CRITICAL POINT APPROACHES FOR SECOND-ORDER DYNAMIC STURM–LIOUVILLE BOUNDARY VALUE PROBLEMS

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Abstract. Economics is a direction in which there becomes visible to be many occasions for applications of time scales. The time scales approach will not only unify the standard discrete and continuous models in economics, but also, for example, authorizes for payments that reach unequally spaced points in time. We are going to study dynamic optimization problems from economics, construct a time scales model, and apply variational methods and critical point theory to obtain the existence of solutions. We derive several conditions ensuring existence of solutions of dynamic Sturm–Liouville boundary value problems. Variational methods are utilized in the proofs. We discuss the existence of at least one, three and infinitely many solutions for the problems under different conditions on the data. Examples are also given to illustrate the main results.

Contents

1. Introduction

Differential equations always play an essential role in the field of mathematics and calculus. In order to calculate the rate of change in one quantity with respect to the other one, we need differential equations. As differential equations are used in many fields like physics, chemistry, biology, economics and finance, so the existence of solutions of differential equations has been investigated by many researchers recently. Finding solution is not a single process but a set of processes where

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some binary operations, repetitions of the operations, checking the consistency of functions within the interval etc. take place. Here it becomes very important that the time taken to execute the differential equations problem must be least. Thus a vital need of pipelining performance improvement approaches, parallel processing etc. is demanded. There are many places where differential equations are used in the critical situation for example weather forecasting, treatment of diseases like analysis of heart beat, infection evolution, cancer etc. Therefore, there must be quick output from differential equations required which is acquired by the parallel approach to solve the differential equations faster than the sequential approach [27, 56, 60].

Time scales theory was created by Hilger [48] in 1988, and it serves to unify continuous and discrete analysis. Moreover, basic elements of variational calculus on time scales were introduced in 2004 [9]. Time scales theory has received considerable attention due to its potential applications in the study of epidemic models, population models, finance, stock market, economics [8], and heat transfer, and it has since then been further developed by many authors in several different directions, e.g., [31, 49, 55].

In economic modeling, either continuous timing or discrete timing is present, and there is not a common view among economists on which representation of time is better for economic models [32]. Concurrently, many results regarding differential equations may carry over to related results for difference equations, while other results seem to be completely different in nature from their continuous analogs [21].

Recently, BVPs for dynamic equations on time scales have been extensively studied by many researchers. Various methods and techniques have been applied, such as methods of lower and upper solutions, cone-theoretic fixed point theorems, variational methods, and coincidence degree theory.

The general assumption that economic processes are either entirely continuous or entirely discrete, while advantageous for traditional mathematical approaches, may sometimes be unsuitable, because in reality many economic occurrences present both continuous and discrete components. In biology, an intimate example is a seasonal breeding population in which generations do not overlap [32]. A similar example in economics is the seasonally changing investment and revenue in which seasons play an important effect on this kind of economic activity. Therefore, there is a great need to find a more adaptable mathematical structure to precisely model the dynamical blend of such systems, so that they are exactly described and better understood. To meet this requirement, an emerging, progressive and modern area of mathematics, known as dynamic equations on time scales, has been introduced.

In the last decades, this kind of equation arises in numerous problems in finance, economics and management. Research on this field and its applications have become a notable endeavor among researchers in mathematical finance, optimal control and differential equation. Recently, mathematical modelling and computer simulation have become important in all scientific research. In order to gain both quantitative and qualitative features, one should resort to nonlinear modelling frameworks being increasingly employed to explore the intricate dynamics of complex systems which are capable of exhibiting rich features like self-organization and multiple equilibria.

Many classical results of variational calculus such as sufficient and necessary conditions for optimality have been generalized to arbitrary time scales. Because many economic and finance models are dynamic models, the results of time scale calculus are directly applicable to economics as well, see [14,15,19,21,37]. Solutions of ordinary differential equations, such as initial value problems and boundary value problems, have been studied and published during the past two decades on time scales. In 2002, Hoffacker [52] and Ahlbrandt and Morian [7] demonstrated the related ideas to the multivariate case and studied partial dynamic equations on time scales. Notations and definitions on multivariate time scales calculus can be found in Bohner and Guseinov [16, 18]. For a more current reference for the multivariate case we refer to [17]. Jackson [53] extended the existing ideas of the time scales calculus [7] to the multivariate case. The method of generalized Laplace transform on time scales is applied to find solutions of the homogeneous and nonhomogeneous heat and wave equations. Recent developments in the method of finding solutions have aroused further interest in the discussion of partial dynamic equations on time scales. Very recently, the study of boundary value problems for dynamic equations on time scales develops at relatively rapid rate. By applying various methods and techniques, such as the cone theoretic fixed point theorems, the method of upper and lower solutions, coincidence degree theory, variational methods, a series of existence results of solutions or positive solutions have been established in the literature, see [3, 5, 6, 10, 28, 30, 35, 43, 57, 63, 65, 66, 68] and the references therein. For example, in [5,6], Agarwal et al. respectively studied the following equation on time scales

$$\begin{cases}-u^{\Delta\Delta}(t) = f(t, u^{\sigma}(t)), \qquad t \in [0, T]_{\mathbb{T}}\end{cases}$$

or

$$\begin{cases}-u^{\Delta\Delta}(t) = f(\sigma(t), u^{\sigma}(t)), \qquad t \in [0, T]_{\mathbb{T}}\end{cases}$$

with Dirichlet boundary condition, and established some existence criteria of single and multiple positive solutions by using variational techniques. Çetin and Topal in [28] by using the Krasnosel'skii fixed point theorem, obtained existence of at least one or two symmetric positive solutions of the above problem on time scales. Zhang and Sun in [66] by using variational method and critical point theory, obtained that the boundary value problem has solutions for ν being in some different intervals for eigenvalue boundary value problems on time scales. Eckhardt and Teschl in [30] established the connection between Sturm–Liouville equations on time scales and Sturm–Liouville equations with measure valued coefficients. Based on this connection, they generalized several results for Sturm–Liouville equations on time scales, which have been obtained by various authors in the past. Thiramanus and Tariboon in [63] by using the Krasnosel'skii fixed point theorem, the Avery–Henderson fixed point theorem and the Leggett–Williams fixed point theorem, obtained some results for the existence of at least one, two or three positive solutions of m-point integral boundary value problems for nonlinear second order p-Laplacian dynamic equations on time scales. Zhou and Li in [68], by using variational methods and critical point theory, obtained the existence of nontrivial periodic solutions for a class of p-Laplacian systems on time scales.

Also, Sturm–Liouville equations on time scales have attracted substantial interest, see for example [3, 30, 35, 36, 50, 51, 57, 65, 66, 68]. Zhang and Sun in [66], using critical point theory and variational methods, have established existence of solutions for $(P_{\nu,0})$. First, they have ensured an existence interval for ν such that $(P_{\nu,0})$ possesses one or two solutions. Then, under entirely different assumptions on f and by employing a three critical point theorem, they have derived some sufficient conditions for existence of at least three solutions for $(P_{\nu,0})$ when ν is located in a certain interval. In [30], the authors established a connection between dynamic Sturm–Liouville equations and corresponding equations with measure-valued coefficients. Based on this, they generalized several known results for dynamic Sturm–Liouville equations. In [57], Ozkan considered a boundary value problem involving a dynamic Sturm–Liouville equation and boundary conditions depending on a spectral parameter, and he also introduced an operator formulation for the problem and gave several properties of eigenvalues and eigenfunctions. Finally, for finite time scales, he derived the exact number of eigenvalues of the problem. In [65], existence of at least one and at least two positive solutions for $(P_{1,0})$ was obtained. In [61], for a periodic time scale, Su and Feng have studied a dynamic second-order p-Laplacian equation together with certain boundary value conditions. They used the three critical point theorem, the least action principle, and the saddle point theorem in order to obtain existence of at least one or at least three distinct periodic solutions. Existence results on periodic time scales, by establishing a suitable variational setting, were also proved in [68] for a class of dynamic p-Laplacian systems.

Here we are going to discuss the existence of one solution and multiple solutions for dynamic Sturm–Liouville boundary value problems which turns out as an optimization problems on time scales which arises in economics and finance, using variational methods and critical point theory under different assumptions. To apply variational methods to our models first we construct related energy functionals and by assuming appropriate hypotheses on the data we prove the existence of critical points of the energy functionals and we show that the critical points are solutions of our models. Moreover, we overcome some technical difficulties treating the problems. First, under an asymptotical behavior of the nonlinear datum at zero we discuss the existence of solutions for our models. After that, we obtain the existence of at least three distinct nonnegative solutions for the double eigenvalue models by utilizing a critical point result. Requiring an additional asymptotical behaviour of the data at zero, nontriviality of the solution can also be achieved under appropriate assumptions. Moreover, we investigate the existence of solutions for $\nu \to 0^+$. Then, we study the existence of solutions for the second order Sturm–Liouville type boundary value problem on time scales, under an appropriate oscillating behaviour of the nonlinear term f, and we determine the exact collections of the parameter ν in which the problem for every non negative arbitrary function g which is measurable in $[0, T]$ and of class $C^1(\mathbb{R})$ satisfying a certain growth at infinity, choosing ζ sufficiently small, admits infinitely many solutions. Replacing the oscillating behaviour condition at infinity, with a similar one at zero, we obtain a sequence of pairwise distinct solutions that converges to zero. We also list some consequences the main results. The applicability of our results is illustrated by several examples. The main references are the papers [10, 41, 43].

2. Time scale calculations and notations

This section is devoted to introduce some basic notations and results on time scale.

Let be given any time scale $\mathbb{T}$, namely, a closed and nonempty subset of the real numbers. In particular, $\mathbb{T} = \mathbb{Z}$ (all integers) and $\mathbb{T} = \mathbb{R}$ (all real numbers) are two examples of time scales, and so-called dynamic equations on these time scales correspond to difference equations and differential equations, respectively.

Definition 1. One defines the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\zeta : \mathbb{T} \to \mathbb{R}^+$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \ \rho(t) = \sup\{s \in \mathbb{T} : s < t\},\ \ \zeta(t) = \sigma(t) - t \text{ for every } t \in \mathbb{T},$$

respectively. If $\sigma(t) = t$, then t is called right dense (otherwise: right scattered), and if $\rho(t) = t$, then t is called left dense (otherwise: left scattered). Denote $y^\sigma(t) = y(\sigma(t))$ and $\sigma^2(T) = \sigma(\sigma(T))$. We are going to give some calculus on time scales which can be found in [21]. Let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* (or simply: *differentiable*) at $t \in \mathbb{T}$ provided there exists an α such that for all ε there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in \mathcal{N}.$$

In this case we denote the α by $f^\Delta(t)$, and if f is differentiable for every $t \in \mathbb{T}$, then f is said to be *differentiable* on $\mathbb{T}$ and f^Δ is a new function defined on $\mathbb{T}$. If f is differentiable at $t \in \mathbb{T}$, then it is easy to see that

$$f^\Delta(t) = \begin{cases} \lim_{s \to t, s \in \mathbb{T}} \frac{f(t) - f(s)}{t - s} & \text{if} \quad \zeta(t) = 0 \\ \frac{f(\sigma(t)) - f(t)}{\zeta(t)} & \text{if} \quad \zeta(t) > 0. \end{cases} \tag{1}$$

To illustrate the idea, we now give another formula, which holds whenever f is differentiable at $t \in \mathbb{T}$:

$$f(\sigma(t)) = f(t) + \zeta(t) f^\Delta(t). \tag{2}$$

When applying formula (2), we do not need to distinguish between the two cases $\zeta(t) = 0$ and $\zeta(t) > 0$. Formula (2) holds in both of these cases. Two further examples of such formulas are the product rule for the derivative of the product of two differentiable functions f and g:

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f(\sigma(t)) g^\Delta(t) \tag{3}$$

and the quotient rule for the derivative of the quotient of two differentiable functions f and $g \neq 0$:

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t) g(t) - f(t) g^\Delta(t)}{g(t) g(\sigma(t))}. \tag{4}$$

Clearly, $1^\Delta = 0$ and $t^\Delta = 1$, so we can use (3) to find

$$(t^2)^\Delta = (t \cdot t)^\Delta = t + \sigma(t),$$

and we can use (4) to find

$$\left(\frac{1}{t}\right)^\Delta = -\frac{1}{t\sigma(t)}.$$

Other formulas may be obtained likewise.

Here, F is called an antiderivative of a function f defined on $\mathbb{T}$ if $F^\Delta = f$ holds on $\mathbb{T}$. In this case we define an integral by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \quad \text{where} \quad s, t \in \mathbb{T}.$$

An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a polynomial (or any "nice" formula of a function) which is an antiderivative of t (where $\mathbb{T}$ is an arbitrary time scale). The role of t^2 is therefore played in the time scales calculus by

$$\int_0^t \sigma(\tau)\Delta\tau \quad \text{and} \quad \int_0^t \tau\Delta\tau.$$

Note that both integrals exist as the functions σ and identity are both continuous.

For $f \in L^1_\Delta([t_1,t_2)_\mathbb{T})$, we denote for convenience $\int_{t_1}^{t_2} f(s)\Delta s = \int_{[t_1,t_2)\cap\mathbb{T}} f(s)\Delta s$. In order to study our problems on time scales, the variational setting is the space

$$\mathcal{H} := H^1_\Delta([0,\sigma^2(T)]) =$$

$$\left\{u : [0,\sigma^2(T)] \to \mathbb{R} :\ u \in AC[0,\sigma^2(T)] \text{ and } \ u^\Delta \in L^2_\Delta([0,\sigma^2(T)))\right\}.$$

Then $H^1_\Delta([0,\sigma^2(T)])$ is a Hilbert space with the inner product,

$$(u,v)_{H^1_\Delta} = \int_0^{\sigma^2(T)} u(t)v(t)\Delta t + \int_0^{\sigma^2(T)} u^\Delta(t)v^\Delta(t)\Delta t$$

(see [67]), and let $\|\cdot\|_{H^1_\Delta}$ be the norm induced by the inner product $(\cdot,\cdot)_{H^1_\Delta}$. For every $u,v \in H^1_\Delta([0,\sigma^2(T)])$, we define

$$(u,v)_0 = \int_0^{\sigma^2(T)} p(t)u^\Delta(t)v^\Delta(t)\Delta t + \int_0^{\sigma(T)} q(t)u^\sigma(t)v^\sigma(t)\Delta t$$
$$+\beta_1 p(0)u(0)v(0) + \beta_2 p(\sigma(T))u(\sigma^2(T))v(\sigma^2(T))$$

where

$$\beta_1 = \begin{cases} \dfrac{\alpha_1}{\alpha_2}, & \text{if } \alpha_2 > 0,\\ 0, & \text{if } \alpha_2 = 0 \end{cases} \tag{5}$$

and

$$\beta_2 = \begin{cases} \dfrac{\alpha_3}{\alpha_4}, & \text{if } \alpha_4 > 0,\\ 0, & \text{if } \alpha_4 = 0. \end{cases} \tag{6}$$

We let $\|u\|_0$ be the norm induced by the inner product $(u,v)_0$.

Lemma 2. *[66, Lemmas 2.1, 2.2 and 4.2] The immersion $H^1_\Delta([0,\sigma^2(T)]) \hookrightarrow C([0,\sigma^2(T)])$ is compact. If $u \in H^1_\Delta([0,\sigma^2(T)])$, then*

$$|u(t)| \le \sqrt{2}\max\left\{(\sigma^2(T))^{\frac12}, (\sigma^2(T))^{-\frac12}\right\}\|u\|_{H^1_\Delta} \text{ for every } t \in [0,\sigma^2(T)].$$

If $\alpha_2,\ \alpha_4 > 0$ or $q(t) > 0$ for every $t \in [0,T]$, then for every $u \in H^1_\Delta([0,\sigma^2(T)])$, $|u(t)| \le C\|u\|_0$ for every $t \in [0,\sigma^2(T)]$, where $C = \min\{M_1, M_2, M_3\}$ and

$$M_1 = \sqrt{2}\max\left\{\frac{1}{\sqrt{\beta_1 p(0)}}, \frac{\sqrt{\sigma^2(T)}}{\min_{t\in[0,\sigma(T)]} p(t)}\right\},$$

$$M_2 = \sqrt{2}\max\left\{\frac{1}{\sqrt{\beta_2 p(0)}}, \frac{\sqrt{\sigma^2(T)}}{\min_{t\in[0,\sigma(T)]} p(t)}\right\},$$

$$M_3 = \sqrt{2}\max\left\{\frac{\sqrt{\sigma(T)}}{\min_{t\in[0,T]} q(t)}, \frac{\sqrt{\sigma^2(T)}}{\min_{t\in[0,\sigma(T)]} p(t)}\right\},$$

and where $\dfrac{1}{0} = +\infty$.

3. The existence of one solution

3.1. **Main results.** Let be given any time scale $\mathbb{T}$, namely, a closed and nonempty subset of the real numbers. Let $T > 0$ be fixed and suppose $0, T \in \mathbb{T}$. In this contribution, we study the second-order dynamic Sturm–Liouville BVP (boundary value problem)

$$\begin{cases} -(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = f(t, x^{\sigma}(t)), \quad t \in [0,T]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^{\Delta}(0) = 0, \quad \alpha_3 x(\sigma^2(T)) + \alpha_4 x^{\Delta}(\sigma(T)) = 0, \end{cases} \tag{P^f}$$

where

$$p \in \mathrm{C}^1([0,\sigma(T)]_{\mathbb{T}}, (0,\infty)), \quad q \in \mathrm{C}([0,T]_{\mathbb{T}}, [0,\infty)), \quad f \in \mathrm{C}([0,T]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R}),$$
$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2 \geq 0, \quad \alpha_1 + \alpha_3, \alpha_3 + \alpha_4 > 0.$$

In this section, we discuss existence of *at least one* nontrivial solution of the second-order dynamic Sturm–Liouville boundary value problem (P^f), under a certain asymptotical assumption of the nonlinearity at zero (Theorem 4). In Theorem 5, we present an application of Theorem 4. Moreover, we give some observations and remarks on our results. As a special case of our result, we present Theorem 12, in the case when the function f does not depend on time. Finally, we offer Example 13, in which all hypotheses of Theorem 12 are satisfied. In this section, we prove the existence of at least one nontrivial solution for (P^f).

The main argument in our results is a famous variational principle by Ricceri [59, Theorem 2.1], in the special form given by Bonanno and Molica Bisci in [25]. This principle has been extensively applied to a variety of problems, and we refer to [1, 2, 12, 13, 33, 46, 47].

Theorem 3. *Assume $\mathcal{B}$ is a real and reflexive Banach space. Let be given two Gâteaux-differentiable functionals $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{B} \to \mathbb{R}$ so that $\mathcal{J}_1$ is strongly continuous, sequentially weakly lower semicontinuous, and coercive in $\mathcal{B}$, and $\mathcal{J}_2$ is sequentially weakly upper semicontinuous in $\mathcal{B}$. Define the functional I_ν by $I_\nu := \mathcal{J}_1 - \nu\mathcal{J}_2$, $\nu \in \mathbb{R}$. Moreover, for any $s > \inf_{\mathcal{B}} \mathcal{J}_1$, define the function φ by*

$$\varphi(s) := \inf_{x \in \mathcal{J}_1^{-1}(-\infty,s)} \frac{\sup_{y \in \mathcal{J}_1^{-1}(-\infty,s)} \mathcal{J}_2(y) - \mathcal{J}_2(x)}{s - \mathcal{J}_1(x)}.$$

Then, for any $s > \inf_{\mathcal{B}} \mathcal{J}_1$ and any $\nu \in \left(0, \dfrac{1}{\varphi(s)}\right)$, the restriction of the functional I_ν to $\mathcal{J}_1^{-1}(-\infty, s)$ has a global minimum, which is a critical point (more precisely, a local minimum) of I_ν in $\mathcal{B}$.

We now give our main result for (P^f).

Theorem 4. *Assume*

$$\sup_{\theta>0} \frac{\theta^2}{F_\theta} > 2C^2, \quad \textit{where} \quad F_\theta = \int_0^{\sigma(T)} \max_{|\xi|\leq\theta} F(t,\xi)\Delta t. \tag{S}$$

Then, (P^f) has at least one solution in $\mathcal{H}$.

Proof. We will apply Theorem 3 to (P^f). Let $\mathcal{B} = \mathcal{H}$. Let us introduce the two functionals $\mathcal{J}_1, \mathcal{J}_2$ by

$$\mathcal{J}_1(x) = \frac{1}{2}\|x\|_0^2 \tag{7}$$

and

$$\mathcal{J}_2(x) = \int_0^{\sigma(T)} F(t, x^\sigma(t))\Delta t \tag{8}$$

for $x \in \mathcal{B}$. We define

$$I(x) = \mathcal{J}_1(x) - \mathcal{J}_2(x) \quad \text{for} \quad x \in \mathcal{B}.$$

We now show that $\mathcal{J}_1$ and $\mathcal{J}_2$ fulfill the conditions assumed in Theorem 3. As $\mathcal{B}$ is compactly embedded in $(\mathrm{C}^0([0,T]_{\mathbb{T}}), \mathbb{R})$, it is easy to see that $\mathcal{J}_2$ is Gâteaux differentiable, and the Gâteaux derivative of $\mathcal{J}_2$ at $x \in \mathcal{B}$ is $\mathcal{J}_2'(x) \in \mathcal{B}^*$ given as

$$\mathcal{J}_2'(x)(y) = \int_0^{\sigma(T)} f(t, x^\sigma(t)) y^\sigma(t)\Delta t \quad \text{for all} \quad y \in \mathcal{B},$$

and $\mathcal{J}_2$ is sequentially weakly upper semicontinuous. Additionally, $\mathcal{J}_1$ is also Gâteaux differentiable, and the Gâteaux derivative of $\mathcal{J}_1$ at $x \in \mathcal{B}$ is $\mathcal{J}_1'(x) \in \mathcal{B}^*$ given as

$$\begin{aligned}
\mathcal{J}_1'(x)(y) &= \frac{\mathrm{d}}{\mathrm{d}\nu}\mathcal{J}_1(x+\nu y)\Big|_{\nu=0} = \frac{\mathrm{d}}{2\mathrm{d}\nu}\|x+\nu y\|_0^2\Big|_{\nu=0} \\
&= \frac{\mathrm{d}}{2\mathrm{d}\nu}\Bigg\{\int_0^{\sigma^2(T)} p(t)(x^\Delta(t)+\nu y^\Delta(t))^2\Delta t + \int_0^{\sigma(T)} q(t)(x^\sigma(t)+\nu y^\sigma(t))^2\Delta t \\
&\quad + \beta_1 p(0)(x(0)+\nu y(0))^2 + \beta_2 p(\sigma(T))(x(\sigma^2(T))+\nu y(\sigma^2(T)))^2\Bigg\}\Bigg|_{\nu=0} \\
&= \int_0^{\sigma^2(T)} p(t)(x^\Delta(t)+\nu y^\Delta(t))y^\Delta(t)\Delta t + \int_0^{\sigma(T)} q(t)(x^\sigma(t)+\nu y^\sigma(t))y^\sigma(t)\Delta t \\
&\quad + \beta_1 p(0)(x(0)+\nu y(0))y(0) + \beta_2 p(\sigma(T))(x(\sigma^2(T))+\nu y(\sigma^2(T)))x(\sigma^2(T))\Big|_{\nu=0} \\
&= \int_0^{\sigma^2(T)} p(t)x^\Delta(t)y^\Delta(t)\Delta t + \int_0^{\sigma(T)} q(t)x^\sigma(t)y^\sigma(t)\Delta t \\
&\quad + \beta_1 p(0)x(0)y(0) + \beta_2 p(\sigma(T))x(\sigma^2(T))y(\sigma^2(T))
\end{aligned}$$

for all $y \in \mathcal{B}$. Furthermore, see that $\mathcal{J}_1$ is coercive and sequentially weakly lower semicontinuous. From (S), we can find $\bar{\theta} > 0$ satisfying

$$\frac{\bar{\theta}^2}{F_{\bar{\theta}}} > 2C^2. \tag{9}$$

Define

$$s = \frac{\bar{\theta}^2}{2C^2}.$$

If $x \in \mathcal{J}_1^{-1}(-\infty, s)$, then $\mathcal{J}_1(x) < s$, that is, $\frac{1}{2}\|x\|_0^2 < s$. Hence, by Lemma 2, we obtain $|x(t)| \le C\sqrt{2s} = \bar{\theta}$ for all $t \in [0, \sigma^2(T)]_{\mathbb{T}}$. So

$$\sup_{\mathcal{J}_1(x)<s} \mathcal{J}_2(x) \le F_{\bar{\theta}}.$$

From the above, since $0 \in \mathcal{J}_1^{-1}(-\infty, s)$ and $\mathcal{J}_1(0) = \mathcal{J}_2(0) = 0$, we get

$$\varphi(s) \;=\; \inf_{x \in \mathcal{J}_1^{-1}(-\infty,s)} \frac{(\sup_{y \in \mathcal{J}_1^{-1}(-\infty,s)} \mathcal{J}_2(y)) - \mathcal{J}_2(x)}{s - \mathcal{J}_1(x)}$$

$$\leq \frac{\sup_{x\in\mathcal{J}_1^{-1}(-\infty,s)}\mathcal{J}_2(x)}{s}$$
$$\leq 2C^2\frac{F_{\bar\theta}}{\bar\theta^2}.$$

Thus, it follows that

$$\varphi(s) \leq 2C^2\frac{F_{\bar\theta}}{\bar\theta^2}. \tag{10}$$

Therefore, by (9) and (10), we get $\varphi(s) < 1$. Hence, as $1 \in \left(0, \frac{1}{\varphi(s)}\right)$, Theorem 3 implies that I has at least one critical point (more precisely, local minimum) $\tilde{x} \in \mathcal{J}_1^{-1}(-\infty, s)$. Thus, using that the solutions of (P^f) are exactly the critical points of I, we obtain the claim. □

We remark that Theorem 4 can also be used to ensure the existence of a solution for the parametric problem

$$\begin{cases} -(px^\Delta)^\Delta(t) + q(t)x^\sigma(t) = \nu f(t, x^\sigma(t)), & t \in [0,T]_\mathbb{T}, \\ \alpha_1 x(0) - \alpha_2 x^\Delta(0) = 0, & \alpha_3 x(\sigma^2(T)) + \alpha_4 x^\Delta(\sigma(T)) = 0, \end{cases} \tag{P^f_ν}$$

where $\nu > 0$ is a parameter. This result about (P^f_ν) is given as follows.

Theorem 5. *For all*

$$\nu \in \left(0, \frac{1}{2C^2}\sup_{\theta>0}\frac{\theta^2}{F_\theta}\right),$$

(P^f_ν) admits a solution $x_\nu \in \mathcal{H}$.

Proof. Let ν be in the stated interval. Suppose $\mathcal{J}_1$ and $\mathcal{J}_2$ are as in (7) and (8). Define

$$I_\nu(x) = \mathcal{J}_1(x) - \nu\mathcal{J}_2(x) \quad \text{for all} \quad x \in \mathcal{H}.$$

Thus, we obtain the existence of $\bar\theta > 0$ satisfying

$$2C^2\nu < \frac{\bar\theta^2}{F_{\bar\theta}}.$$

Put

$$s = \frac{\bar\theta^2}{2C^2}.$$

Using the notations from Theorem 4, we get

$$\varphi(s) \leq 2C^2\frac{F_{\bar\theta}}{\bar\theta^2} < \frac{1}{\nu}.$$

Then, as $\nu \in \left(0, \frac{1}{\varphi(s)}\right)$, Theorem 3 implies that I_ν has at least one critical point (more precisely, local minimum) $x_\nu \in \mathcal{J}_1^{-1}(-\infty, s)$, and recalling that critical points of I_ν are solutions of (P^f_ν), we have the conclusion. □

3.2. **Remarks, Applications, Example.** We give some implications of the above results.

Remark 6. We remark that, in general, I_ν may be unbounded in $\mathcal{H}$. Indeed, e.g., when $f(\xi) = 1 + |\xi|^{\gamma-2}\xi$ for $\xi \in \mathbb{R}$ with $\gamma > 2$, for any $x \in \mathcal{H} \setminus \{0\}$ and $\upsilon \in \mathbb{R}$, we get

$$I_\nu(\upsilon x) = \mathcal{J}_1(\upsilon x) - \nu \int_0^{\sigma(T)} F(\upsilon x(t))\Delta t \leq \frac{\upsilon^2}{2}\|x\|_0^2 - \nu\upsilon|\sigma(T)| - \nu\frac{\upsilon^\gamma}{\gamma}|\sigma(T)|^\gamma \to -\infty$$

as $\upsilon \to \infty$. Therefore, the condition [58, (I_2), Theorem 2.2] is not fulfilled. Hence, we cannot use the direct minimization approach to find the critical points of I_ν.

Now we show that I_ν associated with (P_ν^f) is, in general, not coercive. For example, when $F(\xi) = |\xi|^s$ for $\xi \in \mathbb{R}$ with $s \in (2, \infty)$, for any $x \in \mathcal{H} \setminus \{0\}$ and $\upsilon \in \mathbb{R}$, we get

$$\begin{aligned} I_\nu(\upsilon x) &= \mathcal{J}_1(\upsilon x) - \nu \int_0^{\sigma(T)} F(\upsilon x(t))\Delta t \\ &\leq \frac{\upsilon^2}{2}\|x\|_0^2 - \nu\upsilon^s|\sigma(T)|^s \to -\infty \end{aligned}$$

as $\upsilon \to -\infty$.

Remark 7. If in Theorem 4, $f(t, \xi) \geq 0$ for almost every $(t, \xi) \in [0, T]_{\mathbb{T}} \times \mathbb{R}$, then (S) becomes the simpler form

$$\sup_{\theta>0} \frac{\theta^2}{\int_0^{\sigma(T)} F(t, \theta)\Delta t} > 2C^2. \tag{S_ν}$$

Additionally, if

$$\limsup_{\theta\to\infty} \frac{\theta^2}{\int_0^{\sigma(T)} F(t, \theta)\Delta t} > 2C^2,$$

then (S_ν) automatically holds.

Remark 8. Assume $\bar{\theta} > 0$ is fixed and

$$\frac{\bar{\theta}^2}{F_{\bar{\theta}}} > 2C^2.$$

Then the result of Theorem 5 holds with $\|x_\nu\|_0 \leq \bar{\theta}$.

Remark 9. If, in Theorem 5, $f(t, 0) \neq 0$ for almost every $t \in [0, T]_{\mathbb{T}}$, then the solution that is obtained is obviously nontrivial. But the nontriviality of this solution can also be verified when $f(t, 0) = 0$ for almost every $t \in [0, T]_{\mathbb{T}}$, requiring the following additional condition at zero: There exist an open set $\emptyset \neq D \subseteq (0, T)_{\mathbb{T}}$ and $B \subset D$ with positive Lebesgue measure such that

$$\limsup_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{t\in B} F(t, \xi)}{|\xi|^2} = \infty \quad \text{and} \quad \liminf_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{t\in D} F(t, \xi)}{|\xi|^2} > -\infty.$$

To see this, let $0 < \bar{\nu} < \nu^*$, where

$$\nu^* = \frac{1}{2C^2} \sup_{\theta>0} \frac{\theta^2}{F_\theta}.$$

Then, we obtain the existence of $\bar{\theta} > 0$ satisfying

$$2C^2\bar{\nu} < \frac{\bar{\theta}^2}{F_\theta}.$$

According to Theorem 3, for each $\nu \in (0, \bar{\nu})$, I_ν possesses a critical point $x_\nu \in \mathcal{J}_1^{-1}(-\infty, s_\nu)$, where $s_\nu = \dfrac{\bar{\theta}^2}{2C^2}$. In particular, x_ν is a global minimum of the restriction of I_ν to $\mathcal{J}_1^{-1}(-\infty, s_\nu)$. We proceed to show that x_ν is nontrivial. To do so, we prove

$$\limsup_{\|x\|\to 0^+} \frac{\mathcal{J}_2(x)}{\mathcal{J}_1(x)} = \infty. \tag{11}$$

According to our assumptions at zero, we can find $\zeta > 0$ and κ and a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero, satisfying

$$\lim_{n\to\infty} \frac{\operatorname{ess\,inf}_{t\in B} F(t, \xi_n)}{|\xi_n|^2} = \infty$$

and

$$\operatorname*{ess\,inf}_{t\in D} F(t, \xi) \geq \kappa|\xi|^2 \quad \text{for all} \quad \xi \in [0, \zeta].$$

Now, take $C \subset B$ of positive measure and $y \in \mathcal{H}$ with

(i) $y(t) \in [0, 1]$ for all $t \in [0, T]_{\mathbb{T}}$,
(ii) $y(t) = 1 \in \mathbb{R}$ for all $t \in C$,
(iii) $y(t) = 0$ for all $t \in (0, T)_{\mathbb{T}} \setminus D$.

Take $Y > 0$ and let $\eta > 0$ be such that

$$Y < \frac{\eta \operatorname{meas}(C) + \kappa \int_{D\setminus C} |y(t)|^2 \Delta t}{\frac{1}{2}\|y\|_0^2}.$$

Then, there exists $n_0 \in \mathbb{N}$ with $\xi_n < \zeta$ and

$$\operatorname*{ess\,inf}_{t\in B} F(t, \xi_n) \geq \eta|\xi_n|^2$$

for all $n > n_0$. Next, for all $n > n_0$, by using the properties of y (i.e., $0 \leq \xi_n y(t) < \zeta$ for large enough n), we get

$$\begin{aligned} \frac{\mathcal{J}_2(\xi_n y)}{\mathcal{J}_1(\xi_n y)} =& \frac{\int_C F(t, \xi_n)\Delta t + \int_{D\setminus C} F(t, \xi_n y(t))\Delta t}{\mathcal{J}_1(\xi_n y)} \\ >& \frac{\eta \operatorname{meas}(C) + \kappa \int_{D\setminus C} |y(t)|^2 \Delta t}{\frac{1}{2}\|y\|_0^2} > Y. \end{aligned}$$

As Y is arbitrary, we obtain

$$\lim_{k\to\infty} \frac{\mathcal{J}_2(\xi_n y)}{\mathcal{J}_1(\xi_n y)} = \infty,$$

from which (11) follows. Thus, there exists $\{w_n\} \subset \mathcal{H}$ that converges strongly to zero, $w_n \in \mathcal{J}_1^{-1}(-\infty, s)$, and

$$I_\nu(w_n) = \mathcal{J}_1(w_n) - \nu\mathcal{J}_2(w_n) < 0.$$

As x_ν is a global minimum of the restriction of I_ν to $\mathcal{J}_1^{-1}(-\infty, s)$, we conclude

$$I_\nu(x_\nu) < 0, \tag{12}$$

and thus x_ν is nontrivial.

From (12), we also get that the map

$$(0, \nu^*) \ni \nu \mapsto I_\nu(x_\nu) \tag{13}$$

is negative. Further, we have

$$\lim_{\nu \to 0^+} \|x_\nu\|_0 = 0.$$

In fact, by noting that $\mathcal{J}_1$ is coercive and that for $\nu \in (0, \nu^*)$, $x_\nu \in \mathcal{J}_1^{-1}(-\infty, s)$, one obtains existence of a constant $\mathcal{L} > 0$ with $\|x_\nu\| \leq \mathcal{L}$ for all $\nu \in (0, \nu^*)$. This implies existence of $\mathcal{M} > 0$ satisfying

$$\left| \int_0^{\sigma(T)} f(t, x_\nu^\sigma(t)) x_\nu^\sigma(t) \Delta t \right| \leq \mathcal{M} \|x_\nu\|_0 \leq \mathcal{M}\mathcal{L} \tag{14}$$

for every $\nu \in (0, \nu^*)$. Since x_ν is a critical point of I_ν, we obtain $I_\nu'(x_\nu)(y) = 0$ for any $y \in \mathcal{B}$ and all $\nu \in (0, \nu^*)$. In particular, $I_\nu'(x_\nu)(x_\nu) = 0$, that is,

$$\mathcal{J}_1'(x_\nu)(x_\nu) = \nu \int_0^{\sigma(T)} f(t, x_\nu^\sigma(t)) x_\nu^\sigma(t) \Delta t \tag{15}$$

for every $\nu \in (0, \nu^*)$. Then, since

$$0 \leq \|x_\nu\|_0^2 \leq \mathcal{J}_1'(x_\nu)(x_\nu),$$

by using (15), it is concluded that

$$0 \leq \|x_\nu\|_0^2 \leq \mathcal{J}_1'(x_\nu)(x_\nu) \leq \nu \int_0^{\sigma(T)} f(t, x_\nu^\sigma(t)) x_\nu^\sigma(t) \Delta t$$

for any $\nu \in (0, \nu^*)$. Letting $\nu \to 0^+$, by (14), we have $\lim_{\nu \to 0^+} \|x_\nu\|_0 = 0$. One has

$$\lim_{\nu \to 0^+} \|x_\nu\|_\infty = 0. \tag{16}$$

Finally, we demonstrate that the map

$$\nu \mapsto I_\nu(x_\nu)$$

strictly decreases in $(0, \nu^*)$. To do so, we note that

$$I_\nu(x) = \nu \left(\frac{\mathcal{J}_1(x)}{\nu} - \mathcal{J}_2(x) \right) \quad \text{for all} \quad x \in \mathcal{H}. \tag{17}$$

Fix $0 < \nu_1 < \nu_2 < \nu^*$ and let x_{ν_1}, x_{ν_2} be the global minima of I_{ν_1}, I_{ν_2}, restricted to $\mathcal{J}_1(-\infty, s)$. Put

$$m_{\nu_i} = \left(\frac{\mathcal{J}_1(x_{\nu_i})}{\nu_i} - \mathcal{J}_2(x_{\nu_i}) \right) = \inf_{y \in \mathcal{J}_1^{-1}(-\infty, s)} \left(\frac{\mathcal{J}_1(y)}{\nu_i} - \mathcal{J}_2(y) \right)$$

for $i = 1, 2$. Then, (13) and (17), since $\nu > 0$, yield

$$m_{\nu_i} < 0 \text{ for } i = 1, 2. \tag{18}$$

Moreover,

$$m_{\nu_2} \leq m_{\nu_1} \tag{19}$$

due to $0 < \nu_1 < \nu_2$. Then by considering (17)–(19) and again since $0 < \nu_1 < \nu_2$, we get

$$I_{\nu_2}(x_{\nu_2}) = \nu_2 m_{\nu_2} \leq \nu_2 m_{\nu_1} < \nu_1 m_{\nu_1} = I_{\nu_1}(x_{\nu_1}),$$

so that the map $\nu \mapsto I_\nu(x_\nu)$ strictly decreases in $\nu \in (0, \nu^*)$. As $\nu < \nu^*$ is arbitrary, we get $\nu \mapsto I_\nu(x_\nu)$ strictly decreases in $(0, \nu^*)$.

Remark 10. We remark that Theorem 5 is a bifurcation result, i.e., $(0,0)$ belongs to the closure of

$$\big\{(x_\nu,\nu)\in\mathcal{H}\times(0,\infty):\ x_\nu \text{ is a nontrivial solution of } (P_\nu^f)\big\}$$

in $\mathcal{H}\times\mathbb{R}$. Indeed, we know that

$$\|x_\nu\|_0\to 0 \quad\text{as}\quad \nu\to 0.$$

Thus, there are two sequences $\{x_j\}$ in $\mathcal{H}$ and $\nu>0$ (here $x_j=x_\nu$) such that

$$\nu_i\to 0^+ \quad\text{and}\quad \|x_j\|_0\to 0$$

as $j\to\infty$. Additionally, we point out that since the map

$$(0,\nu^*)\ni\nu\mapsto I_\nu(x_\nu)$$

is strictly decreasing, for every $\nu_1,\nu_2\in(0,\nu^*)$ with $\nu_1\neq\nu_2$, the solutions x_{ν_1} and x_{ν_2} are different.

Remark 11. If $f\geq 0$, then also the solution in Theorem 5 is nonnegative. To show this, assume x_0 is a nontrivial solution of (P_ν^f). Suppose

$$\mathcal{A}=\{t\in(0,T]_{\mathbb{T}}:\ x_0(t)<0\}$$

has positive measure. Define $\bar{y}(t)=\min\{0,x_0(t)\}$ for $t\in[0,T]_{\mathbb{T}}$. We obtain $\bar{y}\in\mathcal{H}$ and

$$\begin{aligned}&\int_0^{\sigma^2(T)} p(t)x_0^\Delta(t)\bar{y}^\Delta(t)\Delta t+\int_0^{\sigma(T)} q(t)x_0^\sigma(t)\bar{y}^\sigma(t)\Delta t+\beta_1 p(0)x_0(0)\bar{y}(0)\\&+\beta_2 p(\sigma(T))x_0(\sigma^2(T))\bar{y}(\sigma^2(T))-\nu\int_0^{\sigma(T)} f(t,x_0^\sigma(t))\bar{y}^\sigma(t)\Delta t=0.\end{aligned}$$

Thus, since f is assumed to be nonnegative, we get

$$\begin{aligned}0\leq&\|x_0\|_{\mathcal{A}}^2\leq\int_0^{\sigma^2(T)} p(t)x_0^\Delta(t)x_0^\Delta(t)\Delta t+\int_0^{\sigma(T)} q(t)x_0^\sigma(t)x_0^\sigma(t)\Delta t\\&+\beta_1 p(0)x_0(0)x_0(0)+\beta_2 p(\sigma(T))x_0(\sigma^2(T))x_0(\sigma^2(T))\\=&\nu\int_0^{\sigma(T)} f(t,x_0^\sigma(t))x_0^\sigma(t)\Delta t\leq 0.\end{aligned}$$

Hence, $x_0=0$ in $\mathcal{A}$, and this is a contradiction.

The next theorem is concerned with a particular case of our results.

Theorem 12. *Assume $f:\mathbb{R}\to[0,\infty)$ is continuous and define $F(\xi)=\int_0^\xi f(s)\mathrm{d}s$ for $\xi\in\mathbb{R}$. If*

$$\lim_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}=\infty,$$

then, for all

$$\nu\in\Lambda=\left(0,\frac{1}{2\sigma(T)C^2}\sup_{\theta>0}\frac{\theta^2}{F(\theta)}\right),$$

the problem

$$\begin{cases}-(px^\Delta)^\Delta(t)+q(t)x^\sigma(t)=\nu f(x^\sigma(t)), & t\in[0,T]_{\mathbb{T}},\\ \alpha_1 x(0)-\alpha_2 x^\Delta(0)=0, & \alpha_3 x(\sigma^2(T))+\alpha_4 x^\Delta(\sigma(T))=0\end{cases}$$

has a nontrivial solution $x_\nu \in \mathcal{H}$ *satisfying*

$$\lim_{\nu\to 0^+} \|x_\nu\|_0 = 0,$$

and

$$\nu \to \frac{1}{2}\|x\|_0^2 - \nu \int_0^{\sigma(T)} F(x(t))\Delta t$$

is strictly decreasing in Λ *and negative.*

Finally, we offer an example illustrating Theorem 12.

Example 13. Let $\mathbb{T} = \left\{\frac{4}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ and $T = 1$. Consider

$$\begin{cases} -x^{\Delta\Delta}(t) = \nu f(x^\sigma(t)), \quad t \in [0,1]_{\mathbb{T}}, \\ x(0) - 2x^\Delta(0) = 0, \quad x^\Delta(\frac{4}{3}) = 0, \end{cases} \tag{20}$$

where

$$f(\xi) = \frac{3}{64}\left(2\xi + 2\tan(\xi)\sec^2(\xi) + e^\xi\right) \quad \text{for all} \quad \xi \in \mathbb{R}.$$

Then (20) is in the form of (P_ν^f) with

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 0, \quad \alpha_4 = 1, \quad p(t) \equiv 1, \quad q(t) \equiv 0.$$

We calculate

$$\beta_1 = \frac{1}{2}, \quad \beta_2 = 0, \quad \sigma(1) = \frac{4}{3}, \quad \sigma^2(1) = 2,$$
$$\underline{p} = 1, \quad \underline{q} = 0, \quad M_1 = M_2 = 2, \quad M_3 = \infty, \quad C = 2,$$

and

$$F(\xi) = \frac{3}{64}\left(\xi^2 + \tan^2(\xi) + e^\xi - 1\right) \quad \text{for all} \quad \xi \in \mathbb{R}.$$

Note that we clearly have (using L'Hôpital's rule)

$$\lim_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = \infty.$$

Hence, all assumptions in Theorem 12 are fulfilled. Note that

$$\sup_{\theta>0} \frac{\theta^2}{\theta^2 + \tan^2(\theta) + e^\theta - 1} \approx 0.347529 \geq 0.3475$$

so that, by Theorem 12, for all $\nu \in (0, 0.695)$ problem (20) has a nontrivial solution $x_\nu \in \mathrm{H}^1_\Delta([0,\sigma^2(1)]_{\mathbb{T}})$ satisfying

$$\lim_{\nu\to 0^+} \|x_\nu\|_0 = 0,$$

and the map

$$\nu \mapsto \frac{1}{2}\|x\|_0^2 - \nu \int_0^{\frac{4}{3}} F(x(t))\Delta t$$

is strictly decreasing in $(0, 0.695)$ and negative.

4. The existence of three solutions

4.1. **Main results.** Let $\emptyset \neq \mathbb{T} \subset \mathbb{R}$, called a time scale, be given. For example, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ are time scales that correspond to differential and difference equations, respectively. Let $S > 0$ be fixed and suppose $0, S \in \mathbb{T}$. Consider the second-order dynamic Sturm–Liouville boundary value problem

$$\begin{cases} -(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = \nu f(t, x^{\sigma}(t)) + \zeta g(t, x^{\sigma}(t)), & t \in [0, S]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^{\Delta}(0) = 0, \quad \alpha_3 x(\sigma^2(S)) + \alpha_4 x^{\Delta}(\sigma(S)) = 0, \end{cases} \tag{$P_{\nu,\zeta}$}$$

where $p \in \mathrm{C}^1([0, \sigma(S)]_{\mathbb{T}}, (0, \infty))$, $q \in \mathrm{C}([0, S]_{\mathbb{T}}, [0, \infty))$, $f, g \in \mathrm{C}([0, S]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$, $\nu > 0$ and $\zeta \geq 0$ are real parameters, $\alpha_i \geq 0$ for $i = 1, 2, 3, 4$ and $\alpha_1 + \alpha_2 \geq 0$, $\alpha_3 + \alpha_4 > 0$, $\alpha_1 + \alpha_3 > 0$.

In the section, see Theorem 17 below, we obtain the existence of at least three distinct nonnegative solutions for $(P_{\nu,\zeta})$ by utilizing a critical point result due to Bonanno and Candito [23, Theorem 3.3]. Demanding an additional asymptotical behaviour of the data at zero, nontriviality of the solution can also be achieved under appropriate assumptions, see Remark 18. Moreover, existence of solutions for $\nu \to 0^+$ is investigated, see Remark 19. Theorem 20 follows from Theorem 17. As a special case of Theorem 20, we present Theorems 21 and 22. Next, we offer Example 23, in which the assumptions of Theorem 22 are satisfied. Finally, in Theorem 24, existence of at least four distinct nontrivial solutions of $(P_{\nu,0})$ is discussed.

The main tool to derive existence of at least three solutions for $(P_{\nu,\zeta})$ is the following three critical point theorem due to Bonanno and Candito. For $X \neq 0$, $\mathcal{J}_1, \mathcal{J}_2 : X \to \mathbb{R}$, and $r, r_1, r_2 > \inf_X \mathcal{J}_1$, $r_2 > r_1$, $r_3 > 0$, define

$$\varphi(r) := \inf_{x \in \mathcal{J}_1^{-1}(-\infty, r)} \frac{\sup_{y \in \mathcal{J}_1^{-1}(-\infty, r)} \mathcal{J}_2(y) - \mathcal{J}_2(x)}{r - \mathcal{J}_1(x)},$$

$$\beta(r_1, r_2) := \inf_{x \in \mathcal{J}_1^{-1}(-\infty, r_1)} \sup_{y \in \mathcal{J}_1^{-1}[r_1, r_2)} \frac{\mathcal{J}_2(y) - \mathcal{J}_2(x)}{\mathcal{J}_1(y) - \mathcal{J}_1(x)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2 + r_3)} \mathcal{J}_2(x)}{r_3},$$

and

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

Theorem 14 (See [23, Theorem 3.3])**.** *Let be given a reflexive real Banach space X. Suppose $\mathcal{J}_1 : X \to \mathbb{R}$ is a convex, coercive, and continuously Gâteaux-differentiable functional such that its Gâteaux derivative admits a continuous inverse on X^*. Assume that $\mathcal{J}_2 : X \to \mathbb{R}$ is a continuously Gâteaux-differentiable functional such that its Gâteaux derivative is compact. Assume*

- (a_1) $\inf_X \mathcal{J}_1 = \mathcal{J}_1(0) = \mathcal{J}_2(0) = 0$,
- (a_2) *for every $x_1, x_2 \in X$ such that $\mathcal{J}_2(x_1) \geq 0$ and $\mathcal{J}_2(x_2) \geq 0$, one has*

$$\inf_{s \in [0,1]} \mathcal{J}_2(sx_1 + (1 - s)x_2) \geq 0.$$

Suppose that there exist $r_1, r_2, r_3 > 0$ with $r_1 < r_2$ and

- (a_3) $\varphi(r_1) < \beta(r_1, r_2)$,
- (a_4) $\varphi(r_2) < \beta(r_1, r_2)$,
- (a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for any $\nu \in \left(\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}\right)$, $\mathcal{J}_1 - \nu\mathcal{J}_2$ *admits at least three distinct critical points* x_1, x_2, x_3 *such that*

$$x_1 \in \mathcal{J}_1^{-1}((-\infty, r_1)), \quad x_2 \in \mathcal{J}_1^{-1}([r_1, r_2)), \quad x_3 \in \mathcal{J}_1^{-1}((-\infty, r_2 + r_3)).$$

We refer the reader to [1, 12, 13, 20, 24, 38–40, 42, 54] for situations of successful employments of results such as Theorem 14 in order to prove existence of three solutions for various boundary value problems.

For each $x \in \mathcal{H}$, define the functionals $\mathcal{J}_1$ and $\mathcal{J}_2$ by

$$\mathcal{J}_1(x) = \frac{1}{2}\|x\|_0^2 \tag{21}$$

and

$$\mathcal{J}_2(x) = \int_0^{\sigma(S)} F(t, x^\sigma(t))\Delta t + \frac{\zeta}{\nu}\int_0^{\sigma(S)} G(t, x^\sigma(t))\Delta t \tag{22}$$

where

$$F(t,\xi) = \int_0^\xi f(t,s)\mathrm{d}s \quad \text{for} \quad (t,\xi) \in [0,S]_{\mathbb{T}} \times \mathbb{R}$$

and

$$G(t,\xi) = \int_0^\xi g(t,s)\mathrm{d}s \quad \text{for} \quad (t,\xi) \in [0,S]_{\mathbb{T}} \times \mathbb{R}.$$

Define also

$$I_\nu = \mathcal{J}_1(x) - \nu\mathcal{J}_2(x).$$

The following auxiliary result is used later.

Lemma 15. *Let* $T : \mathcal{H} \to \mathcal{H}^*$ *be defined by* $T(x)(y) = (x,y)_0$. *Then* T *possesses a continuous inverse on* E^*.

Proof. Note that

$$\lim_{\|x\|_0\to\infty} \frac{T(x)(x)}{\|x\|_0} = \lim_{\|x\|_0\to\infty} \frac{(x,x)_0}{\|x\|_0} = \lim_{\|x\|_0\to\infty} \|x\|_0 = \infty.$$

Thus, the map T is coercive. Now, we will prove that T is strictly monotone:

$$\begin{aligned} T(x)(x-y) - T(y)(x-y) &= (x, x-y)_0 - (y, x-y)_0 \\ &= (x-y, x-y)_0 = \|x-y\|_0^2. \end{aligned}$$

By [64, Theorem 26.A(d)], T^{-1} exists and is continuous on $\mathcal{H}^*$. □

Proposition 16. $x \in \mathcal{H}$ *is a critical point of* $\mathcal{J}_1 - \nu\mathcal{J}_2$ *iff* x *solves* $(P_{\nu,\zeta})$.

Proof. Suppose $x \in \mathcal{H}$ is a critical point of $\mathcal{J}_1 - \nu\mathcal{J}_2$. Thus, for any $y \in \mathcal{H}$,

$$\langle(\mathcal{J}_1 - \nu\mathcal{J}_2)'(x), y\rangle = 0,$$

that is,

$$\begin{aligned} &\int_0^{\sigma^2(S)} p(t)x^\Delta(t)y^\Delta(t)\Delta t + \int_0^{\sigma(S)} q(t)x^\sigma(t)y^\sigma(t)\Delta t \\ &- \nu\int_0^{\sigma(S)} f(t, x^\sigma(t))y^\sigma(t)\Delta t - \zeta\int_0^{\sigma(S)} g(t, x^\sigma(t))y^\sigma(t)\Delta t \\ &+ \beta_1 p(0)x(0)y(0) + \beta_2 p(\sigma(S))x(\sigma^2(S))y(\sigma^2(S)) = 0. \end{aligned}$$

Simple calculations show that

$$\begin{aligned}
&-\int_0^{\sigma^2(S)}(px^{\Delta})^{\Delta}(t)y^{\sigma}(t)\Delta t\\
&+\int_0^{\sigma(S)}\left[q(t)x^{\sigma}(t)-\nu f(t,x^{\sigma}(t))-\zeta g(t,x^{\sigma}(t))\right]y^{\sigma}(t)\Delta t\\
&+p(0)y(0)\left[\beta_1 x(0)-x^{\Delta}(0)\right]\\
&+y(\sigma^2(S))\left[p(\sigma^2(S))x^{\Delta}(\sigma^2(S))+\beta_2 p(\sigma(S))x(\sigma^2(S))\right]=0.
\end{aligned}\tag{23}$$

Thus, by the fundamental lemma of variational calculus, x satisfies the dynamic equation in $(P_{\nu,\zeta})$. Then (23) becomes

$$\begin{aligned}
&p(0)y(0)\left[\beta_1 x(0)-x^{\Delta}(0)\right]\\
&\quad+y(\sigma^2(S))\left[p(\sigma^2(S))x^{\Delta}(\sigma^2(S))+\beta_2 p(\sigma(S))x(\sigma^2(S))\right]=0.
\end{aligned}$$

By using (5) and (6), we have

$$\begin{aligned}
&p(0)y(0)\left[\alpha_1 x(0)-\alpha_2 x^{\Delta}(0)\right]\\
&\quad+y(\sigma^2(S))\left[p(\sigma^2(S))\alpha_4 x^{\Delta}(\sigma^2(S))+\alpha_3 p(\sigma(S))x(\sigma^2(S))\right]=0
\end{aligned}$$

for all $y\in X$. We now demonstrate that x satisfies the boundary conditions in $(P_{\nu,\zeta})$. Without restricting generality, suppose

$$\alpha_1 x(0)-\alpha_2 x^{\Delta}(0)>0.$$

We let $y(t)=\sigma^2(S)-t$. Then

$$\begin{aligned}
&p(0)y(0)\left[\alpha_1 x(0)-\alpha_2 x^{\Delta}(0)\right]\\
&\quad+y(\sigma^2(S))\left[p(\sigma^2(S))\alpha_4 x^{\Delta}(\sigma^2(S))+\alpha_3 p(\sigma(S))x(\sigma^2(S))\right]\\
&\quad=p(0)\sigma^2(S)\left[\alpha_1 x(0)-\alpha_2 x^{\Delta}(0)\right]>0,
\end{aligned}$$

a contradiction. So x is a solution of $(P_{\nu,\zeta})$. Conversely, if x is a solution of $(P_{\nu,\zeta})$, for any $y\in X$, multiplying $y(t)$ on both sides of the dynamic equation in $(P_{\nu,\zeta})$ and integrating on $[0,\sigma(S)]_{\mathbb{T}}$, in view of the boundary conditions, we observe that x satisfies $\langle(\mathcal{J}_1-\nu\mathcal{J}_2)'(x),y\rangle=0$ for all $y\in X$. □

Next, for our convenience, let

$$G^{\theta}:=\int_{[0,\sigma(S))_{\mathbb{T}}}G(t,\theta)\Delta t\quad\text{for}\quad\theta>0\tag{24}$$

and

$$G_{\eta}:=\sigma(S)\inf_{[0,\sigma(S))_{\mathbb{T}}\times[0,\eta]}G\quad\text{for}\quad\eta>0.\tag{25}$$

For a positive constant d, set

$$K_d=\frac{d^2}{2}\left(\int_0^{\sigma(S)}q(t)\Delta t+\beta_1 p(0)+\beta_2 p(\sigma(S))\right).$$

We fix four positive constants $\theta_1,\theta_2,\theta_3,d$ and define the constant $\delta_{\nu,g}$ by

$$\min\left\{\frac{1}{2C^2}\min\left\{\frac{\theta_1^2-2C^2\nu\int_0^{\sigma(S)}F(t,\theta_1)\Delta t}{G^{\theta_1}},\frac{\theta_2^2-2C^2\nu\int_0^{\sigma(S)}F(t,\theta_2)\Delta t}{G^{\theta_2}},\right.\right.$$

$$\left.\frac{\theta_3^2-\theta_2^2-2C^2\nu\int_0^{\sigma(S)}F(t,\theta_3)\Delta t}{G^{\theta_3}}\right\},\ \frac{K_d-\nu\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t}{G_d-G^{\theta_1}}\Bigg\}. \quad (26)$$

Theorem 17. *Suppose $f:[0,S]_{\mathbb{T}}\times[0,\infty)\to(0,\infty)$ is continuous. Assume the existence of $\theta_1,\theta_2,\theta_3,d>0$ such that*

$$\theta_1<C\sqrt{2K_d}<\theta_2<\theta_3$$

and

(A_1)

$$\max\left\{\frac{\int_0^{\sigma(S)}F(t,\theta_1)\Delta t}{\theta_1^2},\frac{\int_0^{\sigma(S)}F(t,\theta_2)\Delta t}{\theta_2^2},\frac{\int_0^{\sigma(S)}F(t,\theta_3)\Delta t}{\theta_3^2-\theta_2^2}\right\}$$
$$<\frac{1}{2C^2}\frac{\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t}{K_d}.$$

Then, for every

$$\nu\in\Bigg(\frac{K_d}{\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t},$$
$$\frac{1}{2C^2}\min\left\{\frac{\theta_1^2}{\int_0^{\sigma(S)}F(t,\theta_1)\Delta t},\frac{\theta_2^2}{\int_0^{\sigma(S)}F(t,\theta_2)\Delta t},\frac{\theta_3^2-\theta_2^2}{\int_0^{\sigma(S)}F(t,\theta_3)\Delta t}\right\}\Bigg),$$

for every nonnegative continuous function $g:[0,S]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$, there exists $\delta_{\nu,g}>0$ given by (26) such that, for each $\zeta\in[0,\delta_{\nu,g})$, the problem $(P_{\nu,\zeta})$ possesses at least three nonnegative solutions $x_1,x_2,x_3\in\mathcal{H}$ such that

$$\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}}|x_1(t)|<\theta_1,\quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}}|x_2(t)|<\theta_2,\quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}}|x_3(t)|<\theta_3.$$

Proof. We may assume $f(t,x)=f(t,0)$ for all $(t,x)\in[0,S]_{\mathbb{T}}\times(-\infty,0)$. Let $X=\mathcal{H}$, and we consider $\mathcal{J}_1$ and $\mathcal{J}_2$ defined by (21) and (22), respectively. We now prove that $\mathcal{J}_1$ and $\mathcal{J}_2$ fulfill the assumptions of Theorem 14. It is clear that $\mathcal{J}_2$ is differentiable with

$$\mathcal{J}_2'(x)(y)=\int_0^{\sigma(S)}f(t,x^\sigma(t))y^\sigma(t)\Delta t+\frac{\zeta}{\nu}\int_0^{\sigma(S)}g(t,x^\sigma(t))y^\sigma(t)\Delta t$$

for $x,y\in X$. Moreover, $\mathcal{J}_2':X\to X^*$ is compact. Also, $\mathcal{J}_1$ is continuously differentiable with

$$\mathcal{J}_1'(x)(y)=\int_0^{\sigma^2(S)}p(t)x^\Delta(t)y^\Delta(t)\Delta t+\int_0^{\sigma(S)}q(t)x^\sigma(t)y^\sigma(t)\Delta t$$
$$+\beta_1p(0)x(0)y(0)+\beta_2p(\sigma(S))x(\sigma^2(S))y(\sigma^2(S))$$

for $x,y\in X$, while Lemma 15 yields that $\mathcal{J}_1'$ has a continuous inverse on X^*. In addition, $\mathcal{J}_1$ is sequentially weakly lower semicontinuous. Denote

$$r_1:=\frac{\theta_1^2}{2C^2},\quad r_2:=\frac{\theta_2^2}{2C^2},\quad r_3:=\frac{\theta_3^2-\theta_2^2}{2C^2}$$

and $w(t)=d$ for $t\in[0,S]_{\mathbb{T}}$. Clearly, $w\in X$. From (21), we observe that $\mathcal{J}_1(w)=K_d$, and by the condition

$$\theta_1<C\sqrt{2K_d}<\theta_2<\theta_3,$$

we get $r_3 > 0$ and $r_1 < \mathcal{J}_1(w) < r_2$. From the definition of r_1 and Lemma 2, we obtain

$$\begin{aligned}\mathcal{J}_1^{-1}(-\infty, r_1) &\subseteq \left\{x \in X : \|x\|_0 \leq \sqrt{2r_1}\right\} \\ &\subseteq \left\{x \in X : |x(t)| \leq C\sqrt{2r_1} \text{ for all } t \in [0, \sigma^2(S)]_{\mathbb{T}}\right\} \\ &= \left\{x \in X : |x(t)| \leq \theta_1 \text{ for all } t \in [0, \sigma^2(S)]_{\mathbb{T}}\right\},\end{aligned}$$

and since we assumed that f and g are nonnegative, this ensures

$$\begin{aligned}\mathcal{J}_2(x) &\leq \sup_{y \in \mathcal{J}_1^{-1}(-\infty, r_1)} \left[\int_0^{\sigma(S)} F(t, y^\sigma(t))\Delta t + \frac{\zeta}{\nu}\int_0^{\sigma(S)} G(t, y^\sigma(t))\Delta t.\right] \\ &\leq \int_0^{\sigma(S)} F(t, \theta_1)\Delta t + \frac{\zeta}{\nu}\int_0^{\sigma(S)} G(t, \theta_1)\Delta t \\ &= \int_0^{\sigma(S)} F(t, \theta_1)\Delta t + \frac{\zeta}{\nu} G^{\theta_1}\end{aligned}$$

for every $x \in X$ such that $\mathcal{J}_1(x) < r_1$. Thus,

$$\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_1)} \mathcal{J}_2(x) \leq \int_0^{\sigma(S)} F(t, \theta_1)\Delta t + \frac{\zeta}{\nu} G^{\theta_1}. \tag{27}$$

Similarly,

$$\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2)} \mathcal{J}_2(x) \leq \int_0^{\sigma(S)} F(t, \theta_2)\Delta t + \frac{\zeta}{\nu} G^{\theta_2} \tag{28}$$

and

$$\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2 + r_3)} \mathcal{J}_2(x) \leq \int_0^{\sigma(S)} F(t, \theta_3)\Delta t + \frac{\zeta}{\nu} G^{\theta_3}. \tag{29}$$

Therefore, since $0 \in \mathcal{J}_1^{-1}(-\infty, r_1)$ and $\mathcal{J}_1(0) = \mathcal{J}_2(0) = 0$, using (27), one has

$$\varphi(r_1) \leq \frac{\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_1)} \mathcal{J}_2(x)}{r_1} \leq 2C^2 \frac{\int_0^{\sigma(S)} F(t, \theta_1)\Delta t + \frac{\zeta}{\nu} G^{\theta_1}}{\theta_1^2} < \frac{1}{\nu}$$

because $\zeta < \frac{1}{2C^2} \frac{\theta_1^2 - 2C^2\nu \int_0^{\sigma(S)} F(t, \theta_1)\Delta t}{G^{\theta_1}}$ due to (26), and using (28), one has

$$\varphi(r_2) \leq \frac{\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2)} \mathcal{J}_2(x)}{r_2} \leq 2C^2 \frac{\int_0^{\sigma(S)} F(t, \theta_2)\Delta t + \frac{\zeta}{\nu} G^{\theta_2}}{\theta_2^2} < \frac{1}{\nu}$$

because $\zeta < \frac{1}{2C^2} \frac{\theta_2^2 - 2C^2\nu \int_0^{\sigma(S)} F(t, \theta_2)\Delta t}{G^{\theta_2}}$ due to (26), and using (29), one has

$$\gamma(r_2, r_3) \leq \frac{\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2 + r_3)} \mathcal{J}_2(x)}{r_3} \leq 2C^2 \frac{\int_0^{\sigma(S)} F(t, \theta_3)\Delta t + \frac{\zeta}{\nu} G^{\theta_3}}{\theta_3^2 - \theta_2^2} < \frac{1}{\nu}$$

because $\zeta < \frac{\theta_3^2 - \theta_2^2 - 2C^2\nu \int_0^{\sigma(S)} F(t, \theta_3)\Delta t}{G^{\theta_3}}$ due to (26). Also, taking into account (25), we get

$$\begin{aligned}\mathcal{J}_2(w) &= \int_0^{\sigma(S)} F(t, w(t))\Delta t + \frac{\zeta}{\nu}\int_0^{\sigma(S)} G(t, w(t))\Delta t \\ &\geq \int_0^{\sigma(S)} F(t, w(t))\Delta t + \sigma(S)\frac{\zeta}{\nu} \inf_{[0, \sigma(S)]_{\mathbb{T}} \times [0, d]} G\end{aligned}$$

$$= \int_0^{\sigma(S)} F(t,d)\Delta t + \frac{\zeta}{\nu} G_d.$$

Hence, if $x \in \mathcal{J}_1^{-1}(-\infty, r_1)$, then, taking (27) into account,

$$\mathcal{J}_2(w) - \mathcal{J}_2(x) \geq \int_0^{\sigma(S)} (F(t,d) - F(t,\theta_1))\Delta t + \frac{\zeta}{\nu}(G_d - G^{\theta_1})$$

and

$$0 < \mathcal{J}_1(w) - \mathcal{J}_1(x) \leq \mathcal{J}_1(w) = K_d,$$

so that

$$\beta(r_1, r_2) \geq \frac{\int_0^{\sigma(S)} (F(t,d) - F(t,\theta_1))\Delta t + \frac{\zeta}{\nu}(G_d - G^{\theta_1})}{K_d} > \frac{1}{\nu}$$

because $\zeta < \dfrac{K_d - \nu \int_0^{\sigma(S)} (F(t,d) - F(t,\theta_1))\Delta t}{G_d - G^{\theta_1}}$ due to (26). Altogether, we get

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Next, we illustrate that $\mathcal{J}_2$ fulfills (a_2) of Theorem 14. Suppose x_1, x_2 are local minima for I_ν. Then x_1, x_2 are critical points for I_ν and hence nonnegative solutions of $(P_{\nu,\zeta})$. Thus, it follows that

$$(\nu f + \zeta g)(t, sx_1 + (1-s)x_2) \geq 0$$

for all $s \in [0,1]$, and therefore, $\mathcal{J}_2(sx_1 + (1-s)x_2) \geq 0$ for all $s \in [0,1]$. By utilizing Theorem 14, we get that for every

$$\nu \in \Bigg(\frac{K_d}{\int_0^{\sigma(S)} (F(t,d) - F(t,\theta_1))\Delta t}, \\ \frac{1}{2C^2} \min \left\{ \frac{\theta_1^2}{\int_0^{\sigma(S)} F(t,\theta_1)\Delta t}, \frac{\theta_2^2}{\int_0^{\sigma(S)} F(t,\theta_2)\Delta t}, \frac{\theta_3^2 - \theta_2^2}{\int_0^{\sigma(S)} F(t,\theta_3)\Delta t} \right\} \Bigg)$$

and $\zeta \in [0, \delta_{\nu,g})$, the functional I_ν has three critical points x_i, $i = 1,2,3$, in X such that $\mathcal{J}_1(x_1) < r_1$, $\mathcal{J}_1(x_2) < r_2$, and $\mathcal{J}_1(x_3) < r_2 + r_3$, that is,

$$\max_{t \in [0,\sigma^2(S)]_{\mathbb{T}}} |x_1(t)| < \theta_1, \quad \max_{t \in [0,\sigma^2(S)]_{\mathbb{T}}} |x_2(t)| < \theta_2, \quad \max_{t \in [0,\sigma^2(S)]_{\mathbb{T}}} |x_3(t)| < \theta_3.$$

The proof is complete. □

Remark 18. If $f(t,0) \neq 0$ or $g(t,0) \neq 0$ for some $t \in [0,S]_{\mathbb{T}}$, then the solutions obtained in Theorem 17 are nontrivial. Moreover, nontriviality can be demonstrated also if $f(t,0) = 0$ for some $t \in [0,S]_{\mathbb{T}}$, requiring an extra condition at zero, namely the existence of $\emptyset \neq D \subseteq [0,S]_{\mathbb{T}}$ (D open) and $B \subset D$ such that

$$\limsup_{\xi \to 0^+} \frac{\inf_{t \in B} F(t,\xi)}{|\xi|^2} = \infty \quad \text{and} \quad \liminf_{\xi \to 0^+} \frac{\inf_{t \in D} F(t,\xi)}{|\xi|^2} > -\infty. \tag{30}$$

To see this, let $0 < \overline{\nu} < \nu^*$, where

$$\nu^* = \frac{1}{2C^2} \min \left\{ \frac{\theta_1^2}{\int_0^{\sigma(S)} F(t,\theta_1)\Delta t}, \frac{\theta_2^2}{\int_0^{\sigma(S)} F(t,\theta_2)\Delta t}, \frac{\theta_3^2 - \theta_2^2}{\int_0^{\sigma(S)} F(t,\theta_3)\Delta t} \right\}.$$

Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be as given in (21) and (22), respectively. Because of Theorem 14, for all $\nu \in \left(\frac{K_d}{\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t}, \overline{\nu}\right)$, there exist three critical points $x_{1\nu}$, $x_{2\nu}$ and $x_{3\nu}$ of the functional $I_\nu = \mathcal{J}_1(x) - \nu\mathcal{J}_2(x)$ such that

$$x_{1\nu} \in \mathcal{J}_1^{-1}(-\infty, r_{1\nu}),\ x_{2\nu} \in \mathcal{J}_1^{-1}(-\infty, r_{2\nu}),\ x_{3\nu} \in \mathcal{J}_1^{-1}(-\infty, r_{3\nu}),$$

where

$$r_{1\nu} = \frac{\theta_{1\nu}^2}{2C^2},\ r_{2\nu} = \frac{\theta_{2\nu}^2}{2C^2},\ r_{3\nu} = \frac{\theta_{3\nu}^2 - \theta_{2\nu}^2}{2C^2}.$$

In particular, $x_{i\nu}$ for $i = 1,2,3$ is a global minimum of the restriction of I_ν to $\mathcal{J}_1^{-1}(-\infty, r_{i\nu})$ for $i = 1,2,3$. We will prove that $x_{1\nu}$ cannot be trivial. Let us show that

$$\limsup_{\|x\|\to 0^+} \frac{\mathcal{J}_2(x)}{\mathcal{J}_1(x)} = \infty. \tag{31}$$

According to (30), we may $\tau > 0$ and κ and a sequence $\{\xi_n\} \subset \mathbb{R}^+$ that converges to zero such that, for all $\xi \in [0,\tau]$,

$$\lim_{\xi\to 0^+} \frac{\inf_{t\in B} F(t,\xi_n)}{|\xi_n|^2} = \infty \quad \text{and} \quad \inf_{t\in D} F(t,\xi) > \kappa|\xi|^2.$$

We consider $E \subset B$ of positive measure and $y \in X$ satisfying

(k$_1$) $y(t) \in [0,1]$ for all $t \in [0,S]_{\mathbb{T}}$,
(k$_2$) $y(t) = 1$ for all $t \in E$,
(k$_3$) $y(t) = 0$ for all $t \in [0,S]_{\mathbb{T}} \setminus D$.

Finally, let $M > 0$ and let $\eta > 0$ such that

$$M < 2\frac{\operatorname{meas}(E) + mp\kappa\int_{D\setminus E}|y(t)|\Delta t}{K_y},$$

where

$$K_y := \int_0^{\sigma^2(S)} p(t)|y^\Delta(t)|^2\Delta t + \int_E q(t)\Delta t + \int_{D\setminus E} q(t)|y^\sigma(t)|^2\Delta t$$
$$+ \beta_1 p(0)y^2(0) + \beta_2 p(\sigma(S))y^2(\sigma^2(S)).$$

Then, there is $n_0 \in \mathbb{N}$ such that

$$\xi_n < \tau \quad \text{and} \quad \inf_{t\in B} F(t,\xi_n) \geq \kappa|\xi_n|^p$$

for all $n > n_0$. Now, for every $n > n_0$, by taking into account $0 \leq \xi_n y(t) < \tau$ for sufficiently large n, since g is nonnegative, one has

$$\begin{aligned}
\frac{\mathcal{J}_2(\xi_n y)}{\mathcal{J}_1(\xi_n y)} &= \frac{\int_E F(t,\xi_n)\Delta t + \int_{D\setminus E} F(t,\xi_n y(t))\Delta t + \frac{\zeta}{\nu}\int_0^{\sigma(S)} G(t,\xi_n y(t))\Delta t}{\mathcal{J}_1(\xi_n y)} \\
&\geq \frac{\int_E F(t,\xi_n)\Delta t + \int_{D\setminus E} F(t,\xi_n y(t))\Delta t + \frac{\zeta}{\nu}G_\tau}{\mathcal{J}_1(\xi_n y)} \\
&\geq \frac{\eta\operatorname{meas}(E) + \kappa\int_{D\setminus E}|y(t)|\Delta t}{K_y} > M.
\end{aligned}$$

Hence, by the arbitrariness of M, we obtain

$$\lim_{n\to\infty} \frac{\mathcal{J}_2(\xi_n y)}{\mathcal{J}_1(\xi_n y)} = \infty,$$

from which (31) follows. Thus, there exists $\{\omega_n\} \subset X$ that converges strongly to zero such that $\omega_n \in \mathcal{J}_1^{-1}(-\infty, r_{1\nu})$ and

$$I_\nu(\omega_n) = \mathcal{J}_1(\omega_n) - \nu\mathcal{J}_2(\omega_n) < 0.$$

Since $x_{1\nu}$ is a global minimum of the restriction of I_ν to $\mathcal{J}_1^{-1}(-\infty, r_{1\nu})$, we get

$$I_\nu(x_{1\nu}) < 0, \tag{32}$$

which means $x_{1\nu}$ is nontrivial. By the same arguments, we see that $x_{2\nu}$ and $x_{3\nu}$ are nontrivial. If we assume that there exist $\emptyset \neq D \subseteq [0, S]_{\mathbb{T}}$ (D open) and $B \subset D$ such that

$$\limsup_{\xi\to 0^+} \frac{\inf_{t\in B} G(t,\xi)}{|\xi|^2} = \infty \text{ and } \liminf_{\xi\to 0^+} \frac{\inf_{t\in D} G(t,\xi)}{|\xi|^2} > -\infty$$

instead of (30), respectively, then the solutions are again nontrivial.

Remark 19. Using (32), we obtain negativity of the map

$$\Lambda := \left(\frac{K_d}{\int_0^{\sigma(S)} (F(t,d) - F(t,\theta_1))\Delta t}, \nu^* \right) \ni \nu \mapsto I_\nu(x_{i\nu}), \quad i = 1,2,3. \tag{33}$$

Also, one has

$$\lim_{\nu\to 0^+} \|x_{i\nu}\| = 0, \quad i = 1,2,3.$$

Indeed, recalling that $\mathcal{J}_2$ is coercive and for all $\nu \in \Lambda$, for the solution $x_{i\nu} \in \mathcal{J}_1^{-1}(-\infty, r_{i\nu})$, $i = 1,2,3$, we get the existence of $L > 0$ satisfying $\|x_{i\nu}\| \leq L$, $i = 1,2,3$ for all $\nu \in \Lambda$. Then, there exists $N > 0$ with

$$\left| \int_0^{\sigma(S)} f(t, x_{i\nu}^\sigma(t)) x_{i\nu}^\sigma(t)\Delta t + \frac{\zeta}{\nu} \int_0^{\sigma(S)} g(t, x_{i\nu}^\sigma(t)) x_{i\nu}^\sigma(t)\Delta t \right| \leq N\|x_{i\nu}\| \leq NL \tag{34}$$

for $i = 1,2,3$, for every $\nu \in \Lambda$. Since $x_{i\nu}$, $i = 1,2,3$ is a critical point of I_ν, we have $I'_\nu(x_{i\nu})(y) = 0$, $i = 1,2,3$, for all $y \in X$ and all $\nu \in \Lambda$. Hence, $I'_\nu(x_{i\nu})(x_{i\nu}) = 0$, $i = 1,2,3$, that is,

$$\mathcal{J}'_1(x_{i\nu})(x_{i\nu}) = \nu \int_0^{\sigma(S)} f(t, x_{i\nu}^\sigma(t)) x_{i\nu}^\sigma(t)\Delta t + \zeta \int_0^{\sigma(S)} g(t, x_{i\nu}^\sigma(t)) x_{i\nu}^\sigma(t)\Delta t$$

for all $\nu \in \Lambda$. Then, it follows

$$0 \leq \mathcal{J}'_1(x_{i\nu})(x_{i\nu}) = \nu \int_0^{\sigma(S)} f(t, x_{i\nu}^\sigma(t)) x_{i\nu}^\sigma(t)\Delta t + \zeta \int_0^{\sigma(S)} g(t, x_{i\nu}^\sigma(t)) x_{i\nu}^\sigma(t)\Delta t$$

for all $\nu \in \Lambda$. Letting $\nu \to 0^+$ by (34), we obtain

$$\lim_{\nu\to 0^+} \|x_{i\nu}\| = 0, \quad i = 1,2,3.$$

This gives the desired conclusion. Finally, we show that the map

$$\nu \mapsto I_\nu(x_{i\nu}), \quad i = 1,2,3$$

strictly decreases in $\nu \in \Lambda$. For any $x \in X$, we have

$$I_\nu(x) = \nu \left(\frac{\mathcal{J}_1(x)}{\nu} - \mathcal{J}_2(x) \right). \tag{35}$$

Take $0 < \nu_1 < \nu_2 < \nu^*$ and let $x_{i\nu_j}$ be the global minimum of I_{ν_j} restricted to $\mathcal{J}_1(-\infty, r_{i\nu_j})$ for $i = 1,2,3$, $j = 1,2$. Also, put

$$m_{i\nu_j} = \frac{\mathcal{J}_1(x_{i\nu_j})}{\nu_i} - \mathcal{J}_2(x_{i\nu_j}) = \inf_{y \in \mathcal{J}_1^{-1}(-\infty, r_{i\nu_j})} \left(\frac{\mathcal{J}_1(y)}{\nu_j} - \mathcal{J}_2(y) \right)$$

for every $i = 1,2,3$, $j = 1,2$. Then, (33) in conjunction with (35) and $\nu > 0$ yields

$$m_{i\nu_j} < 0 \quad \text{for} \quad i = 1,2,3, \quad j = 1,2. \tag{36}$$

Moreover,

$$m_{i\nu_2} < m_{i\nu_1}, \quad i = 1,2,3 \tag{37}$$

due to the fact that $0 < \nu_1 < \nu_2$. Then, by (35)–(37) and again by the fact that $0 < \nu_1 < \nu_2$, we get

$$I_{\nu_2}(x_{i\nu_2}) = \nu_2 m_{i\nu_2} \le \nu_2 m_{i\nu_1} < \nu_1 m_{i\nu_1}, \quad i = 1,2,3$$

so that the map $\nu \mapsto I_\nu(x_{i\nu})$, $i = 1,2,3$, decreases strictly in $\nu \in \Lambda$. Since $\nu < \nu^*$ is arbitrary, we obtain that $\nu \mapsto I_\nu(x_{i\nu})$ decreases strictly in $\nu \in \Lambda$.

For positive constants θ_1, θ_4, and d, set

$$\begin{aligned} \delta'_{\nu,g} := \min \Bigg\{ & \frac{1}{2C^2} \min \Bigg\{ \frac{\theta_1^2 - \nu \int_a^{\sigma(S)} F(t,\theta_1)\Delta t}{G^{\theta_1}}, \\ & \frac{\theta_4^2 - 2\nu \int_0^{\sigma(S)} F(t, \frac{1}{\sqrt{2}}\theta_4)\Delta t}{2G^{\frac{1}{\sqrt{2}}\theta_4}}, \frac{\theta_4^2 - 2\nu \int_0^{\sigma(S)} F(t,\theta_4)\Delta t}{2G^{\theta_4}} \Bigg\}, \\ & \frac{K_d - \nu \int_0^{\sigma(S)} (F(t,d) - F(t,\theta_1))\Delta t}{G_d - G^{\theta_1}} \Bigg\}. \end{aligned} \tag{38}$$

Theorem 20. *Let $f : [0,S]_{\mathbb{T}} \times [0,\infty) \to [0,\infty)$ be continuous. Assume the existence of $\theta_1, \theta_4, d > 0$ such that*

$$\theta_1 < C\sqrt{2K_d} < \frac{\theta_4}{\sqrt{2}}$$

and

$$(\mathrm{A}_2) \ \max\left\{ \frac{\int_0^{\sigma(S)} F(t,\theta_1)\Delta t}{\theta_1^2}, \ \frac{2\int_0^{\sigma(S)} F(t,\theta_4)\Delta t}{\theta_4^2} \right\} < \tfrac{1}{4C^2} \frac{\int_0^{\sigma(S)} F(t,d)\Delta t}{K_d}.$$

Then, for every

$$\nu \in \left(\frac{K_d}{\int_0^{\sigma(S)} F(t,d)\Delta t}, \frac{1}{4C^2} \min\left\{ \frac{\theta_1^2}{\int_0^{\sigma(S)} F(t,\theta_1)\Delta t}, \frac{\theta_4^2}{2\int_0^{\sigma(S)} F(t,\theta_4)\Delta t} \right\} \right)$$

and every continuous $g : [0,S]_{\mathbb{T}} \times \mathbb{R} \to [0,\infty)$, there is $\delta'_{\nu,g} > 0$ defined by (38) such that, for all $\zeta \in [0, \delta'_{\nu,g})$, $(P_{\nu,\zeta})$ admits at least three nonnegative solutions $x_1, x_2, x_3 \in \mathcal{H}$ satisfying

$$\max_{t\in[0,S]_{\mathbb{T}}} |x_1(t)| < \theta_1, \ \max_{t\in[0,S]_{\mathbb{T}}} |x_2(t)| < \frac{\theta_4}{\sqrt{2}}, \ \max_{t\in[0,S]_{\mathbb{T}}} |x_3(t)| < \theta_4.$$

Proof. Choose $\theta_2 = \frac{\theta_4}{\sqrt{2}}$ and $\theta_3 = \theta_4$. So, from (A$_2$) one has

$$\begin{aligned}\frac{\int_0^{\sigma(S)} F(t,\theta_2)\Delta t}{\theta_2^2} &= \frac{2\int_0^{\sigma(S)} F(t,\frac{\theta_4}{\sqrt{2}})\Delta t}{\theta_4^2} \leq \frac{2\int_a^{\sigma(S)} F(t,\theta_4)\Delta t}{\theta_4^2} \\ &< \frac{1}{4C^2}\frac{\int_0^{\sigma(S)} F(t,d)\Delta t}{K_d}\end{aligned} \tag{39}$$

and

$$\frac{\int_0^{\sigma(S)} F(t,\theta_3)\Delta t}{\theta_3^2-\theta_2^2} = \frac{2\int_0^{\sigma(S)} F(t,\theta_4)\Delta t}{\theta_4^2} < \frac{1}{4C^2}\frac{\int_0^{\sigma(S)} F(t,d)\Delta t}{K_d}. \tag{40}$$

Moreover, taking into account that $\theta_1 < C\sqrt{2K_d} < \dfrac{\theta_4}{\sqrt{2}}$, by using (A$_2$), we have

$$\begin{aligned}\frac{1}{2C^2}\frac{\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t}{K_d} &> \frac{1}{2C^2}\frac{\int_0^{\sigma(S)} F(t,d)\Delta t}{K_d} - \frac{\int_0^{\sigma(S)} F(t,\theta_1)\Delta t}{\theta_1^2} \\ &> \frac{1}{2C^2}\left(\frac{\int_0^{\sigma(S)} F(t,d)\Delta t}{K_d} - \frac{2C^2}{4C^2}\frac{\int_0^{\sigma(S)} F(t,d)\Delta t}{K_d}\right) \\ &= \frac{1}{4C^2}\frac{\int_0^{\sigma(S)} F(t,d)\Delta t}{K_d}.\end{aligned}$$

Hence, from (A$_2$), (39), and (40), we observe that (A$_1$) of Theorem 17 is fulfilled, completing the proof. □

The following two results are special cases of Theorem 20.

Theorem 21. *Let $f_1 \in L^1([0,S]_{\mathbb{T}})$ and $f_2 \in C(\mathbb{R})$. Put $\tilde{F}(\xi) = \int_0^\xi f_2(s)\mathrm{d}s$, $\xi \in \mathbb{R}$, and assume the existence of $\theta_1, \theta_4, d > 0$ with*

$$\theta_1 < C\sqrt{2K_d} < \frac{\theta_4}{\sqrt{2}}$$

and

(A$_3$) *$f_1(t) \geq 0$ for all $t \in [0,S]_{\mathbb{T}}$ and $f_2(\xi) \geq 0$ for each $\xi \in [0,\infty)$,*
(A$_4$) $\max\left\{\dfrac{\tilde{F}(\theta_1)}{\theta_1^2}, \dfrac{2\tilde{F}(\theta_4)}{\theta_4^2}\right\} < \dfrac{1}{4C^2}\dfrac{\tilde{F}(d)}{K_d}$.

Then, for every

$$\nu \in \left(\frac{K_d}{\tilde{F}(d)\int_0^{\sigma(S)} f_1(t)\Delta t}, \frac{1}{4C^2\int_0^{\sigma(S)} f_1(t)\Delta t}\min\left\{\frac{\theta_1^2}{\tilde{F}(\theta_1)}, \frac{\theta_4^2}{2\tilde{F}(\theta_4)}\right\}\right)$$

and every continuous $g : [0,S]_{\mathbb{T}} \times \mathbb{R} \to [0,\infty)$, whenever

$$\begin{aligned}\zeta \in \Bigg[0, \min\Bigg\{\frac{1}{2C}\min\Bigg\{&\frac{\theta_1^2 - \nu\tilde{F}(\theta_1)\int_0^{\sigma(S)} f_1(t)\Delta t}{G^{\theta_1}}, \\ &\frac{\theta_4^2 - 2\nu\tilde{F}(\frac{\theta_4}{\sqrt{2}})\int_0^{\sigma(S)} f_1(t)\Delta t}{2G^{\frac{\theta_4}{\sqrt{2}}}}, \frac{\theta_4^2 - 2\nu\tilde{F}(\theta_4)\int_0^{\sigma(S)} f_1(t)\Delta t}{2G^{\theta_4}}\Bigg\},\end{aligned}$$

$$\left.\frac{K_d - \nu \int_0^{\sigma(S)} f_1(t)\Delta t - \left(\tilde{F}(d) - \tilde{F}(\theta_1)\right)}{G_d - G^{\theta_1}}\right\}\Bigg],$$

the problem

$$\begin{cases} -(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = \nu f_1(t) f_2(x^{\sigma}(t)) + \zeta g(t, x^{\sigma}(t)), & t \in [0, S]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^{\Delta}(0) = 0, \quad \alpha_3 x(\sigma^2(S)) + \alpha_4 x^{\Delta}(\sigma(S)) = 0 \end{cases} \tag{P_ν}$$

admits at least three nonnegative solutions $x_1, x_2, x_3 \in \mathcal{H}$ *satisfying*

$$\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_1(t)| < \theta_1, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_2(t)| < \frac{\theta_4}{\sqrt{2}}, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_3(t)| < \theta_4.$$

Proof. Put $f(t,x) = f_1(t) f_2(x)$ for $(t,x) \in [0,S]_{\mathbb{T}} \times \mathbb{R}$. Since $F(t,x) = f_1(t)\tilde{F}(x)$ for all $(t,x) \in [0,S]_{\mathbb{T}} \times \mathbb{R}$, from ($A_4$), we obtain ($A_2$). □

Theorem 22. *Assume the existence of* $\theta_1, \theta_4, d > 0$ *with*

$$\theta_1 < C\sqrt{2K_d} < \frac{\theta_4}{\sqrt{2}}$$

and

(A_5) $f(\xi) \geq 0$ *for all* $\xi \in [0, \infty)$,

(A_6) $\max\left\{\frac{F(\theta_1)}{\theta_1^2}, \frac{2F(\theta_4)}{\theta_4^2}\right\} < \frac{1}{4C^2}\frac{F(d)}{K_d}$.

Then, for every

$$\nu \in \left(\frac{K_d}{\sigma(S)F(d)}, \frac{1}{4C^2\sigma(S)} \min\left\{\frac{\theta_1^2}{F(\theta_1)}, \frac{\theta_4^2}{2F(\theta_4)}\right\}\right)$$

and every continuous $g : [0,S]_{\mathbb{T}} \times \mathbb{R} \to [0, \infty)$, *whenever*

$$\zeta \in \left[0, \min\left\{\frac{1}{2C^2} \min\left\{\frac{\theta_1^2 - \nu\sigma(S)F(\theta_1)}{G^{\theta_1}}, \frac{\theta_4^2 - 2\nu\sigma(S)F(\frac{1}{\sqrt{2}}\theta_4)}{2G^{\frac{1}{\sqrt{2}}\theta_4}}\right\}, \frac{K_d - \nu\sigma(S)(F(d) - F(\theta_1))}{G_d - G^{\theta_1}}\right\}\right],$$

the problem

$$\begin{cases} -(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = \nu f(x^{\sigma}(t)) + \zeta g(t, x^{\sigma}(t)), & t \in [0, S]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^{\Delta}(0) = 0, \quad \alpha_3 x(\sigma^2(S)) + \alpha_4 x^{\Delta}(\sigma(S)) = 0 \end{cases} \tag{41}$$

admits at least three nonnegative solutions $x_1, x_2, x_3 \in \mathcal{H}$ *satisfying*

$$\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_1(t)| < \theta_1, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_2(t)| < \frac{1}{\sqrt{2}}\theta_4, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_3(t)| < \theta_4.$$

In the following example, all assumptions of Theorem 22 are fulfilled.

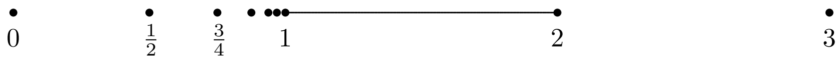
Example 23. Consider the nontrivial time scale (see Figure 1)

$$\mathbb{T} = \left\{1 - \frac{1}{2^n} : n \in \mathbb{N}_0\right\} \cup [1, 2] \cup \{3\}.$$

Let $p(t) \equiv 1$ and $q(t) \equiv 1$ on $\mathbb{T}$. Let

$$S = 1, \quad \alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 1, \quad \alpha_4 = 4,$$

FIGURE 1. $\mathbb{T}$ in Example 23



and

$$f(\xi)=\begin{cases}13\xi^{12}, & \xi\le 1,\\ \frac{13}{\xi}, & \xi>1.\end{cases}$$

Hence, we have

$$\sigma(S)=\sigma^2(S)=1,\quad \beta_1=\frac{1}{2},\quad \beta_2=\frac{1}{4},\quad C=\min\left\{2,2\sqrt{2},\sqrt{2}\right\}=\sqrt{2},$$

and

$$F(\xi)=\begin{cases}\xi^{13}, & \xi\le 1,\\ 1+13\ln(\xi), & \xi>1.\end{cases}$$

We now choose

$$\theta_1=10^{-8},\quad \theta_4=10^3,\quad \text{and}\quad d=10.$$

Now, it is easy to check that all assumptions of Theorem 22 are fulfilled. Thus, for every

$$\nu\in\left(\frac{87.5}{1+13\ln 10},\frac{10^6}{16(1+39\ln 10)}\right)$$

and every continuous $g:[0,1]_{\mathbb{T}}\times\mathbb{R}\to[0,\infty)$, whenever

$$\zeta\in\left[0,\min\left\{\frac{1}{4}\min\left\{\frac{10^{-16}-\nu 10^{-104}}{G^{10^{-8}}},\frac{10^6-2\nu\left(1+13\ln\left(\frac{1000}{\sqrt{2}}\right)\right)}{2G^{\frac{1000}{\sqrt{2}}}}\right\},\right.\right.$$
$$\left.\left.\frac{87.5-\nu\left(1+13\ln 10-10^{-104}\right)}{G_{10}-G^{10^{-8}}}\right\}\right],$$

the problem

$$\begin{cases}-x^{\Delta\Delta}(t)+x^{\sigma}(t)=\nu f(x^{\sigma}(t))+\zeta g(t,x^{\sigma}(t)), & t\in[0,1]_{\mathbb{T}},\\ x\left(\frac{1}{2}\right)=2x^{\Delta}(0),\quad x(1)+4x^{\Delta}(1)=0\end{cases}\tag{42}$$

admits at least three nonnegative solutions x_1, x_2, and x_3 satisfying

$$\max_{t\in[0,1]_{\mathbb{T}}}|x_1(t)|<10^{-8},\ \max_{t\in[0,1]_{\mathbb{T}}}|x_2(t)|<\frac{1000}{\sqrt{2}},\ \max_{t\in[0,1]_{\mathbb{T}}}|x_3(t)|<10^3.$$

Our final result is concerned with the case $\zeta=0$.

Theorem 24. *Let $f:[0,S]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$ be continuous such that $\xi f(t,\xi)>0$ for all $(t,\xi)\in[0,S]_{\mathbb{T}}\times(\mathbb{R}\setminus\{0\})$. Suppose*

(A_7) $\lim_{\xi\to 0}\frac{f(t,\xi)}{|\xi|}=\lim_{|\xi|\to\infty}\frac{f(t,\xi)}{|\xi|}=0.$

Then, for all

$$\nu > \nu^{**} := \max\left\{ \inf_{d>0} \frac{K_d}{\int_0^{\sigma(S)} F(t,d)\Delta t},\ \inf_{d<0} \frac{K_d}{\int_0^{\sigma(S)} F(t,d)\Delta t} \right\},$$

the problem $(P_{\nu,0})$ *admits at least four distinct nontrivial solutions.*

Proof. Put

$$f_1(t,\xi) = \begin{cases} f(t,\xi) & \text{if } (t,\xi) \in [0,S]_{\mathbb{T}} \times [0,\infty), \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_2(t,\xi) = \begin{cases} -f(t,-\xi) & \text{if } (t,\xi) \in [0,S]_{\mathbb{T}} \times [0,\infty), \\ 0 & \text{otherwise} \end{cases}$$

and set $F_1(t,\xi) := \int_0^t f_1(t,s)\mathrm{d}s$ for every $(t,\xi) \in [0,S]_{\mathbb{T}} \times \mathbb{R}$. Take $\nu > \nu^{**}$ and $d > 0$ with $\nu > \frac{K_d}{\int_0^{\sigma(S)} F_1(t,d)\Delta t}$. From

$$\lim_{t\to 0^+} \frac{f_1(t,\xi)}{|\xi|} = \lim_{t\to\infty} \frac{f_1(t,\xi)}{|\xi|} = 0,$$

there exist θ_1 and θ_4 such that

$$\theta_1 < C\sqrt{2K_d} < \frac{\theta_4}{\sqrt{2}},\quad \frac{\int_a^{\sigma(S)} F_1(t,\theta_1)\Delta t}{\theta_1^2} < \frac{1}{4C^2\nu},\quad \frac{\int_0^{\sigma(S)} F_1(t,\theta_4)\Delta t}{\theta_4^2} < \frac{1}{8C^2\nu}.$$

Then, (A_2) in Theorem 20 is fulfilled, and

$$\nu \in \left(\frac{K_d}{\int_0^{\sigma(S)} F_1(t,d)\Delta t}, \frac{1}{4C^2} \min\left\{ \frac{\theta_1^2}{\int_0^{\sigma(S)} F_1(t,\theta_1)\Delta t}, \frac{\theta_4^2}{2\int_0^{\sigma(S)} F_1(t,\theta_4)\Delta t} \right\} \right).$$

Hence, the problem $(P_{\nu,0}^{f_1})$ admits two positive solutions x_1 and x_2, and they are positive solutions of $(P_{\nu,0})$. Next, by the same arguments, from

$$\lim_{t\to 0^+} \frac{f_2(t,\xi)}{|\xi|} = \lim_{\xi\to\infty} \frac{f_2(t,\xi)}{|\xi|} = 0,$$

we guarantee existence of two positive solutions x_3 and x_4 for $(P_{\nu,0}^{f_2})$. Clearly, $-x_3$ and $-x_4$ are negative solutions of $(P_{\nu,0})$, and the proof is complete. □

Remark 25. We remark that in Theorem 24, f is not assumed to be symmetric. But, if $f \not\equiv 0$ is odd and continuous satisfying $f(t,\xi) \geq 0$ for all $(t,\xi) \in [0,S]_{\mathbb{T}} \times [0,\infty)$, then (A_7) may be substituted by

(A_8) $\lim_{\xi\to 0^+} \frac{f(t,\xi)}{\xi} = \lim_{\xi\to\infty} \frac{f(t,\xi)}{\xi} = 0$,

guaranteeing existence of at least four distinct nontrivial solutions of $(P_{\nu,0})$ for every $\nu > \inf_{d>0} \frac{K_d}{\int_0^{\sigma(S)} F(t,d)\Delta t}$.

5. The existence of infinitely many solutions

5.1. **Main results.** Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$. In particular, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ are examples of time scales corresponding to differential and difference equations, respectively. Let $T > 0$ be fixed and suppose $0, T \in \mathbb{T}$. The aim of this paper is to investigate the existence of infinitely many solutions for the following second order Sturm–Liouville type boundary value problem on time scales:

$$\begin{cases} -(pu^{\Delta})^{\Delta}(t) + q(t)u^{\sigma}(t) = \nu f(t, u^{\sigma}(t)) + \zeta g(t, u^{\sigma}(t)), & t \in [0,T]_{\mathbb{T}}, \\ \alpha_1 u(0) - \alpha_2 u^{\Delta}(0) = 0, \qquad \alpha_3 u(\sigma^2(T)) + \alpha_4 u^{\Delta}(\sigma(T)) = 0, \end{cases} \tag{$P^{f,g}_{\nu,\zeta}$}$$

where $p \in C^1([0,\sigma(T)],(0,+\infty))$, $q \in C([0,T],[0,+\infty))$, $f, g \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $\nu > 0$, $\zeta \geq 0$, $\alpha_i \geq 0$, for $i = 1,2,3,4$, $\sigma(0) = 0$, $\sigma(T) = T$ and $\alpha_1 + \alpha_2 \geq 0, \alpha_3 + \alpha_4 > 0, \alpha_1 + \alpha_3 > 0$.

In this section, we study the existence of solutions for the second order Sturm–Liouville type boundary value problem on time scales $(P^{f,g}_{\nu,\zeta})$ which turns out as an optimization problem on time scales which arises in economics and finance. In fact, employing a smooth version of [25, Theorem 2.1], under an appropriate oscillating behaviour of the nonlinear term f, we determine the exact collections of the parameter ν in which the problem $(P^{f,g}_{\nu,\zeta})$ for every non negative arbitrary function $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ which is measurable in $[0,T]$ and of class $C^1(\mathbb{R})$ satisfying a certain growth at infinity, choosing ζ sufficiently small, admits infinitely many solutions (Theorem 27). Replacing the oscillating behaviour condition at infinity, by a similar one at zero, we achieve a sequence of pairwise distinct solutions which converges to zero (Theorem 32). We also list some consequences the main results. The applicability of our results is illustrated by an example.

We formulate our main results discussing the existence of infinitely many solutions for the problem $(P^{f,g}_{\nu,\zeta})$. Our main tool to ensure the results is a smooth version [25, Theorem 2.1] which is a more precise version of Ricceri's variational principle [59, Theorem 2.5] that we now recall here.

Theorem 26. *Let X be a reflexive real Banach space, let $\mathcal{J}_1, \mathcal{J}_2 : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that $\mathcal{J}_1$ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and $\mathcal{J}_2$ is sequentially weakly upper semicontinuous. For every $r > \inf_X \mathcal{J}_1$, let us put*

$$\varphi(r) := \inf_{u \in \mathcal{J}_1^{-1}(-\infty,r)} \frac{\sup_{u \in \mathcal{J}_1^{-1}(-\infty,r)} \mathcal{J}_2(u) - \mathcal{J}_2(u)}{r - \mathcal{J}_1(u)}$$

and

$$\theta := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \mathcal{J}_1)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \mathcal{J}_1$ and every $\nu \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $I_\nu = \mathcal{J}_1 - \nu\mathcal{J}_2$ to $\mathcal{J}_1^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_ν in X.*

(b) *If $\theta < +\infty$ then, for each $\nu \in \left]0, \frac{1}{\theta}\right[$, the following alternative holds: either*

(b_1) *I_ν possesses a global minimum,*
or
(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_ν such that*

$$\lim_{n\to+\infty} \mathcal{J}_1(u_n) = +\infty.$$

(c) *If $\delta < +\infty$ then, for each $\nu \in \left]0, \frac{1}{\delta}\right[$, the following alternative holds:*
(c_1) *there is a global minimum of $\mathcal{J}_1$ which is a local minimum of I_ν,*
(c_2) *there is a sequence of pairwise distinct critical points (local minima) of I_ν which weakly converges to a global minimum of $\mathcal{J}_1$.*

We refer the interested reader to the papers [29, 34, 44, 45] in which Theorem 26 has been successfully employed to discuss the existence of infinitely many solutions for boundary value problems.

For convenience, put

$$A = \liminf_{\xi\to+\infty} \frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x|\le\xi} F(t,x)\Delta t}{\xi^2},$$

$$B = \frac{2}{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))} \limsup_{\xi\to+\infty} \frac{\int_0^{\sigma(T)} F(t,\xi)\Delta t}{\xi^2},$$

$$\nu_1 = \frac{1}{B}$$

and

$$\nu_2 = \frac{1}{2C^2 A}.$$

Theorem 27. *Assume that*

(B$_1$) *$F(t,x) \ge 0$ for each $(t,x) \in [0,T] \times [0,+\infty)$;*
(B$_2$)

$$A < \frac{1}{2C^2} B.$$

Then, for each $\nu \in]\nu_1, \nu_2[$ for every nonnegative arbitrary function $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ which is measurable in $[0,T]$ and of class $C^1(\mathbb{R})$ satisfying the condition

$$g_\infty := \limsup_{\xi\to+\infty} \frac{\int_0^{\sigma(T)} \sup_{|x|\le\xi} G(t,x)\Delta t}{\xi^2} < +\infty \tag{43}$$

and for every $\zeta \in [0, \zeta_{g,\nu}[$ where

$$\zeta_{g,\nu} := \frac{1}{2C^2 g_\infty}\left(1 - 2\nu C^2 A\right), \tag{44}$$

the problem $(P^{f,g}_{\nu,\zeta})$ has an unbounded sequence of solutions in $H^1_\Delta([0,\sigma^2(T)])$.

Proof. Our aim is to apply Theorem 26 to the problem $(P^{f,g}_{\nu,\zeta})$. Take $X = H^1_\Delta([0,\sigma^2(T)])$ and let the functionals $\mathcal{J}_1, \mathcal{J}_2$ for every $u \in X$, defined by

$$\mathcal{J}_1(u) = \frac{1}{2}\|u\|_0^2 \tag{45}$$

and

$$\mathcal{J}_2(u) = \int_0^{\sigma(T)} F(t, u^\sigma(t))\Delta t + \frac{\zeta}{\nu}\int_0^{\sigma(T)} G(t, u^\sigma(t))\Delta t.$$

Let us prove that the functionals $\mathcal{J}_1$ and $\mathcal{J}_2$ satisfy the required conditions in Theorem 26. It is well known that $\mathcal{J}_2$ is a differentiable functional whose differential at the point $u \in X$ is

$$\mathcal{J}_2'(u)(v) = \int_0^{\sigma(T)} f(t, u^\sigma(t))v^\sigma(t)\Delta t + \frac{\zeta}{\nu}\int_0^{\sigma(T)} g(t, u^\sigma(t))v^\sigma(t)\Delta t$$

for every $v \in X$, as well as is sequentially weakly upper semicontinuous. Moreover, $\mathcal{J}_1$ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\mathcal{J}_1'(u)(v) = &\int_0^{\sigma^2(T)} p(t)u^\Delta(t)v^\Delta(t)\Delta t + \int_0^{\sigma(T)} q(t)u^\sigma(t)v^\sigma(t)\Delta t\\ &+\beta_1 p(0)u(0)v(0) + \beta_2 p(\sigma(T))u(\sigma^2(T))v(\sigma^2(T))\end{aligned}$$

for every $v \in X$. Moreover, $\mathcal{J}_1$ is sequentially weakly lower semicontinuous and coercive. Therefore, we observe that the regularity assumptions on $\mathcal{J}_1$ and $\mathcal{J}_2$, as requested in Theorem 26, are verified. Let $\{\xi_n\}$ be a real sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n = +\infty$, and

$$A = \lim_{n\to+\infty} \frac{\int_0^{\sigma(T)} \sup_{|x|\le\xi_n} F(t,x)\Delta t}{\xi_n^2}.$$

Put

$$r_n = \frac{\xi_n^2}{2C^2}.$$

If $u \in \mathcal{J}_1^{-1}(-\infty, r)$, then $\mathcal{J}_1(u) < r_n$, that is $\frac{1}{2}\|u\|_0^2 < r_n$. Hence, by Lemma 2, we have $|u(t)| \le C\sqrt{2r_n} = \xi_n$ for every $t \in [0, \sigma^2(T)]$. So

$$\sup_{\mathcal{J}_1(u)<r_n} \mathcal{J}_2(u) \le \int_0^{\sigma(T)} \sup_{|x|\le\xi_n} F(t,x)\Delta t.$$

Therefore, since $0 \in \mathcal{J}_1^{-1}(-\infty, r_n)$ and $\mathcal{J}_1(0) = \mathcal{J}_2(0) = 0$, one has

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\mathcal{J}_1^{-1}(-\infty,r_n)} \frac{(\sup_{u\in\mathcal{J}_1^{-1}(-\infty,r_n)}\mathcal{J}_2(u)) - \mathcal{J}_2(u)}{r_n - \mathcal{J}_1(u)} \le \frac{\sup_{u\in\mathcal{J}_1^{-1}(-\infty,r_n)}\mathcal{J}_2(u)}{r_n}\\ &= \frac{\sup_{u\in\mathcal{J}_1^{-1}(-\infty,r_n)} \int_0^{\sigma(T)} \left[F(t,u(t)) + \frac{\zeta}{\nu}G(t,u(t))\right]\Delta t}{r_n}\\ &\le \frac{\int_0^{\sigma(T)} \sup_{|x|\le\xi_n} F(t,x)\Delta t}{r_n} + \frac{\frac{\zeta}{\nu}\int_0^{\sigma(T)} \sup_{|x|\le\xi_n} G(t,x)\Delta t}{r_n}\\ &= 2C^2\frac{\int_0^{\sigma(T)} \sup_{|x|\le\xi_n} F(t,x)\Delta t}{\xi_n^2} + 2\frac{\zeta}{\nu}C^2\frac{\int_0^{\sigma(T)} \sup_{|x|\le\xi_n} G(t,x)\Delta t}{\xi_n^2}\end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, from the assumption (B_2) and the condition (43) one has

$$\theta \leq \liminf_{n\to+\infty} \varphi(r_n) \leq 2C^2 \left(A + \frac{\zeta}{\nu} g_\infty \right) < +\infty.$$

Now, let $\{\eta_n\}$ be positive real sequences and for all $n \in \mathbb{N}$, and

$$\lim_{n\to+\infty} \eta_n = +\infty.$$

Define $w_n(t) = \eta_n$ for all $t \in [0, T]$. Clearly, $w_n \in X$, from (45), we have

$$\mathcal{J}_1(w_n) = \left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T)) \right) \frac{\eta_n^2}{2}. \tag{46}$$

On the other hand, since g is non negative, bearing the assumption (B_1) in mind, from (45) one has

$$\begin{aligned}
\mathcal{J}_2(w_n) &= \int_0^{\sigma(T)} F(t, \eta_n)\Delta t + \frac{\zeta}{\nu} \int_0^{\sigma(T)} G(t, \eta_n)\Delta t \\
&\geq \int_0^{T} F(t, \eta_n)\Delta t.
\end{aligned}$$

Then,

$$\begin{aligned}
I_\nu(w_n) =& \mathcal{J}_1(w_n) - \nu \mathcal{J}_2(w_n) \\
\leq & \left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T)) \right) \frac{\eta_n^2}{2} - \nu \int_0^{\sigma(T)} F(t, \eta_n)\Delta t.
\end{aligned}$$

Now, consider the following cases:
If $B < +\infty$, let $\epsilon \in \left] 0, B - \frac{1}{\nu} \right[$. There exists ν_ϵ such that

$$\int_0^{\sigma(T)} F(t, \eta_n) dt > (B - \epsilon) \left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T)) \right) \frac{\eta_n^2}{2}$$

for all $n > \nu_\epsilon$, and so

$$\begin{aligned}
I_\nu(w_n) &< \left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T)) \right) \frac{\eta_n^2}{2} - \nu \int_0^{\sigma(T)} F(t, w_n(t))\Delta t \\
&= \left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T)) \right) \frac{\eta_n^2}{2} (1 - \nu(B - \epsilon)).
\end{aligned}$$

Since $1 - \nu(B - \epsilon) < 0$, and taking (46) into account, one has

$$\lim_{n\to+\infty} I_\nu(w_n) = -\infty.$$

If $B = +\infty$, fix $N > \frac{1}{\nu}$. There exists ν_N such that

$$\int_0^{\sigma(T)} F(t, \eta_n)\Delta t > N \left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T)) \right) \frac{\eta_n^2}{2}$$

for all $n > \nu_N$, and moreover,

$$I_\nu(w_n) < \left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T)) \right) \frac{\eta_n^2}{2} (1 - \nu N).$$

Since $1-\nu N<0$, arguing as before, we have

$$\lim_{n\to+\infty} I_\nu(w_n)=-\infty.$$

Notice that

$$\left]\frac{1}{B},\frac{1}{2C^2A}\right[\subset\left]0,\frac{1}{\theta}\right[,$$

and I_ν does not possess a global minimum, from part (b) of Theorem 26, there exists an unbounded sequence $\{u_n\}$ of critical points which are the solutions of $(P^{f,g}_{\nu,\zeta})$. So, the conclusion is achieved. $\square$

We present an example to illustrate Theorem 27 as follows.

Example 28. Let $\mathbb{T}=\{\frac{4}{n}:n=1,2,\ldots\}\cup\{0\}$ and $T=1$. Consider the problem

$$\begin{cases}-u^{\Delta\Delta}(t)=\nu f(u^\sigma(t)), & t\in[0,1]_{\mathbb{T}},\\ u(0)-2u^\Delta(0)=0, & u^\Delta(\frac{4}{3})=0,\end{cases}\tag{47}$$

where $f(\xi)=2\xi+40\xi\sin^2(e^\xi-1)+40\xi^2e^\xi\sin(e^\xi-1)\cos(e^\xi-1)$ and $g(\xi)=\frac{7}{5}\sqrt[5]{\xi^2}$ for every $\xi\in\mathbb{R}$. By the expressions of f and g, we have $F(\xi)=\xi^2\left(1+20\sin^2(e^\xi-1)\right)$ and $G(\xi)=\sqrt[5]{\xi^7}$ for every $\xi\in\mathbb{R}$. We observe that $C=2$. By simple calculations, we see that

$$\liminf_{\xi\to+\infty}\frac{\sup_{|x|\le\xi}F(x)}{\xi^2}=\liminf_{\xi\to+\infty}\frac{\xi^2\left(1+20\sin^2(e^\xi-1)\right)}{\xi^2}=\liminf_{\xi\to+\infty}\frac{\xi^2(1+0)}{\xi^2}=1,$$

$$\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}=\limsup_{\xi\to+\infty}\frac{\xi^2\left(1+20\sin^2(e^\xi-1)\right)}{\xi^2}=\limsup_{\xi\to+\infty}\frac{\xi^2(1+20)}{\xi^2}=21,$$

$$g_\infty:=\limsup_{\xi\to+\infty}\frac{\sup_{|x|\le\xi}G(x)}{\xi^2}=\limsup_{\xi\to+\infty}\frac{\sqrt[5]{\xi^7}}{\xi^2}=\limsup_{\xi\to+\infty}\frac{1}{\xi^{\frac{3}{5}}}=0<+\infty,$$

$$\beta_1=\begin{cases}\dfrac{\alpha_1}{\alpha_2}=\dfrac{1}{2}, & \text{if } \alpha_2>0,\\ 0, & \text{if } \alpha_2=0,\end{cases}$$

$$\beta_2=\begin{cases}\dfrac{\alpha_3}{\alpha_4}=0, & \text{if } \alpha_4>0,\\ 0, & \text{if } \alpha_4=0,\end{cases}$$

$$M_1=\sqrt{2}\max\left\{\frac{1}{\sqrt{\beta_1p(0)}},\frac{\sqrt{\sigma^2(T)}}{\min_{t\in[0,\sigma(T)]}p(t)}\right\}=\sqrt{2}\max\left\{\frac{1}{\sqrt{\frac{1}{2}}},\frac{\frac{4}{3}}{1}\right\},$$

$$M_2=\sqrt{2}\max\left\{\frac{1}{\sqrt{\beta_2p(0)}},\frac{\sqrt{\sigma^2(T)}}{\min_{t\in[0,\sigma(T)]}p(t)}\right\}=\sqrt{2}\max\left\{\frac{1}{0},\frac{\frac{4}{3}}{1}\right\},$$

$$M_3=\sqrt{2}\max\left\{\frac{\sqrt{\sigma(T)}}{\min_{t\in[0,T]}q(t)},\frac{\sqrt{\sigma^2(T)}}{\min_{t\in[0,\sigma(T)]}p(t)}\right\}=\sqrt{2}\max\left\{\frac{\sqrt{\frac{4}{3}}}{0},\frac{\frac{4}{3}}{1}\right\},$$

and where $\frac{1}{0} = +\infty$,

$$A = \liminf_{\xi \to +\infty} \frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x|\le\xi} F(x)\Delta t}{\xi^2} = \frac{4}{3} \liminf_{\xi\to+\infty} \frac{\sup_{|x|\le\xi} F(x)}{\xi^2} = \frac{4}{3},$$

$$B = \frac{2}{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))} \limsup_{\xi\to+\infty} \frac{\int_0^{\sigma(T)} F(t,\xi)\Delta t}{\xi^2}$$

$$= \frac{2}{0 + \dfrac{1}{2} + 0}(\frac{4}{3} \times 21) = 112$$

and $C = \min\{M_1, M_2, M_3\} = \sqrt{2}\dfrac{1}{\sqrt{\dfrac{1}{2}}} = 2$. We clearly see that all assumptions

$$A = \frac{4}{3} < \frac{112}{8} = \frac{1}{2C^2} B.$$

are satisfied. Then, for every $\nu \in \left(\dfrac{1}{112}, \dfrac{3}{32}\right)$ and for each $\zeta \in [0, +\infty)$ the problem (47) admits a sequence of solutions which is unbounded in $H^1_\Delta([0, \sigma^2(1)])$.

Remark 29. Under the conditions $A = 0$ and $B = +\infty$, Theorem 27 deduces that for every $\nu > 0$ and for each

$$\zeta \in \left[0, \frac{1}{2C^2 g_\infty}\right[$$

the problem $(P^{f,g}_{\nu,\zeta})$ admits infinitely many solutions in $H^1_\Delta([0, \sigma^2(T)])$. Moreover, if $g_\infty = 0$, the result holds for every $\nu > 0$ and $\zeta \ge 0$.

Remark 30. Put

$$\hat{\nu}_1 = \nu_1$$

and

$$\hat{\nu}_2 = \frac{1}{\lim_{n\to+\infty} \dfrac{\int_0^{\sigma(T)} \sup_{|x|\le c_n} F(t,x)\Delta t - \int_0^{\sigma(T)} F(t,b_n)\Delta t}{\dfrac{c_n^2}{2C^2} - \dfrac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2} b_n^2}}.$$

We explicitly observe that the assumption (A_2) in Theorem 27 could be replaced by the following more general condition:
(B_3) there exist two sequence $\{c_n\}$ with $\{b_n\}$ for all $n \in \mathbb{N}$ and

$$b_n^p < \frac{1}{2C^2\left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))\right)} c_n^p$$

for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} c_n = +\infty$ such that

$$\lim_{n\to+\infty} \frac{\int_0^{\sigma(T)} \sup_{|x|\le c_n} F(t,x)\Delta t - \int_0^{\sigma(T)} F(t,b_n)\Delta t}{\dfrac{c_n^2}{2C^2} - \dfrac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T)}{2} b_n^2}$$

$$< \frac{2}{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))} \limsup_{n\to+\infty} \frac{\int_0^{\sigma(T)} F(t,\eta_n)\Delta t}{\eta_n^2}.$$

Obviously, from (A_3) we obtain (A_2), by choosing $b_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (B_3) instead of (B_2) and set

$$r_n = \frac{c_n^2}{2C^2}$$

for all $n \in \mathbb{N}$, by the same arguing as inside in Theorem 27, we obtain

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\mathcal{J}_1^{-1}(-\infty,r_n)} \frac{(\sup_{u\in\mathcal{J}_1^{-1}(-\infty,r_n)} \mathcal{J}_2(u)) - \mathcal{J}_2(u)}{r_n - \mathcal{J}_1(u)} \\ &\leq \frac{\sup_{u\in\mathcal{J}_1^{-1}(-\infty,r_n)} \mathcal{J}_2(u) - \left[\int_0^{\sigma(T)} F(t,u(t))\Delta t + \frac{\zeta}{\nu}\int_0^{\sigma(T)} G(t,u(t))\Delta t\right]}{r_n - \mathcal{J}_1(u)} \\ &\leq \frac{\int_0^{\sigma(T)} \sup_{|x|\leq c_n} F(t,x)\Delta t - \int_0^{\sigma(T)} F(t,b_n)\Delta t}{\frac{c_n^2}{2C^2} - \frac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2} b_n^p}.\end{aligned}$$

We have the same conclusion as in Theorem 27 with Λ replaced by $\Lambda' :=]\hat{\nu}_2, \hat{\nu}_2[$.

Here we point out the following consequence of Theorem 27.

Corollary 31. *Assume that (A_1) holds and*

(B_4) $\liminf_{\xi\to+\infty} \frac{\int_0^{\sigma(T)} \sup_{|x|\leq\xi} F(t,x)\Delta t}{\xi^2} < \frac{1}{2C^2}$;

(A_5) $\limsup_{\xi\to+\infty} \frac{\int_0^{\sigma(T)} F(t,\xi)\Delta t}{\xi^2} > \frac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2}$.

Then, for every nonnegative arbitrary function $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ which is measurable in $[0,T]$ and of class $C^1(\mathbb{R})$ satisfying the condition (43) and for every $\zeta \in [0, \zeta_{g,1}[$ where

$$\zeta_{g,1} := \frac{1}{2C^2 g_\infty}\left(1 - 2C^2 A\right),$$

the problem

$$\begin{cases} -(pu^\Delta)^\Delta(t) + q(t)u^\sigma(t) = f(t,u^\sigma(t)) + \zeta g(t,u^\sigma(t)), & t \in [0,T], \\ \alpha_1 u(0) - \alpha_2 u^\Delta(0) = 0, \quad \alpha_3 u(\sigma^2(T)) + \alpha_4 u^\Delta(\sigma(T)) = 0, \end{cases} \qquad (P_{1,\zeta}^{f,g})$$

has an unbounded sequence of solutions in $H_\Delta^1([0,\sigma^2(T)])$.

In the same way as in the proof of Theorem 27 but using conclusion [(c) of Theorem 26 instead of [(b), we will obtain the following result.

Theorem 32. *Assume that all the hypotheses of Theorem 27 hold except for Assumption (B_2). Suppose that*

(B_5)

$$\bar{A} < \frac{1}{2C^2\left(\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))\right)}\bar{B}$$

where

$$\bar{A} = \liminf_{\xi\to 0^+} \frac{\int_0^{\sigma(T)} \sup_{|x|\leq\xi} F(t,x)\Delta t}{\xi^2}$$

and

$$\bar{B} = \frac{2}{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))} \limsup_{\xi \to 0^+} \frac{\int_0^{\sigma(T)} F(t,\xi)\Delta t}{\xi^2}.$$

Then, for each $\nu \in]\nu_3, \nu_4[$ *where*

$$\nu_3 := \frac{1}{\bar{B}}$$

and

$$\nu_4 := \frac{1}{2C^2 \bar{A}}$$

for every nonnegative arbitrary function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ *which is measurable in* $[0,T]$ *and of class* $C^1(\mathbb{R})$ *satisfying the condition*

$$g_0 := \limsup_{\xi \to 0^+} \frac{\int_0^{\sigma(T)} \sup_{|x| \le \xi} G(t,x)\Delta t}{\xi^2} < +\infty \tag{48}$$

and for every $\zeta \in [0, \zeta_{g_0,\nu}[$ *where*

$$\zeta_{g_0,\nu} := \frac{1}{2C^2 g_0}\left(1 - 2\nu C^2 \liminf_{\xi \to 0^+} \frac{\int_0^{\sigma(T)} \sup_{|x| \le \xi} F(t,x)\Delta t}{\xi^2}\right), \tag{49}$$

the problem $(P^{f,g}_{\nu,\zeta})$ *has a sequence of pairwise distinct solutions which strongly converges to* 0 *in* $H^1_\Delta([0,\sigma^2(T)])$.

Proof. We take $\mathcal{J}_1$ and $\mathcal{J}_2$ as in the proof of Theorem 27 and put $I_{\bar\nu}(u) = \mathcal{J}_1(u) - \bar\nu \mathcal{J}_2(u)$ for every $u \in H^1_\Delta([0,\sigma^2(T)])$. Since

$$\frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x| \le \xi} [F(t,x) + \frac{\bar\zeta}{\bar\nu} G(t,x)]\Delta t}{\xi^2} \le \frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x| \le \xi} F(t,x)\Delta t}{\xi^2} + \frac{\bar\zeta}{\bar\nu} \frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x| \le \xi} G(t,x)\Delta t}{\xi^2},$$

taking into account (48) one has

$$\liminf_{\xi \to 0^+} \frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x| \le \xi} [F(t,u(t)) + \frac{\bar\zeta}{\bar\nu} G(t,u(t))]\Delta t}{\xi^2} \le \liminf_{\xi \to 0^+} \frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x| \le \xi} F(t,x)\Delta t}{\xi^2} + \frac{\bar\zeta}{\bar\nu} g_0.$$

We verify that $\delta < +\infty$. For this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n \to +\infty} \frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x| \le \xi_n} \left[F(t,x) + \frac{\bar\zeta}{\bar\nu} G(t,x)\right]\Delta t}{\xi_n^2} < +\infty.$$

Put

$$\bar{A} = \lim_{n\to+\infty} \frac{\displaystyle\int_0^{\sigma(T)} \sup_{|x|\le\xi_n} F(t,x)\Delta t}{\xi_n^2}$$

and

$$r_n = \frac{\xi_n^2}{2C^2}$$

for every $n \in \mathbb{N}$. Therefore, from assumption (B_5) and the condition (48) one has

$$\delta \le \liminf_{n\to+\infty} \varphi(r_n) \le 2C^2 \left(\bar{A} + \frac{\bar{\zeta}}{\bar{\nu}} g_0\right) < +\infty.$$

Let us show that the functional $I_{\bar{\nu}}$ does not have a local minimum at zero. For this, let $\{\eta_n\}$ be a sequence of positive such that $\eta_n \to 0^+$ as $n \to +\infty$. Put

$$\bar{B} = \frac{2}{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))} \lim_{n\to 0^+} \frac{\int_0^{\sigma(T)} F(t,\eta_n)\Delta t}{\eta_n^2}. \tag{50}$$

Let $\{w_n\}$ be a sequence in $H^1_\Delta([0,\sigma^2(T)])$ with $w_n(t) = \eta_n$ for all $t \in [0,T]$. Moreover, since g is non negative, from the assumption (B_1) we obtain

$$\begin{aligned}\mathcal{J}_2(w_n) &= \int_0^{\sigma(T)} F(t,\eta_n)\Delta t + \frac{\bar{\zeta}}{\bar{\nu}} \int_0^{\sigma(T)} G(t,\eta_n)\Delta t \\ &\ge \int_0^{\sigma(T)} F(t,\eta_n)\Delta t.\end{aligned}$$

Then,

$$\begin{aligned}I_{\bar{\nu}}(w_n) &= \mathcal{J}_1(w_n) - \bar{\nu}\mathcal{J}_2(w_n) \\ &\le \frac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2}\eta_n^2 - \bar{\nu}\int_0^{\sigma(T)} F(t,\eta_n)\Delta t.\end{aligned}$$

Consider the following cases.
If $\bar{B} < +\infty$, let $\varepsilon \in]0, \bar{B} - \frac{1}{\bar{\nu}}[$. By (50), there exists ν_ε such that

$$\int_0^{\sigma(T)} F(t,\eta_n)\Delta t > (\bar{B}-\varepsilon)\frac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2}\eta_n^2$$

for all $n > \nu_\varepsilon$, hence

$$\begin{aligned}I_\nu(w_n) &< \frac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2}\eta_n^2 - \bar{\nu}(\bar{B}-\varepsilon)\int_0^{\sigma(T)} F(t,w_n(t))\Delta t \\ &= \frac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2}\eta_n^2(1-\bar{\nu}(\bar{B}-\varepsilon)).\end{aligned}$$

Since $1 - \bar{\nu}(\bar{B}-\varepsilon) < 0$, and by considering (46), one has

$$\lim_{n\to+\infty} I_{\bar{\nu}}(w_n) = 0.$$

If $\bar{B} = +\infty$, fix $N_0 > \frac{1}{\bar{\nu}}$. There exists ν_{N_0} such that

$$\int_0^{\sigma(T)} F(t,\eta_n)\Delta t > N_0 \frac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2}\eta_n^2$$

for all $n > \nu_{N_0}$, and moreover,

$$I_{\bar{\nu}}(w_n) < \frac{\int_0^{\sigma(T)} q(t)\Delta t + \beta_1 p(0) + \beta_2 p(\sigma(T))}{2} \eta_n^2 (1 - \bar{\nu} N_0).$$

Since $1 - \bar{\nu} N_0 < 0$, and as above, we can say

$$\lim_{n \to +\infty} I_{\bar{\nu}}(w_n) = 0.$$

Since $I_{\bar{\nu}} = 0$, this implies that the functional $I_{\bar{\nu}}$ does not have a local minimum at zero. Hence, part (c) of Theorem 26 ensures that there exists a sequence $\{u_n\}$ in $H^1_\Delta([0, \sigma^2(T)])$ of critical points of $I_{\bar{\nu}}$ such that $\|u_n\| \to 0$ as $n \to \infty$, and the proof is complete. □

6. Conclusion

Optimization problem consists of maximizing or minimizing a real function. Optimization problems are ubiquitous in the mathematical modeling of real world systems and cover a very broad range of applications. These applications arise in economics and finance. Global optimization has its focus on finding the maximum or minimum over all input values. On the other hand, the calculus on time scales is a powerful tool to unify discrete and continuous analysis and is also applicable to any field in which dynamic processes can be described with discrete or continuous models, such as economics and finance. In this thesis, we have searched for the existence of local minima for the Euler functionals corresponding to a adynamic Sturm–Liouville boundary value problem on time scale, and we have proved the existence of one, three and a sequence of solutions for the problems employing variational methods and critical point theory. We have presented examples to illustrate the abstract results.

References

[1] Ghasem A. Afrouzi, Martin Bohner, Giuseppe Caristi, Shapour Heidarkhani, and Shahin Moradi. An existence result for impulsive multi-point boundary value systems using a local minimization principle. *J. Optim. Theory Appl.*, 177(1):1–20, 2018.

[2] Ghasem A. Afrouzi, Armin Hadjian, and Giovanni Molica Bisci. Some remarks for one-dimensional mean curvature problems through a local minimization principle. *Adv. Nonlinear Anal.*, 2(4):427–441, 2013.

[3] Ravi P. Agarwal, Martin Bohner, and Patricia J. Y. Wong. Sturm-Liouville eigenvalue problems on time scales. *Appl. Math. Comput.*, 99(2-3):153–166, 1999.

[4] Ravi P. Agarwal, Victoria Otero-Espinar, Kanishka Perera, and Dolores R. Vivero. Basic properties of Sobolev's spaces on time scales. *Adv. Difference Equ.*, Art. ID 38121, 14 pages, 2006.

[5] Ravi P. Agarwal, Victoria Otero-Espinar, Kanishka Perera, and Dolores R. Vivero. Existence of multiple positive solutions for second order nonlinear dynamic BVPs by variational methods. *J. Math. Anal. Appl.*, 331(2):1263–1274, 2007.

[6] Ravi P. Agarwal, Victoria Otero-Espinar, Kanishka Perera, and Dolores R. Vivero. Multiple positive solutions of singular Dirichlet problems on time scales via variational methods. *Nonlinear Anal.*, 67(2):368–381, 2007.

[7] Calvin D. Ahlbrandt and Christina Morian. Partial differential equations on time scales. *J. Comput. Appl. Math.*, 141: 35–55, 2002.

[8] Ferhan M. Atici, Daniel C. Biles, and Alex Lebedinsky. An application of time scales to economics. *Math. Comput. Modelling*, 43(7-8):718–726, 2006.

[9] Martin Bohner. Calculus of variations on time scales. *Dynam. Systems Appl.*, 13(3-4):339–349, 2004.

[10] David Barilla, Martin Bohner, Shapour Heidarkhani, Shahin Moradi. Existence results for dynamic SturmLiouville boundary value problems via variational methods. *Applied Mathematics and Computation.*, 2020. To appear.

[11] Martin Bohner, Giuseppe Caristi, Shapour Heidarkhani, and Farahnaz Ayazi. A critical point approach for a second-order dynamic Sturm–Liouville boundary value problem with p-Laplacian. *Appl. Math. Comput.*, 2020. To appear.

[12] Martin Bohner, Giuseppe Caristi, Shapour Heidarkhani, and Shahin Moradi. A critical point approach to boundary value problems on the real line. *Appl. Math. Lett.*, 76:215–220, 2018.

[13] Martin Bohner, Giuseppe Caristi, Shapour Heidarkhani, and Shahin Moradi. Existence of at least one homoclinic solution for a nonlinear second-order difference equation. *Int. J. Nonlinear Sci. Numer. Simul.*, 20(3-4):433–439, 2019.

[14] Martin Bohner, Gregory Gelles, and Julius Heim. Multiplier-accelerator models on time scales. *Int. J. Stat. Econ.*, 4(S10):1–12, 2010.

[15] Martin Bohner and Gregory M. Gelles. Risk aversion and risk vulnerability in the continuous and discrete case. *Decis. Econ. Finance*, 35(1):1–28, 2012.

[16] Martin Bohner and Svetlin G. Georgiev. Multiple integration on time scales. *Dyn. Syst. Appl.*, 14: (2005) 579-606.

[17] Martin Bohner and Svetlin G. Georgiev. *Multivariable Dynamic Calculus on Time Scales,* Springer, (2016).

[18] Martin Bohner and Svetlin G. Georgiev. Partial differentiation on time scales. *Dyn. Syst. Appl.*, **13** (2004) 351-379.

[19] Martin Bohner, Julius Heim, and Ailian Liu. Qualitative analysis of a Solow model on time scales. *J. Concr. Appl. Math.*, 13(3-4):183–197, 2015.

[20] Martin Bohner, Shapour Heidarkhani, Amjad Salari, and Giuseppe Caristi. Existence of three solutions for impulsive multi-point boundary value problems. *Opuscula Math.*, 37(3):353–379, 2017.

[21] Martin Bohner and Allan Peterson. *Dynamic equations on time scales.* Birkhäuser Boston, Inc., Boston, MA, 2001. An introduction with applications.

[22] Martin Bohner and Allan Peterson. *Advances in dynamic equations on time scales.* Birkhäuser Boston, Inc., Boston, MA, 2003.

[23] Gabriele Bonanno and Pasquale Candito. Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities. *J. Differential Equations*, 244(12):3031–3059, 2008.

[24] Gabriele Bonanno and Beatrice Di Bella. A boundary value problem for fourth-order elastic beam equations. *J. Math. Anal. Appl.*, 343(2):1166–1176, 2008.

[25] Gabriele Bonanno and Giovanni Molica Bisci. Infinitely many solutions for a boundary value problem with discontinuous nonlinearities. *Bound. Value Probl.*, Art. ID 670675, 20 pages, 2009.

[26] Jifeng Chu, Shapour Heidarkhani, Amjad Salari, and Giuseppe Caristi. Weak solutions and energy estimates for singular p-Laplacian type equations.*J. Dyn. Control Syst.*, **24** (2018) 51-63.

[27] S. Dubey, S. Prajapat, R. Verma, R. Jhaggar, *Solution of differential equations by parallel processing and analysis of performance improvement,* International Journal of Scientific Research in Computer Science and Engineering **5** (2017) 57-62.

[28] E. Çetin, F.S. Topal, *Symmetric positive solutions of fourth order boundary value problems for an increasing homeomorphism and homomorphism on time scales,* Appl. Math. Lett. **24** (2011) 87-92.

[29] Giuseppina D'Aguì, Shapour Heidarkhani, Angela Sciammetta. *Infinitely many solutions for a class of quasilinear two point boundary value problems,* Electron. J. Qual. Theory Differ. Equ. 2015, No. 8, 1-15.

[30] Jonathan Eckhardt and Gerald Teschl. Sturm-Liouville operators on time scales. *J. Difference Equ. Appl.*, 18(11):1875–1887, 2012.

[31] Rui A. C. Ferreira, Agnieszka B. Malinowska, and Delfim F. M. Torres. Optimality conditions for the calculus of variations with higher-order delta derivatives. *Appl. Math. Lett.*, 24(1):87–92, 2011.

[32] Giancarlo Gandolfo.Economic dynamics. *North-Holland Publishing Co.*, 1997.

[33] Marek Galewski and Giovanni Molica Bisci. Existence results for one-dimensional fractional equations. *Math. Methods Appl. Sci.*, 39(6):1480–1492, 2016.

[34] John R. Graef, Shapour Heidarkhani, and Lingju Kong. *Infinitely many solutions for problems of multi point boundary value equations,* Topol. Meth. Nonl. Anal. **42** (2013) 105-118.

[35] Gusein Sh. Guseinov. An expansion theorem for a Sturm-Liouville operator on semi-unbounded time scales. *Adv. Dyn. Syst. Appl.*, 3(1):147–160, 2008.

[36] Gusein Sh. Guseinov. Eigenfunction expansions for a Sturm-Liouville problem on time scales. *Int. J. Difference Equ.*, 2(1):93–104, 2007.

[37] Małgorzata Guzowska, Agnieszka B. Malinowska, and Moulay Rchid Sidi Ammi. Calculus of variations on time scales: applications to economic models. *Adv. Difference Equ.*, 2015:203, 15 pages, 2015.

[38] Shapour Heidarkhani, Ghasem A. Afrouzi, Giuseppe Caristi, Johnny Henderson, and Shahin Moradi. A variational approach to difference equations. *J. Difference Equ. Appl.*, 22(12):1761–1776, 2016.

[39] Shapour Heidarkhani, Ghasem A. Afrouzi, Shahin Moradi, and Giuseppe Caristi. Existence of three solutions for multi-point boundary value problems. *J. Nonlinear Funct. Anal.*, 2017:Art. ID 47, 1–19, 2017.

[40] Shapour Heidarkhani, Anderson L. A. De Araujo, Ghasem A. Afrouzi, and Shahin Moradi. Multiple solutions for Kirchhoff-type problems with variable exponent and nonhomogeneous Neumann conditions. *Math. Nachr.*, 291(2-3):326–342, 2018.

[41] Shapour Heidarkhani, Shahin Moradi, and Giuseppe Caristi. Existence results for a dynamic SturmLiouville boundary value problem on time scales. *Optim. Lett.*, (2020). https://doi.org/10.1007/s11590-020-01646-4

[42] Shapour Heidarkhani, Shahin Moradi, and Stepan A. Tersian. Three solutions for second-order boundary-value problems with variable exponents. *Electron. J. Qual. Theory Differ. Equ.*, Paper No. 33, 19 pages, 2018.

[43] Shapour Heidarkhani, Martin Bohner, Giuseppe Caristi, and Farahnaz Ayazi. A critical point approach for a second order dynamic Sturm–Liouville boundary value problem with p-Laplacian. *Applied Mathematics and Computation*, 2020, 125521.

[44] S. Heidarkhani, J. Henderson, *Infinitely many solutions for nonlocal elliptic problems of* $(p_1,\dots,p_n)$*-Kirchhoff type,* Electron. J. Differential Equations 2012, No. 69, 1–15.

[45] Shapour Heidarkhani, Yulin Zhao, Giuseppe Caristi, Ghasem A. Afrouzi, and Shahin Moradi. Infinitely many solutions for perturbed impulsive fractional differential problems. *Appl. Anal.* **96** (2017) 1401-1424.

[46] Shapour Heidarkhani, Ghasem A. Afrouzi, Massimiliano Ferrara, Giuseppe Caristi, and Shahin Moradi. Existence results for impulsive damped vibration systems. *Bull. Malays. Math. Sci. Soc.*, 41(3):1409–1428, 2018.

[47] Shapour Heidarkhani, Ghasem A. Afrouzi, Shahin Moradi, Giuseppe Caristi, and Bin Ge. Existence of one weak solution for $p(x)$-biharmonic equations with Navier boundary conditions. *Z. Angew. Math. Phys.*, 67(3):Art. 73, 13, 2016.

[48] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[49] Roman Hilscher and Vera Zeidan. Weak maximum principle and accessory problem for control problems on time scales. *Nonlinear Anal.*, 70(9):3209–3226, 2009.

[50] Adil Huseynov. Eigenfunction expansion associated with the one-dimensional Schrödinger equation on semi-infinite time scale intervals. *Rep. Math. Phys.*, 66(2):207–235, 2010.

[51] Adil Huseynov. Limit point and limit circle cases for dynamic equations on time scales. *Hacet. J. Math. Stat.*, 39(3):379–392, 2010.

[52] Joan Hoffacker, *Basic partial dynamic equations on time scales,* J. Differ. Equ. Appl. **8** (2002) 307-319.

[53] B. Jackson, *Partial dynamic equations on time scales,* J. Comput. Appl. Math. **186** (2006) 391-415.

[54] Lingju Kong. Existence of solutions to boundary value problems arising from the fractional advection dispersion equation. *Electron. J. Differential Equations*, No. 106, 15 pages, 2013.

[55] Agnieszka B. Malinowska and Delfim F. M. Torres. Natural boundary conditions in the calculus of variations. *Math. Methods Appl. Sci.*, 33(14):1712–1722, 2010.

[56] Chikhaoui Oussama, Gressier Jérémie, and Grondin Gilles. *Spectral volume method: application to Euler equations and performance appraisal,* In: Fifth European Conference on Computational Fluid Dynamics–ECCOMAS CFD 2010, 14-17 Jun 2010, Lisbon, Portugal.

[57] A. Sinan Ozkan. Sturm-Liouville operator with parameter-dependent boundary conditions on time scales. *Electron. J. Differential Equations*, Paper No. 212, 10 pages, 2017.

[58] Paul H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[59] Biagio Ricceri. A general variational principle and some of its applications. *J. Comput. Appl. Math.*, 113(1-2):401–410, 2000. Fixed point theory with applications in nonlinear analysis.

[60] W.C. Rheinboldt, B. Simeon, *Performance analysis of some methods for solving Euler–Lagrange equations,* Appl. Math. Lett. **8** (1995) 77-82.

[61] You-Hui Su and Zhaosheng Feng. Variational approach for a p-Laplacian boundary value problem on time scales. *Appl. Anal.*, 97(13):2269–2287, 2018.

[62] Hong-Rui Sun and Wan-Tong Li. Positive solutions for nonlinear three-point boundary value problems on time scales. *J. Math. Anal. Appl.*, 299(2):508–524, 2004.

[63] Phollakrit Thiramanus and Jessada Tariboon. Positive solutions of m-point integral boundary value problems for second order p-Laplacian dynamic equations on time scales. *Adv. Difference Equ.* **2013** (2013) 206.

[64] Eberhard Zeidler. *Nonlinear functional analysis and its applications. II/B*. Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.

[65] Quan-Guo Zhang, Xi-Ping He, and Hong-Rui Sun. Positive solutions for Sturm-Liouville BVPs on time scales via sub-supersolution and variational methods. *Bound. Value Probl.*, 2013:123, 12 pages, 2013.

[66] Quan-Guo Zhang and Hong-Rui Sun. Variational approach for Sturm-Liouville boundary value problems on time scales. *J. Appl. Math. Comput.*, 36(1-2):219–232, 2011.

[67] Jianwen Zhou and Yongkun Li. Sobolev's spaces on time scales and its applications to a class of second order Hamiltonian systems on time scales. *Nonlinear Anal.*, 73(5):1375–1388, 2010.

[68] Jianwen Zhou and Yongkun Li. Variational approach to a class of p-Laplacian systems on time scales. *Adv. Difference Equ.*, 2013:297, 16 pages, 2013.

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