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From Lagrangian to Quantum Mechanics with Symmetries

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Abstract. We present an old and regretfully forgotten method by Jacobi which allows one to find many Lagrangians of simple classical models and also of nonconservative systems. We underline that the knowledge of Lie symmetries generates Jacobi last multipliers and each of the latter yields a Lagrangian. Then it is shown that Noether's theorem can identify among those Lagrangians the physical Lagrangian(s) that will successfully lead to quantization. The preservation of the Noether symmetries as Lie symmetries of the corresponding Schrödinger equation is the key that takes classical mechanics into quantum mechanics. Some examples are presented.

1. Introduction

The inverse problem of calculus of variation has attracted a lot of interest since in the second half of the XVIII century Euler [8] and then Lagrange [17] introduced the direct problem, namely the idea of linking the solution of a differential equation to the maximum/minimum of a functional, the celebrated problem of the brachistochrone being indeed the most famous classical example. It will take hundred of pages to cite all the papers and books that have been published since up to date. Most authors mark the birthdate of the inverse problem with the 1887-papers by either Helmholtz [11] or Volterra [33]. Some other especially among the Russian speaking researchers pushes the date slightly back to the 1886-paper by Sonin [32]. Very few recognize the seminal work by Jacobi, namely his 1845-paper [14] and his 1842-1843 Dynamics Lectures published posthumously in 1884 [15], available in English since 2009 [16], where he links his last multiplier to the Lagrangian for any even-order ordinary differential equation (ODE). Actually both Volterra and Sonin recognize the contribution of Jacobi last multiplier in their papers, Sonin more explicitly than Volterra since he showed that his own method involves the Jacobi last multiplier (p.10 in [32]).

The method of Jacobi last multiplier was enhanced when Lie determined the link with his symmetries [18], a link very easy to implement that allows to derive many multipliers and therefore Lagrangians.

It is known that a single second-order ODE admits different many Lagrangians [34], but so far there is not a method that can discern the physical Lagrangian among them, although some were proposed, e.g. [6]. The same is true for systems of second-order ODEs that admit more that one Lagrangian. It has been shown that some systems of second-order ODEs do not admit a Lagrangian [7] although in [27] the Lagrangian of some of those systems were determined by following Bateman's statement [2], namely *finding a set of equations equal in number to a given set, compatible with it and derivable from a variational principle* without recourse to any additional set of equations.

We propose that the physical Lagrangian should be the one that admits the highest possible number of Noether symmetries [26], [23], [24]. It was proven in [10] that the maximal dimension of the Lie symmetry algebra of a system of n ODEs of second order is $n^2 + 4n + 3$, and the highest number of corresponding Noether symmetries is $(n^2 + 3n + 6)/2$. In particular for a single second-order ODE the highest number of Noether symmetries is five [20].

Consequently we conjecture that the passage from a classical system to its quantum analogue should preserve exactly those Noether symmetries, namely the Noether symmetries of the physical Lagrangian shall become the Lie symmetries of the corresponding Schrödinger equation [26], [23], [24].

In this paper after recalling the method of Jacobi last multiplier and its link to Lie symmetries, we present the simple example of the one-dimensional free particle: ten inequivalent[1] Lagrangians are presented, their Noether symmetries identify two Lagrangians that admit the highest number of Noether symmetry: one independent and one dependent on time. Then the corresponding Schrödinger equations are obtained. Also the case of the nonlinear second-order Riccati equation is illustrated and its corresponding Schrödinger equation obtained.

2. Jacobi last multiplier

The method of the Jacobi Last Multiplier [12], [13], [14], [15] provides a means to determine all the solutions of the partial differential equation

$$\mathcal{A}f = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n)\frac{\partial f}{\partial x_i} = 0 \tag{1}$$

or its equivalent associated Lagrange system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \tag{2}$$

In fact, if one knows the Jacobi Last Multiplier and all but one of the solutions, then the last solution can be obtained by a quadrature. The Jacobi Last Multiplier M is given by

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = M\mathcal{A}f, \tag{3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{4}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (1) or, equivalently, first integrals of (2) independent of each other. This means that M is a function of the variables $(x_1, \ldots, x_n)$ and depends on the chosen $n-1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi Last Multiplier are:

[1] Namely they do not differ by a total derivative.

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \ldots, \eta_{n-1}$ of equation (1), then the corresponding Last Multiplier N is linked to M by the relationship:

$$N = M\frac{\partial(\eta_1, \ldots, \eta_{n-1})}{\partial(\omega_1, \ldots, \omega_{n-1})}.$$

(b) Given a non-singular transformation of variables

$$\tau: \quad (x_1, x_2, \ldots, x_n) \longrightarrow (x_1', x_2', \ldots, x_n'),$$

then the Last Multiplier M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x_1', x_2', \ldots, x_n')},$$

where M obviously comes from the $n-1$ solutions of $\mathcal{A}F = 0$ which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1}.

(c) One can prove that each multiplier M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0, \tag{5}$$

or equivalently:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = 0; \tag{6}$$

viceversa every solution M of this equation is a Jacobi Last Multiplier.

(d) If one knows two Jacobi Last Multipliers M_1 and M_2 of equation (1), then their ratio is a solution ω of (1), or, equivalently, a first integral of (2). Naturally the ratio may be quite trivial, namely a constant. Viceversa the product of a multiplier M_1 times any solution ω yields another last multiplier $M_2 = M_1\omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternate formulation in terms of symmetries was provided by Lie [18], [19]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [3]. If we know $n-1$ symmetries of (1)/(2), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \ldots, x_n)\partial_{x_j}, \quad i = 1, n-1, \tag{7}$$

Jacobi's last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}. \tag{8}$$

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation of motion is missing the variable associated with that component, i.e., $\partial a_i/\partial x_i = 0$, the last multiplier is a constant, and any other Jacobi Last Multiplier is a first integral.

Another property of the Jacobi Last Multiplier is its (almost forgotten) relationship with the Lagrangian, $L = L(t, q, \dot{q})$, for any second-order equation

$$\ddot{q} = F(t, q, \dot{q}) \tag{9}$$

i.e. [15] (Lecture 10)[2], [34]

$$M = \frac{\partial^2 L}{\partial \dot{q}^2} \tag{10}$$

where $M = M(t, q, \dot{q})$ satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \frac{\partial F}{\partial \dot{q}} = 0. \tag{11}$$

Then equation (9) becomes the Euler-Lagrange equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0. \tag{12}$$

The proof is based on taking the derivative of (12) with respect to $\dot{q}$ and showing that this yields (11). If one knows a Jacobi last multiplier, then L can be easily obtained by a double integration, i.e.:

$$L = \int \left(\int M \, \mathrm{d}\dot{q} \right) \mathrm{d}\dot{q} + f_1(t, q)\dot{q} + f_2(t, q), \tag{13}$$

where f_1 and f_2 are functions of t and q which have to satisfy a single partial differential equation related to (9) [25]. As it was shown in [25], f_1, f_2 are related to the gauge function $g = g(t, q)$. In fact, we may assume

$$\begin{aligned} f_1 &= \frac{\partial g}{\partial q} \\ f_2 &= \frac{\partial g}{\partial t} + f_3(t, q) \end{aligned} \tag{14}$$

where f_3 has to satisfy the mentioned partial differential equation and g is obviously arbitrary. We remark the importance of the gauge function in order to apply Noether's theorem [21] correctly. Therefore we do not annihilate the gauge function.

In [25] it was shown that if one knows several (at least two) Lie symmetries of the second-order differential equation (9), i.e.

$$\Gamma_j = V_j(t, q)\partial_t + G_j(t, q)\partial_q, \quad j = 1, r, \tag{15}$$

then many Jacobi Last Multipliers could be derived by means of (8), i.e.

$$\frac{1}{M_{nm}} = \Delta_{nm} = \det \begin{bmatrix} 1 & \dot{q} & F(t, q, \dot{q}) \\ V_n & G_n & \dfrac{\mathrm{d}G_n}{\mathrm{d}t} - \dot{q}\dfrac{\mathrm{d}V_n}{\mathrm{d}t} \\ V_m & G_m & \dfrac{\mathrm{d}G_m}{\mathrm{d}t} - \dot{q}\dfrac{\mathrm{d}V_m}{\mathrm{d}t} \end{bmatrix}, \tag{16}$$

with $(n, m = 1, r)$, and therefore many Lagrangians can be obtained by means of (13).

[2] *Jacobi's Lectures on Dynamics* are finally available in English [16].

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3. Lagrangians for the free particle

It is well-known since Lie's seminal work [19] that the equation of a free particle

$$\ddot{q} = 0 \tag{17}$$

admits an eight-dimensional Lie symmetry algebra[3], $sl(3, I\!R)$, generated by the following operators:

$$X_1 = qt\partial_t + q^2\partial_q, \qquad X_2 = q\partial_t, \qquad X_3 = t^2\partial_t + qt\partial_q, \qquad X_4 = q\partial_q,$$
$$X_5 = t\partial_t, \qquad X_6 = \partial_t, \qquad X_7 = t\partial_q, \qquad X_8 = \partial_q. \tag{18}$$

An obvious Jacobi Last multiplier (JLM) of (17) is a constant since $\dot{q}$ does not appear in its right-hand side. This also implies that any JLM is a first integral of (17). We can found this trivial JLM and many others by calculating the inverse of the determinant of the matrix (16) for all the possible combinations of two different operators in (18). It results that ten different JLM and consequently as many Lagrangians, by means of (13), can be obtained[4], i.e.:

$$\begin{aligned}
M_{13} &= -\frac{1}{(t\dot{q}-q)^3} &&\Rightarrow\quad L_{13} = -\frac{1}{2t^2(t\dot{q}-q)} + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{15} &= -\frac{1}{\dot{q}(t\dot{q}-q)^2} &&\Rightarrow\quad L_{15} = \frac{\dot{q}}{q^2}\left(\log(t\dot{q}-q) - \log(\dot{q})\right) + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{16} &= \frac{1}{\dot{q}^2(t\dot{q}-q)} &&\Rightarrow\quad L_{16} = \left(\frac{t\dot{q}}{q^2} - \frac{1}{q}\right)\left(\log(\dot{q}) - \log(t\dot{q}-q)\right) + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{17} &= -\frac{1}{(t\dot{q}-q)^2} &&\Rightarrow\quad L_{17} = -\frac{1}{t^2}\log(t\dot{q}-q) + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{18} &= \frac{1}{\dot{q}(t\dot{q}-q)} &&\Rightarrow\quad L_{18} = -\frac{\dot{q}}{q}\log(\dot{q}) - \left(\frac{1}{t} - \frac{\dot{q}}{q}\right)\log(t\dot{q}-q) + \frac{1}{t}(1+\log(q)) + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{26} &= -\frac{1}{\dot{q}^3} &&\Rightarrow\quad L_{26} = -\frac{1}{2\dot{q}} + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{28} &= \frac{1}{\dot{q}^2} &&\Rightarrow\quad L_{28} = -\log(\dot{q}) + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{38} &= \frac{1}{t\dot{q}-q} &&\Rightarrow\quad L_{38} = \left(\frac{\dot{q}}{t} - \frac{q}{t^2}\right)\left(\log(t\dot{q}-q) - 1\right) + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{48} &= -\frac{1}{\dot{q}} &&\Rightarrow\quad L_{48} = \dot{q}(1-\log(\dot{q})) + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)\\
M_{87} &= 1 &&\Rightarrow\quad L_{87} = \tfrac{1}{2}\dot{q}^2 + \frac{\mathrm{d}g}{\mathrm{d}t}(t,q)
\end{aligned} \tag{19}$$

The ten Lagrangians are NOT linked by the gauge function $g = g(t, q)$. We note that seven of the matrices (16) have determinant equal to zero, i.e.:

$$\Delta_{12}, \qquad \Delta_{14}, \qquad \Delta_{24}, \qquad \Delta_{35}, \qquad \Delta_{37}, \qquad \Delta_{57}, \qquad \Delta_{68}. \tag{20}$$

[3] Also reported in [9].
[4] Since $M_{nm} = -M_{mn}$ we have arbitrarily chosen the sign as we wished.

This is in agreement with the application of Jacobi's method to the linear harmonic oscillator [25] that yielded fourteen different Lagrangians and three matrices having determinant equal to zero.
Applying Noether's theorem yields that both L_{13} and L_{87} admit five Noether point symmetries, the maximum possible [20] in the case of equation (17). The main difference between the two Lagrangians is that one Lagrangian only is independent on time and is the traditional Lagrangian, namely the kinetic energy L_{87}. Moreover the two Lagrangians admit different Noether point symmetries. In fact Lagrangian L_{13} admits the following five Noether symmetries and corresponding first integrals of equation (17)

$$\begin{aligned} X_1 &\implies Int_1 = -\frac{\dot{q}}{q-t\dot{q}}, \\ X_2 &\implies Int_2 = \frac{\dot{q}^2}{2(q-t\dot{q})^2}, \\ X_3 &\implies Int_3 = -\frac{1}{q-t\dot{q}}, \\ X_4 - X_5 &\implies Int_4 = -\frac{\dot{q}}{(q-t\dot{q})^2}, \\ X_7 &\implies Int_7 = -\frac{1}{2(q-t\dot{q})^2}, \end{aligned} \tag{21}$$

while Lagrangian L_{87} admits the following five Noether symmetries and corresponding first integrals of equation (17)

$$\begin{aligned} X_3 &\implies In_3 = -\tfrac{1}{2}(q-t\dot{q})^2, \\ X_4 + 2X_5 &\implies In_4 = -\dot{q}(q-t\dot{q}), \\ X_6 &\implies In_6 = \tfrac{1}{2}\dot{q}^2, \\ X_7 &\implies In_7 = q-t\dot{q}, \\ X_8 &\implies In_8 = -\dot{q}. \end{aligned} \tag{22}$$

4. Schrödinger equations for the free particle

The Schrödinger equation for the free particle is

$$2i\psi_t + \psi_{xx} = 0. \tag{23}$$

We show that this equation can be obtained by considering a generic linear parabolic equation

$$2i\psi_t + f_1(x)\psi_{xx} + f_2(x)\psi_x + f_3(x)\psi = 0 \tag{24}$$

with $f_k, (k = 1, 3)$ functions of x to be determined in such a way that equation (24) admits the following five Lie symmetries[5]

$$\begin{aligned} X_3 &\Rightarrow \Omega_1 = t^2\partial_t + xt\partial_x + \omega_1\partial_\psi, \\ X_4 + 2X_5 &\Rightarrow \Omega_2 = 2t\partial_t + x\partial_x + \omega_2\partial_\psi, \\ X_6 &\Rightarrow \Omega_3 = \partial_t + \omega_3\partial_\psi, \\ X_7 &\Rightarrow \Omega_4 = t\partial_x + \omega_4\partial_\psi, \\ X_8 &\Rightarrow \Omega_5 = \partial_x + \omega_5\partial_\psi. \end{aligned} \tag{25}$$

[5] We have identified q with x.

where $\omega_i = \omega_i(t, x, \psi), (i = 1, 5)$ are functions of t, x, ψ that have to be determined. Equation (24) also admits the following two symmetries

$$\Omega_6 = \psi\partial_\psi, \qquad \Omega_\alpha = \alpha(t, x)\partial_\psi \tag{26}$$

with α any solution of equation (24) itself, since any linear partial differential equation possesses these two symmetries.

Using the interactive REDUCE programs [22], we obtain that

$$f_1 = 1, \quad f_2 = f_3 = 0, \tag{27}$$

and

$$\omega_1 = \tfrac{1}{2}(ix^2 - t)\psi, \quad \omega_2 = \omega_3 = \omega_5 = 0, \quad \omega_4 = ix\psi. \tag{28}$$

Therefore the Noether symmetries admitted by the Lagrangian L_{87}, namely the physical Lagrangian for the free particle, yield the right Schrödinger's equation (23) and thus the correct quantization procedure is achieved.

We now show that a quantum-correct Schrödinger equation can be derived even from the five Noether symmetries admitted by the time-dependent Lagrangian L_{13}. We consider a generic linear partial differential equation

$$f_{11}(t, x)\psi_{tt} + f_{12}(t, x)\psi_{tx} + f_{22}(t, x)\psi_{xx} + f_1(t, x)\psi_t + f_2(t, x)\psi_x + f_0(t, x)\psi = 0 \tag{29}$$

with $f_{rs}, f_r, (r, s = 1, 2), f_0$ functions of t, x to be determined in such a way that equation (29) admits the following five Lie symmetries[6]

$$\begin{aligned}
X_1 &\Rightarrow W_1 = xt\partial_t + x^2\partial_x + w_1\partial_\psi, \\
X_2 &\Rightarrow W_2 = x\partial_t + w_2\partial_\psi, \\
X_3 &\Rightarrow W_3 = t^2\partial_t + xt\partial_x + w_3\partial_\psi, \\
X_4 - X_5 &\Rightarrow W_4 = -t\partial_t + x\partial_x + w_4\partial_\psi, \\
X_7 &\Rightarrow W_5 = t\partial_x + w_5\partial_\psi.
\end{aligned} \tag{30}$$

where $w_i = w_i(t, x, \psi), (i = 1, 5)$ are functions of t, x, ψ that have to be determined. Equation (29) also admits the two symmetries

$$W_6 = \psi\partial_\psi, \qquad W_\beta = \beta(t, x)\partial_\psi \tag{31}$$

with β any solution of equation (29) itself.

Using the interactive REDUCE programs [22], we obtain that equation (29) becomes

$$4t^2\psi_{tt} + 8tx\psi_{tx} + 4x^2\psi_{xx} + 12t\psi_t + 12x\psi_x + 3\psi = 0 \tag{32}$$

with

$$w_1 = -\tfrac{1}{2}x\psi, \quad w_2 = w_4 = w_5 = 0, \quad w_3 = -\tfrac{1}{2}t\psi. \tag{33}$$

Equation (32) is parabolic and therefore can be put into its normal form since its characteristic coordinates are

$$\xi = \frac{x}{t}, \quad x = x, \quad \psi = \phi(\xi, x). \tag{34}$$

[6] We have identified q with x.

Thus equation (32) transforms into

$$4x^2\phi_{xx} + 12x\phi_x + 3\phi = 0 \tag{35}$$

with solution:

$$\phi(\xi, x) = \alpha_1(\xi)x^{-1/2} + \alpha_2(\xi)x^{-3/2}, \tag{36}$$

where α_1, α_2 are arbitrary functions of ξ. This is obviously in agreement with the quantization of the free particle.

We remark that there is some freedom when imposing equation (29) to admit the Lie symmetries (30) but in any possible cases a parabolic equation with characteristic coordinates (34) is always obtained. For example another possibility is

$$4t^4\psi_{tt} + 8t^2x\psi_{tx} + 4t^2x^2\psi_{xx} + 4t^2(3t+x)\psi_t + 4tx(3t+x)\psi_x + (3t^2 + 4tx + x^2)\psi = 0 \tag{37}$$

with

$$\begin{aligned} w_1 = -\tfrac{1}{2}\left(1 + \frac{x}{t}\right)x\psi, \quad w_2 = \frac{x^2}{2t^2}\log(t)\psi, \quad w_3 = -\tfrac{1}{2}(t+x)\psi, \\ w_4 = \frac{x}{t}\log(t)\psi, \quad w_5 = -\tfrac{1}{2}\log(t)\psi. \end{aligned} \tag{38}$$

The normal form of equation (37) is

$$4x^2\phi_{xx} + (12 + 4\xi)x\phi_x + (3 + 4\xi + \xi^2)\phi = 0 \tag{39}$$

with solution:

$$\phi(\xi, x) = \alpha_1(\xi)x^{-3/2-\xi/2} + \alpha_2(\xi)x^{-1/2-\xi/2}, \tag{40}$$

also in agreement with the quantization of the free particle.

5. Schrödinger equation for the second-order Riccati equation

In [28] it was shown that the linearizable second-order Riccati equation, a member of the Riccati-chain [1], i.e.

$$\ddot{x} + 3x\dot{x} + x^3 = 0 \tag{41}$$

possesses many JLM and therefore Lagrangians. In particular the following time-independent Lagrangian[7]

$$Lagr = -\frac{1}{2(\dot{x} + x^2)} \tag{42}$$

was shown to admit five Noether point symmetries, i.e.

$$\Gamma_2 - \Gamma_8, \quad \Gamma_3 - \frac{2}{3}\Gamma_7, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_6 \tag{43}$$

among the following eight Lie symmetries admitted by equation (41):

$$\begin{aligned} \Gamma_1 &= t^3(tx-2)\partial_t - t(xt-2)(x^2t^2 + 2 - 2xt)\partial_x \\ \Gamma_2 &= xt^3\partial_t - (xt-1)(x^2t^2 + 4 - 2xt)\partial_x \\ \Gamma_3 &= xt^2\partial_t - x(x^2t^2 + 2 - 2xt)\partial_x \\ \Gamma_4 &= xt\partial_t - x^2(xt-1)\partial_x \\ \Gamma_5 &= x\partial_t - x^3\partial_x \\ \Gamma_6 &= \partial_t \\ \Gamma_7 &= t\partial_t - x\partial_x \\ \Gamma_8 &= t^2\partial_t - 2(xt-1)\partial_x. \end{aligned} \tag{44}$$

[7] This Lagrangian has also been studied in [5].

Lagrangian (42) was obtained from the inverse of the determinant of the matrix (16) with the two symmetries Γ_5 and Γ_6 and the application of formula (13).

It interesting to show that equation (41) is linked to the following cubic-power dissipative equation

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{t}^2}+\left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}}\right)^3=0 \tag{45}$$

by means of the canonical form of the two-dimensional Lie algebra generated by Γ_5 and Γ_6. In fact this Lie algebra corresponds to Type I in Lie's classification of the real two-dimensional Lie algebras in the plane [19], [3], namely it is abelian and transitive[8] and therefore its canonical coordinates are

$$\partial_{\tilde{t}}, \qquad \partial_{\tilde{x}}. \tag{46}$$

Therefore the following identification

$$\Gamma_5=\partial_{\tilde{t}}, \qquad \Gamma_6=\partial_{\tilde{x}} \tag{47}$$

yields the transformation

$$\tilde{t}=\frac{1}{2x^2}, \qquad \tilde{x}=\frac{tx-1}{x} \tag{48}$$

that takes (41) into (45).

Equation (45) is also linearizable and admits an eight-dimensional Lie symmetry algebra. Work is in progress to quantize n-power dissipative equations with the method illustrated here and to compare it to other proposed method, e.g. [29], [30], [31].

We now quantize the second-order Riccati equation (41) by imposing equation (29) to admit the following five Lie symmetries

$$\begin{aligned}
\Gamma_2-\Gamma_8 \quad &\Rightarrow \quad \Lambda_1=(xt-1)t^2\partial_t-(xt-1)(x^2t^2-2-2xt)\partial_x+\lambda_1\partial_\psi,\\
\Gamma_3-\frac{2}{3}\Gamma_7 \quad &\Rightarrow \quad \Lambda_2=\left(xt-\frac{2}{3}\right)t\partial_t-x\left(x^2t^2+\frac{4}{3}-2xt\right)\partial_x+\lambda_2\partial_\psi,\\
\Gamma_4 \quad &\Rightarrow \quad \Lambda_3=xt\partial_t-x^2(xt-1)\partial_x+\lambda_3\partial_\psi,\\
\Gamma_5 \quad &\Rightarrow \quad \Lambda_4=x\partial_t-x^3\partial_x+\lambda_4\partial_\psi,\\
\Gamma_6 \quad &\Rightarrow \quad \Lambda_5=\partial_t+\lambda_5\partial_\psi.
\end{aligned} \tag{49}$$

where $\lambda_i=\lambda_i(t,x,\psi),(i=1,5)$ are functions of t,x,ψ that have to be determined. We remind that equation (29) also admits the two symmetries (31).

Using the interactive REDUCE programs [22], we obtain that equation (29) becomes

$$4\psi_{tt}-8x^2\psi_{tx}+4x^4\psi_{xx}+8x^3\psi_x-3x^2\psi=0 \tag{50}$$

with

$$\begin{aligned}
\lambda_1=-\frac{(tx-1)^3}{2x}\psi, \quad \lambda_2=-\tfrac{1}{2}(tx-1)^2\psi, \quad \lambda_3=-\tfrac{1}{2}(tx-1)x\psi\\
\lambda_4=-\tfrac{1}{2}x^2\psi, \quad \lambda_5=-\frac{tx-1}{x}\psi.
\end{aligned} \tag{51}$$

[8] In [28] a missprint led to the erroneously statement that Γ_5 and Γ_6 generate an intransitive Lie algebra.

Equation (50) is parabolic and therefore can be put into its normal form since its characteristic coordinates are

$$\varrho = t - \frac{1}{x}, \quad x = x, \quad \psi = \phi(\varrho, x). \tag{52}$$

Thus equation (50) transforms into

$$4x^2\phi_{xx} + 8x\phi_x - 3\phi = 0 \tag{53}$$

with solution:

$$\phi(\varrho, x) = \beta_1(\varrho)x^{1/2} + \beta_2(\varrho)x^{-3/2}, \tag{54}$$

where β_1, β_2 are arbitrary functions of ϱ. Is this the correct quantization of equation (41)? It seems so but further insight is needed especially from the experimentalists.

6. Final remarks

In [23] a method was proposed to overcome the deadlock of nonlinear canonical transformations when quantizing with the known procedures [26]. It consists of the following steps to be applied to a classical Lagrangian equation[9]:

- Find the Lie symmetries of the Lagrangian equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Among the Lie symmetries find the Noether symmetries admitted by the given Lagrangian

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting these Noether symmetries as Lie symmetries

$$2i\psi_t + f_1(x)\psi_{xx} + f_2(x)\psi_x + f_3(x)\psi = 0$$

$$\Omega = V_0(t,x)\partial_t + V_1(t,x)\partial_x + G(t,x,\psi)\partial_\psi$$

- Summarizing: quantize preserving the Noether symmetries

In [24] this method has been applied to classical Lagrangian systems. In particular it led to the Schrödinger equation of a known completely integrable and solvable many-body problem, the so-called ‘goldfish’ [4]. In the case of Lagrangian systems the method consists of the following steps[10]:

- Find the Lie symmetries of the Lagrangian system

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

- Among the Lie symmetries find the Noether symmetries admitted by the given Lagrangian

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

[9] The physical Lagrangian turns out to admit the highest number of Noether point symmetries.
[10] The physical Lagrangian turns out to admit the highest number of Noether point symmetries.

- Construct the Schrödinger equation admitting these Noether symmetries as Lie symmetries

$$2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})\psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})\psi_{x_k} + f_3(\underline{x})\psi = 0$$

$$\Omega = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k} + G(t, \underline{x}, \psi)\partial_\psi$$

- Summarizing: quantize preserving the Noether symmetries

In this paper we have proposed a method that takes classical equations into the quantum realm by means of the Schrödinger equation regardless of the number of Lagrangians that may classically exist. The method consists of the following steps:

- Find the Lie symmetries of the given equation

$$\Upsilon = W_0(t, x)\partial_t + W_1(t, x)\partial_x$$

- Among many Lagrangians identify the one[11] that admits the highest number of Noether symmetries among the Lie symmetries

$$\Gamma = V_0(t, x)\partial_t + V_1(t, x)\partial_x, \qquad \Gamma \subset \Upsilon$$

- Construct a linear partial differential equation admitting these Noether symmetries as Lie symmetries
$f_{11}(t, x)\psi_{tt} + f_{12}(t, x)\psi_{tx} + f_{22}(t, x)\psi_{xx} + f_1(t, x)\psi_t + f_2(t, x)\psi_x + f_0(t, x)\psi = 0$

$$\Omega = V_0(t, x)\partial_t + V_1(t, x)\partial_x + G(t, x, \psi)\partial_\psi$$

- Since this equation is parabolic, determine its canonical coordinates in order to transform it into its normal form, namely the Schrödinger equation
- Summarizing: quantize preserving the Noether symmetries.

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