orbits related to the prime numbers,

(ii) break time reversal symmetry, to agree with the Gaussian unitary ensemble statistics,

(iii) be quasi-one dimensional.

The first classical Hamiltonian proposed by Berry, i.e.

$$H_{cl} = xp$$

was then shown by Berry and Keating [3] not to fulfill the requirement (i).

Sierra and collaborators [25], [24] and also Berry and Keating [4] have proposed different modifications of the xp Hamiltonian in order to have bounded classical trajectories and a discrete quantum spectrum. The first such Hamiltonian was [25]

$$H_S = x \left(p + \frac{\ell_p^2}{p} \right). \tag{1}$$

Others followed, in particular

$$H_B = \cosh\left(\frac{x}{R}\right) \left(p + \frac{\ell_p^2}{p} \right). \tag{2}$$

In the present paper we study the Lagrangian equations that derive from those Hamiltonians. In particular, we find their Lie point symmetries, their Jacobi last multipliers, various Lagrangians and the Noether symmetries that they admit. Finally we use those Noether symmetries to straightforwardly construct the Schrödinger equations [18], [19], [20], [7].

The paper is organized in the following way. In the next section we recall the properties of the Jacobi last multiplier, its connection with Lagrangians of second-order equations [12], [27], and the link with Lie symmetries [14], [15], [5]. In section 3 we study the Hamiltonian (1) and in section 4 the Hamiltonian (48). The last section contains some final remarks.

2. Jacobi last multiplier, Lie symmetries, Lagrangians

The method of the Jacobi Last Multiplier [9]-[12] provides a means to determine all the solutions of the partial differential equation

$$\mathcal{A}f = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} = 0 \tag{3}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \tag{4}$$

In fact, if one knows the JLM and all but one of the solutions, then the last solution can be obtained by a quadrature. The JLM M is given by

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = M\mathcal{A}f, \tag{5}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{6}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (3) or, equivalently, first integrals of (4) independent of each other. This means that M is a function of the variables $(x_1, \ldots, x_n)$ and depends on the chosen $n-1$ solutions, in the sense that it varies as they vary. The essential properties of the JLM are:

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \ldots, \eta_{n-1}$ of equation (3), then the corresponding JLM N is linked to M by the relationship:

$$N = M \frac{\partial(\eta_1, \ldots, \eta_{n-1})}{\partial(\omega_1, \ldots, \omega_{n-1})}.$$

(b) Given a non-singular transformation of variables

$$\tau : \quad (x_1, x_2, \ldots, x_n) \longrightarrow (x'_1, x'_2, \ldots, x'_n),$$

then the JLM M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x'_1, x'_2, \ldots, x'_n)},$$

where M obviously comes from the $n-1$ solutions of $\mathcal{A}F = 0$ which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1}.

(c) One can prove that each JLM M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial(M a_i)}{\partial x_i} = 0, \tag{7}$$

or equivalently:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = 0; \tag{8}$$

viceversa every solution M of this equation is a JLM.

(d) If one knows two JLMs M_1 and M_2 of equation (3), then their ratio is a solution ω of (3), or, equivalently, a first integral of (4). Naturally the ratio may be quite trivial, namely a constant. Viceversa the product of a multiplier M_1 times any solution ω yields another JLM $M_2 = M_1 \omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternate formulation in terms of symmetries was provided by Lie [14], [15]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [5]. If we know $n-1$ symmetries of (3)/(4), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \ldots, x_n) \partial_{x_j}, \quad i = 1, n-1, \tag{9}$$

then a JLM is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$
\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}. \tag{10}
$$

Another property of the JLM is its (almost forgotten) relationship with the Lagrangian, $L = L(t, q, \dot{q})$, for any second-order equation

$$
\ddot{q} = F(t, q, \dot{q}) \tag{11}
$$

i.e. [12] (Lecture 10)[1], [27]

$$
M = \frac{\partial^2 L}{\partial \dot{q}^2} \tag{12}
$$

where $M = M(t, q, \dot{q})$ satisfies the following equation

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \frac{\partial F}{\partial \dot{q}} = 0. \tag{13}
$$

Then equation (11) becomes the Euler-Lagrange equation:

$$
-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0. \tag{14}
$$

The proof is based on taking the derivative of (14) with respect to $\dot{q}$ and showing that this yields (13). If one knows a JLM, then L can be easily obtained by a double integration, i.e.:

$$
L = \int \left(\int M \, \mathrm{d}\dot{q} \right) \mathrm{d}\dot{q} + f_1(t, q)\dot{q} + f_2(t, q), \tag{15}
$$

where f_1 and f_2 are functions of t and q which have to satisfy a single partial differential equation related to (11) [21]. As it was shown in [22], f_1, f_2 are related to the gauge function $g = g(t, q)$. In fact, we may assume

$$
\begin{aligned}
f_1 &= \frac{\partial g}{\partial q} \\
f_2 &= \frac{\partial g}{\partial t} + f_3(t, q)
\end{aligned} \tag{16}
$$

where f_3 has to satisfy the mentioned partial differential equation and g is obviously arbitrary.

In [21] it was shown that if one knows several (at least two) Lie symmetries of the second-order differential equation (11), i.e.

$$
\Gamma_j = V_j(t, q)\partial_t + G_j(t, q)\partial_q, \quad j = 1, r, \tag{17}
$$

then many Jacobi Last Multipliers could be derived by means of (10), i.e.

$$
\frac{1}{M_{nm}} = \Delta_{nm} = \det \begin{bmatrix} 1 & \dot{q} & F(t, q, \dot{q}) \\ V_n & G_n & \dfrac{\mathrm{d}G_n}{\mathrm{d}t} - \dot{q}\dfrac{\mathrm{d}V_n}{\mathrm{d}t} \\ V_m & G_m & \dfrac{\mathrm{d}G_m}{\mathrm{d}t} - \dot{q}\dfrac{\mathrm{d}V_m}{\mathrm{d}t} \end{bmatrix}, \tag{18}
$$

with $(n, m = 1, r)$, and therefore many Lagrangians can be obtained by means of (15).

---

[1] *Jacobi's Lectures on Dynamics* are finally available in English [13].

3. The case $H = x(p + \ell_p^2/p)$

Sierra has proposed the following Hamiltonian [25], [24]:

$$H_S = x \left(p + \frac{\ell_p^2}{p} \right) \tag{19}$$

This Hamiltonian yields the Lagrangian equation:

$$\ddot{x} = -4x + 6\dot{x} - \frac{\dot{x}^2}{x}. \tag{20}$$

This equations admits an eight-dimensional Lie point symmetry algebra $sl(3, \mathbb{R})$ generated by

$$\Gamma_1 = \exp(-2t)x^2(\partial_t + 2x\partial_x), \quad \Gamma_2 = \exp(-4t)x^2(\partial_t + x\partial_x)$$
$$\Gamma_3 = \exp(2t)(\partial_t + 2x\partial_x), \quad \Gamma_4 = \partial_t, \quad \Gamma_5 = \exp(-2t)(\partial_t + x\partial_x), \tag{21}$$
$$\Gamma_6 = x\partial_x, \quad \Gamma_7 = \frac{\exp(4t)}{x}\partial_x, \quad \Gamma_8 = \frac{\exp(2t)}{x}\partial_x,$$

which implies that equation (20) is linearizable by means of a point transformation[2] [15]. In order to find the linearizing transformation we have to look for an abelian intransitive subalgebra of $sl(3, \mathbb{R})$ and, following Lie's classification of two-dimensional algebras in the real plane [15], we have to transform it into the canonical form

$$\partial_{\tilde{x}}, \quad \tilde{t}\partial_{\tilde{x}} \tag{22}$$

with $\tilde{x}$ and $\tilde{t}$ the new dependent and independent variables, respectively. We have found that such subalgebra[3] is that generated by Γ_7 and Γ_8. Then it is easy to derive that

$$\tilde{x} = \tfrac{1}{2}\exp(-2t)x^2, \quad \tilde{t} = \exp(2t) \tag{23}$$

and equation (20) is transformed into the free-particle equation, i.e.:

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{t}^2} = 0. \tag{24}$$

Several JLM can be obtained by means (18) and two of the eight symmetries (21). Consequently several Lagrangians can be derived from (15). In particular from Γ_7 and Γ_8 comes the Lagrangian

$$L_{78} = -\exp(-6t)\left(\frac{1}{4}\dot{x}^2 x^2 - \tfrac{1}{2}x^4 \right) \tag{25}$$

that admits the following five Noether symmetries [17]

$$\Gamma_3, \quad \Gamma_4 + \frac{3}{2}\Gamma_6 = \partial_t + \frac{3}{2}x\partial_x, \quad \Gamma_5, \quad \Gamma_7, \quad \Gamma_8. \tag{26}$$

If we quantize by preserving those Noether symmetries then the following Schrödinger equation is obtained:

$$4\mathrm{i}\exp(-2t)x^4\Phi_t + 4\exp(4t)x^2\Phi_{xx} - \left(2\exp(-8t)x^8 + 4\mathrm{i}\exp(-2t)x^4 + 3\exp(4t) \right)\Phi = 0 \tag{27}$$

[2] We prefer to cite the original work by Sophus Lie, although many textbooks/papers on Lie symmetries report his findings.

[3] We may find other such subalgebras but they are obviously related by a point transformation since there exist only one abelian intransitive subalgebra of $sl(3, \mathbb{R})$ [26]. For example, two other abelian intransitive subalgebras are those generated by $< \Gamma_1, \Gamma_3 >$, and $< \Gamma_2, \Gamma_5 >$, respectively.

and its spectrum is obviously a continuum [24]. Indeed equation (27) admits the following five Lie symmetries

$$
\begin{aligned}
X_1 &= \Gamma_3 + \Phi\left(-2\exp(2t) + \frac{\mathrm{i}}{2}x^4\exp(-4t)\right)\partial_\Phi, \\
X_2 &= \Gamma_4 + \frac{3}{2}\Gamma_6, \\
X_3 &= \Gamma_5 - \Phi\left(\frac{1}{2}\exp(-2t) + \frac{\mathrm{i}}{4}x^4\exp(-8t)\right)\partial_\Phi, \\
X_4 &= \Gamma_7 + \Phi\left(-\frac{1}{2x^2}\exp(4t) + \mathrm{i}x^2\exp(-2t)\right)\partial_\Phi, \\
X_5 &= \Gamma_8 + \Phi\left(-\frac{1}{2x^2}\exp(2t) + \frac{\mathrm{i}}{2}x^2\exp(-4t)\right)\partial_\Phi,
\end{aligned}
\tag{28}
$$

and also $\Phi\partial_\Phi$, and $\phi(t,x)\partial_\Phi$ with ϕ any solution of (27).

From Γ_1 and Γ_3 comes another Lagrangian

$$
L_{13} = \frac{1}{4x(2x - \dot{x})}
\tag{29}
$$

that admits the following five Noether symmetries [17]

$$
\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_7.
\tag{30}
$$

If we quantizing by requiring the preservation of the five Noether symmetries (30) then the following parabolic equation is obtained

$$
\Omega_{tt} + 4x\Omega_{xt} + 4x^2\Omega_{xx} + 4x\Omega_x - \Omega = 0
\tag{31}
$$

Its Lie symmetries are generated by[4]

$$
\hat{X}_1 = \Gamma_1 + x^2\Omega\exp(-2t)\partial_\Omega, \quad \hat{X}_2 = \Gamma_2, \quad \hat{X}_3 = \Gamma_3 + \Omega\exp(2t)\partial_\Omega, \quad \hat{X}_4 = \Gamma_4, \quad \hat{X}_5 = \Gamma_7, \tag{32}
$$

and can be reduced to its canonical form by the change of independent variable $x = \zeta\exp(2t)$, i.e.

$$
\Omega_{tt} - \Omega = 0
\tag{33}
$$

that can be solved to give

$$
\Omega(t,\zeta) = f_1(\zeta)\exp(t) + f_2(\zeta)\exp(-t),
\tag{34}
$$

with f_1, f_2 arbitrary functions of ζ. Equation (33) is not surprising. Indeed equation (20) can be transformed into

$$
\ddot{z} = 2(b+3)\dot{z} - (b+2)(b+4)z
\tag{35}
$$

by means of the transformation

$$
z = x^2\exp(bt)
\tag{36}
$$

where b is an arbitrary constant. Substituting $b = -3$ into (35) yields

$$
\ddot{z} = z,
\tag{37}
$$

---

[4] Also $\Omega\partial_\Omega$, $\omega(t,x)\partial_\Omega$ with ω any solution of (31).

which corresponds exactly to (33).

The Lagrangian associated with the Hamiltonian (1) is given by[5] [24]:

$$L_S = -2\ell_p\sqrt{x(x-\dot{x})} \tag{38}$$

This Lagrangian admits three Noether symmetries only, i.e.:

$$\Gamma_4, \quad \Gamma_5, \quad \Gamma_8. \tag{39}$$

These symmetries generate the following family of equations:

$$\ddot{x} = -4x + 6\dot{x} - \frac{\dot{x}^2}{x} + A(x-\dot{x})\sqrt{x(x-\dot{x})} \tag{40}$$

where A is an arbitrary constant. The three symmetries (39) yield three JLM and consequently three Lagrangians. In particular Γ_5 and Γ_8 yields the JLM

$$M_{58} = \frac{\sqrt{x}}{A(x-\dot{x})^{3/2}} \tag{41}$$

by means of (18), and the Lagrangian

$$L_{58} = -2\ell_p\sqrt{x(x-\dot{x})} + \frac{A}{4}\ell_p x^2, \tag{42}$$

by means of (15). Lagrangian L_{58} coincides with L_S if $A = 0$, but in this case M_{58} does not exist since in (18) $\Delta_5 8$ is equal to zero. This means that the JLM that yields the Lagrangian L_S does not come from the two symmetries Γ_5 and Γ_8. If $A \neq 0$ then L_{58} admits only two Noether symmetries, namely Γ_4 and Γ_5.

If we quantizing by requiring the preservation of the three Noether symmetries (39) then the following parabolic equation is obtained

$$4\Psi_{tt} + 8x\Psi_{xt} + 4x^2\Psi_{xx} - 4\Psi_t - 3\Psi = 0 \tag{43}$$

Its Lie symmetries are generated by[6]

$$Y_1 = \Gamma_4, \quad Y_2 = \Gamma_5 - \frac{1}{2}\exp(2t)\Psi\partial_\Psi, \quad Y_3 = \Gamma_8 \tag{44}$$

and can be reduced to its canonical form by the change of independent variable $x = \xi\exp(t)$, i.e.

$$4\Psi_{tt} - 4\Psi_t - 3\Psi = 0 \tag{45}$$

that can be solved to give

$$\Psi(t,\xi) = F_1(\xi)\exp(3t/2) + F_2(\xi)\exp(-t/2), \tag{46}$$

with F_1, F_2 arbitrary functions of ξ. Equation (45) is also not surprising. Substituting $b = -5/2$ into (35) yields

$$\ddot{z} = \dot{z} + \frac{3}{4}z, \tag{47}$$

which corresponds exactly to (45).

---

[5] The reality of L_S implies that $\dot{x} < x$.
[6] Also $\Psi\partial_\Psi$, $\psi(t,x)\partial_\Psi$ with ψ any solution of (43).

4. The case $H = \cosh\left(\frac{x}{R}\right)\left(p + \frac{\ell_p^2}{p}\right)$

Another classical Hamiltonian proposed by Sierra in [24] is

$$H_B = \cosh\left(\frac{x}{R}\right)\left(p + \frac{\ell_p^2}{p}\right) \tag{48}$$

This Hamiltonian yields the Lagrangian equation:

$$\ddot{x} = \frac{1}{R}\tanh\left(\frac{x}{R}\right)\left(-4\cosh^2\left(\frac{x}{R}\right) + 6\dot{x}\cosh\left(\frac{x}{R}\right) - \dot{x}^2\right) \tag{49}$$

Since this equation admits an eight-dimensional Lie symmetry algebra then it is linearizable by means of an abelian intransitive subalgebra and we found that the linearizing transformation is

$$
\begin{aligned}
\tilde{t} &= -\tan\left(\frac{2}{R}\left(-t + R\arctan\left(e^{x/R}\right)\right)\right), \\
\tilde{x} &= \frac{1 - e^{2x/R}}{4\cos(2t/R)(1 - e^{2x/R}) + 8\sin(2t/R)e^{x/R}}
\end{aligned}
\tag{50}
$$

that yields the free-particle equation (24). The Lagrangian that derives from the JLM obtained by means of the abelian intransitive subalgebra is

$$L = \frac{1}{2R\cosh\left(\frac{x}{R}\right)\left(2\cosh\left(\frac{x}{R}\right) - \dot{x}\right)} \tag{51}$$

and admits five Noether symmetries, while the Lagrangian L_B corresponding to the Hamiltonian H_B (48), i.e.

$$L_B = -2\ell_p\sqrt{\cosh\left(\frac{x}{R}\right)^2 - \dot{x}\cosh\left(\frac{x}{R}\right)} \tag{52}$$

admits just three Noether symmetries that actually generate the following family of equations of second order

$$
\begin{aligned}
\ddot{x} &= A\sqrt{\cosh\left(\frac{x}{R}\right)\left(\dot{x} - \cosh\left(\frac{x}{R}\right)\right)}\left(\dot{x} - \cosh\left(\frac{x}{R}\right)\right) \\
&\quad + \frac{1}{R}\tanh\left(\frac{x}{R}\right)\left(-4\cosh^2\left(\frac{x}{R}\right) + 6\dot{x}\cosh\left(\frac{x}{R}\right) - \dot{x}^2\right)
\end{aligned}
\tag{53}
$$

with A an arbitary constant. Here we do not pursue this case any further since the discussion is analogous to that of the previous section but the formula are quite lenghtier. We observe that the linearizing transformation (50) is singular and resembles somehow the transformation between the free-particle and the linear harmonic oscillator: a similar instance recurs also in the case of a Liénard-type nonlinear oscillator [7]. We will address this issue in future work.

5. Final Remarks
We have shown that Lie symmetries may explain what goes wrong/right if one takes a classical (even non-physical) problem into the realm of quantum mechanics.
In fact, the application of Lie symmetries yield that:

- the Lagrangian equation (20) admits an eight dimensional Lie symmetry algebra, therefore is linearizable and can be transformed to either a free-particle by a t-dependent transformation (23) or a variety of linear equations (35);

- there exist many Lagrangians for equation (20) but only one, up to a representation of the two-dimensional abelian intransitive (Type II) subalgebra, admits the maximum number (five) of Noether symmetries;

- those five symmetries lead to the derivation of a time-dependent Schrödinger equation (27) with a continuum spectrum;

- the Hamiltonian (1) introduced by Sierra corresponds to a t-independent Lagrangian (38) which admits three Noether symmetries $\Gamma_4, \Gamma_5, \Gamma_8$: those three symmetries generate the complete symmetry group of the family of equations (40);

- those three symmetries are not admitted by the Lagrangian (42) of equation (40) unless $A = 0$: the Lagrangian (42) can be obtained through the JLM coming from the two symmetries Γ_5 and Γ_8 but not if $A = 0$

- those three symmetries lead to the Schrödinger equation (43) that admits a continuum spectrum.

Indeed we may have transformed one of Berry's quantum obsessions [2] into a Lie symmetry obsession.

Acknowledgments

References

[1] Berry MV (1986) Riemann's zeta function: a model for quantum chaos? in *Quantum Chaos and Statistical Nuclear Physics* ed T H Seligman and H Nishioka vol 263, pp 1–17
[2] Berry MV (2008) Three quantum obsessions *Nonlinearity* **21** T19–T26
[3] Berry MV and Keating JP (1999) $H = xp$ and the Riemann zeros, in *Supersymmetry and Trace Formulae: Chaos and Disorder* ed J P Keating and I V Lerner, Plenum, New York, pp 355–367
[4] Berry MV and Keating JP (2011) A compact hamiltonian with the same asymptotic mean spectral density as the Riemann zeros *J. Phys. A: Math. Theor.* **44** 285203
[5] Bianchi L (1918) *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni* (Pisa, Enrico Spoerri)
[6] Edwards HM (1974) *Riemann's Zeta Function* (New York, Academic Press)
[7] Gubbiotti G and Nucci MC (2013) Noether symmetries and the quantization of a Lienard-type nonlinear oscillator *arXiv:1307.3803*
[8] Jacobi CGJ (1842) Sur un noveau principe de la mécanique analytique *Comptes Rendus du Académie des Sciences de Paris* **15**, 202–205
[9] Jacobi CGJ (1844) Sul principio dell'ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica *Giornale Arcadico di Scienze, Lettere ed Arti* **99** 129–146
[10] Jacobi CGJ (1844) Theoria novi multiplicatoris systemati æquationum differentialum vulgarium applicandi: Pars I *J. Reine Angew. Math.* **27** 199–268
[11] Jacobi CGJ (1845) Theoria novi multiplicatoris systemati æquationum differentialum vulgarium applicandi: Pars II *J. Reine Angew. Math.* **29** 213–279 and 333–376
[12] Jacobi CGJ (1886) *Vorlesungen über Dynamik. Nebst fünf hinterlassenen Abhandlungen desselben herausgegeben von A Clebsch* (Berinl, Druck und Verlag von Georg Reimer)
[13] Jacobi CGJ (2009) *Jacobi's Lectures on Dynamics*, tr. by K. Balagangadharan, ed. by Biswarup Banerjee, Hindustan Book Agency, available through the American Mathematical Society
[14] Lie S (1874) Veralgemeinerung und neue Verwerthung der Jacobischen Multiplicator-Theorie *Forhandlinger I Videnskabs-Selskabet I Christiania* 255–274
[15] Lie S (1912) *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen* (Leipzig: Teubner)
[16] Molina-Vilaplana J and Sierra G (2012) An xp model on AdS2 spacetime *arXiv:1212.2436v2*
[17] Noether E (1918) Invariante Variationsprobleme *Nachr. d. König Gesellsch. d. Wiss. zu Göttingen Math-phys. Klasse* 235–257
[18] Nucci MC (2011) Quantization of classical mechanics: shall we Lie? *Theor.Math. Phys.* **168** 997–1004

[19] Nucci MC (2012) From Lagrangian to Quantum Mechanics with Symmetries *J. Phys.: Conf. Ser.* **380** 012008

[20] Nucci MC (2013) Quantizing preserving Noether symmetries *J. Nonlinear Math. Phys.* **20** 451–463

[21] Nucci MC and Leach PGL (2007) Lagrangians galore *J. Math. Phys.* **48** 123510

[22] Nucci MC and Leach PGL (2008) Gauge variant symmetries for the Schrödinger equation *Il Nuovo Cimento B* **123** 93–101

[23] Sierra G (2007) $H = xp$ with interaction and the Riemann zeros *Nucl. Phys. B* **776** 327–364

[24] Sierra G (2012) General covariant xp models and the Riemann zeros *J. Phys. A:Math. Theor.* **45** 055209

[25] Sierra G and Rodriguez-Laguna J (2011) The H = xp model revisited and the Riemann zeros *Phys. Rev.Lett.* **106** 200201

[26] Winternitz P (2004) Subalgebras of Lie Algebras. Example of $sl(3, \mathbb{R})$ *Centre de Recherches Mathématiques CRM Proceedings and Lecture Notes* **34** 215–227

[27] Whittaker ET (1999) *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge, Cambridge University Press)