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Commutativity and Completeness Degrees of Weakly Complete Hypergroups

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Abstract: We introduce a family of hypergroups, called weakly complete, generalizing the construction of complete hypergroups. Starting from a given group G , our construction prescribes the β -classes of the hypergroups and allows some hyperproducts not to be complete parts, based on a suitably defined relation over G . The commutativity degree of weakly complete hypergroups can be related to that of the underlying group. Furthermore, in analogy to the degree of commutativity, we introduce the degree of completeness of finite hypergroups and analyze this degree for weakly complete hypergroups in terms of their β -classes.

Keywords: hypergroups; complete hypergroup; fundamental relations

MSC: 20N20



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1. Introduction

We refer to hypercompositional algebra as the branch of algebra concerned with hypercompositional structures, that is, algebraic structures where the composition of two elements is a nonempty set rather than a single element [1]. Although hypercompositional algebra differs from classic algebra in its subjects, methods, and goals, the two fields are connected by certain equivalence relations, called fundamental relations [2,3]. Through the fundamental relations, hypercompositional algebra can make use of the wealth of tools typical of traditional algebra.

A fundamental relation is the smallest equivalence relation defined on a hypercompositional structure such that the corresponding quotient is a classic structure whose operational properties are analogous to those of the original structure [4,5]. For example, the quotient of a hypergroup modulo the equivalence β is isomorphic to a group [6–8]. On the other hand, given a group G and a family $\mathfrak{F} = \{A_k\}_{k \in G}$ of nonempty and pairwise disjoint sets, the set $H = \bigcup_{k \in G} A_k$ equipped with the hyperproduct $x \circ y = A_{ij}$, for all $x \in A_i$ and $y \in A_j$, is a hypergroup. Hypergroups built in this way are called complete [4] and have the property that the β -classes are the sets A_k . For any nonempty subset A of a hypergroup (H, \circ) , the set $\mathfrak{C}(A) = \bigcup_{a \in A} \beta(a)$ is the complete closure of A . Hence, a hypergroup (H, \circ) is complete if and only if $x \circ y = \mathfrak{C}(x \circ y)$, for all $x, y \in H$. Complete hypergroups have been the subject of many studies, see, e.g., [9–12], because they have a variety of group-like properties. Notably, in [13], the authors define the commutativity degree of complete hypergroups and characterize it with an identity that is analogous to the class equation for groups. Recall that the commutativity degree of a finite group G was defined by W. Gustafson in [14] as the probability that two randomly chosen elements commute,

$$d(G) = \frac{|\{(x, y) \in G^2 \mid xy = yx\}|}{|G|^2}.$$

Inspired by this concept, in [13] the commutativity degree of a finite hypergroup (H, \circ) is defined as

$$d(H) = \frac{|\{(x, y) \in H^2 \mid x \circ y = y \circ x\}|}{|H|^2}.$$

The probabilistic interpretation of this number is completely analogous to that for groups. In this paper, we define the completeness degree of a finite hypergroup (H, \circ) as the number

$$\Delta(H) = \frac{|\{(x, y) \in H^2 \mid \mathfrak{C}(x \circ y) = x \circ y\}|}{|H|^2},$$

and determine some formulas which allow us to compute the previous numbers for a special class of hypergroups, called weakly complete, that include complete hypergroups.

The plan of this paper is the following: In Section 2, we introduce definitions, notations, and fundamental facts to be used throughout the paper. In Section 3, we give the definition of product-free relations on a group G and study their main characteristics. In particular, we characterize product-free relations that are maximal with respect to inclusion. In Section 4, we present a new construction of hypergroups. These hypergroups are called weakly complete and are defined using a product-free relation I on a group G , a family $\{A_k : k \in G\}$ of nonempty and pairwise disjoint sets and a special family of functions $\{\varphi_{i,j} : (i, j) \in I\}$. The main features of these hypergroups are discussed in this section. The completeness degree $\Delta(H)$ of finite weakly complete hypergroups is defined and analyzed in Section 5. There, we prove lower bounds for $\Delta(H)$ that depend only on the size of the β -classes of H . Finally, in Section 6, we discuss the commutativity degree $d(H)$ of finite weakly complete hypergroups, and establish relations between $d(H)$ and $\Delta(H)$. In particular, in our last theorem we prove that, if the cardinality of A_k does not depend on k , then $|d(H) - \Delta(H)| \leq \frac{1}{4}$.

2. Basic Definitions and Notations

We adopt from known textbooks [1,4,5] standard definitions of basic concepts in hypercompositional algebra, such as semihypergroups and hypergroups. For the reader's convenience, we present below a few concepts that are needed in this work.

Given a semihypergroup (H, \circ) , the relation $\beta \subseteq H \times H$ is defined as $\beta = \cup_{n \geq 1} \beta_n$, where β_1 is the diagonal relation in H and, for every integer $n > 1$, β_n is defined as follows:

$$x \beta_n y \iff \exists z_1, \dots, z_n \in H : \{x, y\} \subseteq z_1 \circ z_2 \circ \dots \circ z_n, \tag{1}$$

see, e.g., [2,3]. This relation is one of the main fundamental relations alluded to in the Introduction. For some special families of semihypergroups, β is transitive; see, e.g., [15,16]. In particular, if (H, \circ) is a hypergroup then β is an equivalence relation, see [7,8], and we have the chain of inclusions

$$\beta_1 \subseteq \beta_2 \subseteq \beta_3 \subseteq \dots \subseteq \beta_n \dots \tag{2}$$

Moreover, the quotient set H/β equipped with the operation $\beta(x) \otimes \beta(y) = \beta(z)$ for all $x, y \in H$ and $z \in x \circ y$, is a group. More precisely, β is the smallest strongly regular equivalence on H such that the quotient H/β is a group [2]. The canonical epimorphism $\pi : H \mapsto H/\beta$ fulfills the identity $\pi(x \circ y) = \pi(x) \otimes \pi(y)$ for all $x, y \in H$, and the kernel $\omega_H = \pi^{-1}(1_{H/\beta})$ of π is the *heart* of (H, \circ) .

Let (H, \circ) be a hypergroup. We say that a nonempty subset $A \subseteq H$ is a complete part if for every $n \geq 1$ and $x_1, x_2, \dots, x_n \in H$,

$$(x_1 \circ x_2 \circ \dots \circ x_n) \cap A \neq \emptyset \implies x_1 \circ x_2 \circ \dots \circ x_n \subseteq A.$$

The complete closure of A is the intersection of all complete parts containing A and is denoted with $\mathfrak{C}(A)$. Using the canonical projection $\pi : H \mapsto H/\beta^*$, the complete closure of A can be characterized as follows:

$$\mathfrak{C}(A) = \pi^{-1}(\pi(A)) = A \circ \omega_H = \omega_H \circ A.$$

A hypergroup (H, \circ) is complete if $x \circ y = \mathfrak{C}(x \circ y)$ for all $x, y \in H$. In other words, (H, \circ) is a complete hypergroup if $x \circ y = \mathfrak{C}(a) = \beta(a)$ for every $(x, y) \in H^2$ and $a \in x \circ y$.

Finally, let G be a group and let $I \subseteq G \times G$ be a binary relation on G . We denote I^T the transpose relation of $I \subseteq G \times G$, that is, $(a, b) \in I \Leftrightarrow (b, a) \in I^T$. Furthermore, we associate with I the span and support sets defined below:

$$\begin{aligned} \text{Span}(I) &= \{ij : (i, j) \in I\}, \\ \text{Supp}(I) &= \{i \in G^* : \exists j \in G^* : (i, j) \in I \text{ or } (j, i) \in I\}. \end{aligned}$$

Here and in the following, G^* denotes the set $G \setminus \{1_G\}$.

3. Product-Free Relations on a Group

The class of complete hypergroups is among the best known in hypergroup theory, and is characterized by the fact that the hyperproduct of any two elements is a β -class. These hypergroups were introduced by P. Corsini in [4] and can be built by considering a group G and a family $\mathfrak{F} = \{A_k\}_{k \in G}$ of nonempty and pairwise disjoint sets. The set $H = \bigcup_{k \in G} A_k$ is endowed with the product $x \circ y = A_{ij}$ for $x \in A_i$ and $y \in A_j$. Then, (H, \circ) is a complete hypergroup and the β -classes of (H, \circ) are the sets A_k . In this section, we introduce a special family of binary relations in a group G . These relations will allow us to define in the next section the class of hypergroups that generalize that of complete hypergroups and is the main subject of this work.

Definition 1. Let G be a group. A binary relation $I \subseteq G \times G$ is called *product-free* or *PF-relation* if, for all $i, j, k \in G$,

$$(i, j) \in I \implies (ij, k) \notin I \text{ and } (k, ij) \notin I.$$

PF_G denotes the family of all PF-relations in the group G . If $I \in \text{PF}_G$ and $(i, j) \in I$, then the elements i, j are different from 1_G . Otherwise, if, for example, $i = 1_G$, then we have the contradiction $(i, j) = (i, 1_G j) = (i, ij) \notin I$. As a consequence, if $|G| = 1$, then PF_G reduces to the empty relation. Hence, if $I \neq \emptyset$, then $|G| \geq 2$ and $I \subseteq G^* \times G^*$.

Our first result provides a characterization of PF-relations in terms of support and span sets. Subsequently, we analyze the structure of PF_G and provide some examples.

Lemma 1. Let $I \subseteq G \times G$. Then, $I \in \text{PF}_G$ if and only if $\text{Supp}(I) \cap \text{Span}(I) = \emptyset$.

Proof. If $x \in \text{Supp}(I) \cap \text{Span}(I)$, then $(x, y) \in I$ or $(y, x) \in I$ for some $y \in G^*$ and there exists $(i, j) \in I$ such that $ij = x$. We obtain $(ij, y) \in I$ or $(y, ij) \in I$, a contradiction. Conversely, if $I \notin \text{PF}_G$, then there exists $(i, j) \in I$ and $y \in G$ such that $(ij, y) \in I$ or $(y, ij) \in I$. However, then we have $ij \in \text{Supp}(I) \cap \text{Span}(I)$. \square

Below we provide a couple of examples of how PF-relations can be built.

Example 1. Let G be a group. For any subset $S \subseteq G^*$, let I_S be the relation

$$I_S = \{(i, j) : i, j \in S, ij \notin S\}.$$

It can be seen that $\text{Supp}(I_S) \subseteq S$ and $\text{Span}(I_S) \subseteq G \setminus S$. Hence, $I_S \in \text{PF}_G$ by Lemma 1. For example, I_{G^*} is the relation consisting of the pairs (x, x^{-1}) for $x \in G^*$. On the other hand, I_{1_G} is the empty relation.

Example 2. Let G and G' be groups. Moreover, let $I \in \text{PF}_G$ and $I' \in \text{PF}_{G'}$. Then, the direct product relation

$$I \otimes I' = \{((a, a'), (b, b')) \mid (a, b) \in I, (a', b') \in I'\}$$

is a PF-relation on the direct product $G \times G'$. Indeed, $\text{Supp}(I \otimes I') = \text{Supp}(I) \times \text{Supp}(I')$ and $\text{Span}(I \otimes I') \subseteq \text{Span}(I) \times \text{Span}(I')$, so the claim follows from Lemma 1.

The following features of PF-relations are self-evident, so we refrain from including a proof.

- Every subset of a PF-relation is a PF-relation.
- If $I_1, I_2 \in \text{PF}_G$, then $I_1 \cap I_2 \in \text{PF}_G$.
- Let G be abelian. Then, $I \in \text{PF}_G$ if and only if $I^T \in \text{PF}_G$.

Hereafter, we show that no PF-relation can contain more than a quarter of all possible pairs of elements in the group. This result will play an important role in the forthcoming sections.

Theorem 1. Let G be a finite group and $I \in \text{PF}_G$. Then, $|I| \leq |G|^2/4$.

Proof. For notational simplicity, let $S = \text{Supp}(I)$. For any element $i \in S$, let $\mathcal{S}(i) = \{j \in G : (i, j) \in I\}$ and $\mathcal{R}(i) = \{ij : j \in \mathcal{S}(i)\}$. Obviously, $\mathcal{S}(i)$ and $\mathcal{R}(i)$ have the same cardinality, since the application $f_i : \mathcal{S}(i) \rightarrow \mathcal{R}(i)$ such that $f_i(j) = ij$ is bijective. Since $\mathcal{R}(i) \subseteq G \setminus S$, we have

$$|\mathcal{R}(i)| \leq |G \setminus S| = |G| - |S|.$$

Moreover,

$$|I| = \left| \bigcup_{i \in S} \mathcal{S}(i) \right| \leq \sum_{i \in S} |\mathcal{S}(i)| = \sum_{i \in S} |\mathcal{R}(i)| \leq |S|(|G| - |S|).$$

To maximize the rightmost quantity, we set $|S| = |G|/2$, and we have the claim. \square

The following example shows that the inequality in the preceding theorem is the best possible, since it can hold as an equality.

Example 3. Let $G = (\mathbb{Z}_m, +)$, where $m \geq 2$ is even. Consider the following relation $I \subset G \times G$:

$$(i, j) \in I \iff i \equiv j \equiv 1 \pmod{2}.$$

It is easy to see that $\text{Span}(I) = \{i \in \mathbb{Z}_m : i \equiv 0 \pmod{2}\}$ and $\text{Supp}(I) = \{i \in \mathbb{Z}_m : i \equiv 1 \pmod{2}\}$. Hence, $I \in \text{PF}_G$ by Lemma 1. Finally, $|\text{Span}(I)| = |\text{Supp}(I)| = m/2$ and $|I| = |G|^2/4$.

Maximal PF-Relations

PF-relations can be semi-ordered by inclusion; hence, it is worth considering maximal elements in PF_G , with regard to their existence and characterization. The existence of maximal relations is shown in the forthcoming result.

Proposition 1. The family PF_G of PF-relations on G has at least one maximal element.

Proof. The family PF_G is nonempty because it contains the empty relation. Moreover, for each chain $\{R_j\}_{j \in J}$ in the partially ordered set (PF_G, \subseteq) , the relation $\widehat{R} = \cup_{j \in J} R_j$ is product free. Indeed, if $(x, y) \in \widehat{R}$ and by chance there exists $z \in G$ such that $(xy, z) \in \widehat{R}$, then there exist $j_1, j_2 \in J$ such that $(x, y) \in R_{j_1}$ and $(xy, z) \in R_{j_2}$. Since $\{R_j\}_{j \in J}$ is a chain, we can assume that $R_{j_1} \subset R_{j_2}$, and so $\{(x, y), (xy, z)\} \subseteq R_{j_2}$, which is impossible because $R_{j_2} \in \text{PF}_G$. Hence, \widehat{R} is an upper bound of $\{R_j\}_{j \in J}$. By Zorn's Lemma, in PF_G there exists a maximal element. \square

Using an argument similar to the previous one, we also have that every PF-relation I on a group G is contained in a maximal PF-relation M . It suffices to apply Zorn’s lemma to the family of PF-relations that contain I . Hence, we have the following result:

Proposition 2. *Let $I \in PF_G$. Then, there exists a maximal PF-relation $M \in PF_G$ such that $I \subseteq M$.*

Remark 1. *Every maximal PF-relation M in an abelian group G is symmetric. Indeed, if $(x, y) \in M$ and $(y, x) \notin M$, then $M \cup \{(y, x)\}$ is a PF-relation and $M \subset M \cup \{(y, x)\}$. The same fact is not true if the group is not abelian, as shown in the following example. Let G be a noncommutative group with two elements $a, b \in G - \{1_G\}$ such that $ab \neq 1_G, a^2 = 1_G$ and $ab \neq ba$, e.g., the symmetric group S_3 . In these hypotheses, $a \neq b$ and the relation $I = \{(a, b), (a, ba)\}$ are product free. If $M \in PF_G$ is maximal and $I \subseteq M$, then we have $(b, a) \notin M$ since $(a, ba) \in M$.*

The empty relation is maximal if and only if G is trivial. In the next result, we give a necessary and sufficient condition for a PF-relation to be maximal.

Theorem 2. *Let G be a group and let $I \in PF_G$. Moreover, let*

$$\bar{I} = \{(x, y) : xy \in \text{Supp}(I) \text{ or } \{x, y\} \cap \text{Span}(I) \neq \emptyset\}.$$

Then, we have

1. $I \cap \bar{I} = \emptyset$;
2. I is maximal if and only if $I \cup \bar{I} = G^* \times G^*$.

Proof. If $I = \emptyset$ then the claim is trivial, so suppose $I \neq \emptyset$. Note that \bar{I} admits the alternative definition

$$\bar{I} = \{(x, y) \in G^* \times G^* \mid \exists (i, j) \in I : xy \in \{i, j\} \text{ or } ij \in \{x, y\}\}.$$

1. Let $(x, y) \in I \cap \bar{I}$. By hypothesis, there exists $(i, j) \in I$ such that $xy \in \{i, j\}$ or $ij \in \{x, y\}$. If $xy = i$ (resp., $xy = j$), then $(xy, j) \in I$ (resp., $(i, xy) \in I$), which contradicts $(x, y) \in I$. Similarly, if $ij = x$ (resp., $ij = y$) then $(ij, y) \in I$ (resp., $(x, ij) \in I$), which contradicts $(i, j) \in I$.

2. By point 1, if $I \cup \bar{I} = G^* \times G^*$, then I is maximal. On the other hand, let I be maximal and $(x, y) \in G^* \times G^*$ with $(x, y) \notin I$. Since $I \cup \{(x, y)\}$ is not a PF-relation, two cases are possible:

- (a) There exist $(i, j) \in I$ and $k \in G^*$ such that $(x, y) = (ij, k)$ or $(x, y) = (k, ij)$.
- (b) There exists $k \in G^*$ such that $(xy, k) \in I$ or $(k, xy) \in I$.

In the first case, we obtain $x = ij$ or $y = ij$; hence, $(x, y) \in \bar{I}$. In the second case, we have $(x, y) \in \bar{I}$ because $xy \in \{xy, k\}$. In both cases, we obtain $I \cup \bar{I} = G^* \times G^*$. \square

Remark 2. *We observe that if I and I' are maximal PF-relations, then the tensor product relation $I \otimes I'$ is not necessarily maximal. For example, let $G = \{1_G, a\}$ and $G' = \{1_{G'}, a', b'\}$ be groups isomorphic to $(\mathbb{Z}_2, +)$ and $(\mathbb{Z}_3, +)$, respectively. Moreover, let $I = \{(a, a)\} \subset G \times G$ and $I' = \{(a', b'), (b', a')\} \subset G' \times G'$. The relations I and I' are maximal PF-relations. However, the tensor product relation $I \otimes I' = \{((a, a'), (a, b')), ((a, b'), (a, a'))\}$ is not maximal because it is contained in the following PF-relation on $G \times G'$:*

$$T = I \otimes I' \cup \{((1_G, a'), (1_G, a'))\}.$$

4. Weakly Complete Hypergroups

In this section, we introduce a new class of hypergroups, whose construction is fundamentally based on PF-relations. We introduce a few auxiliary concepts and notations

for background information. In what follows, we denote $\mathcal{P}^*(X)$ the collection of nonempty subsets of the set X .

Definition 2. Let A, B, C be nonempty sets. A function $\varphi : A \times B \mapsto \mathcal{P}^*(C)$ is a double covering, or bi-covering for short, if for all $a \in A$ and $b \in B$ we have

$$\bigcup_{x \in B} \varphi(a, x) = \bigcup_{x \in A} \varphi(x, b) = C. \tag{3}$$

A bi-covering $\varphi : A \times B \mapsto \mathcal{P}^*(C)$ is called trivial if $\varphi(a, b) = C$ for all $a \in A$ and $b \in B$, and proper if $\varphi(a, b) \subsetneq C$ for all $a \in A$ and $b \in B$.

Example 4. Bi-covering functions can be constructed by considering a group G and three nonempty sets A, B, C of size $\geq |G|$. If $\alpha : A \rightarrow G, \beta : B \rightarrow G$ and $\gamma : C \rightarrow G$ are three surjective functions; then, the function $\varphi : A \times B \rightarrow \mathcal{P}^*(C)$ such that $\varphi(a, b) = \gamma^{-1}(\alpha(a)\beta(b))$, for all $(a, b) \in A \times B$, is bi-covering. Indeed, we trivially have $\bigcup_{x \in B} \varphi(a, x) \subseteq C$, for all $a \in A$. Moreover, if $c \in C$, then, taking $b \in \beta^{-1}(\alpha(a)^{-1}\gamma(c))$, we have $\beta(b) = \alpha(a)^{-1}\gamma(c)$ and we obtain

$$c \in \gamma^{-1}(\gamma(c)) = \gamma^{-1}(\alpha(a)\beta(b)) = \varphi(a, b) \subseteq \bigcup_{x \in B} \varphi(a, x).$$

Hence, $\bigcup_{x \in B} \varphi(a, x) = C$ for all $a \in A$. Analogous arguments prove that $\bigcup_{x \in A} \varphi(x, b) = C$, for all $b \in B$. Thus, φ is a bi-covering. We note in passing that in the previous construction the role of the group G can be played by an arbitrary hypergroup.

Let G be a group and let I be a relation on G . Consider a family $\mathfrak{A} = \{A_k\}_{k \in G}$ of nonempty and pairwise disjoint sets, and let $\mathfrak{I} = \{\varphi_{i,j}\}_{(i,j) \in I}$ be a family of bi-coverings $\varphi_{i,j} : A_i \times A_j \mapsto \mathcal{P}^*(A_{ij})$. In particular, if $I = \emptyset$, then $\mathfrak{I} = \emptyset$. In the set, $H = \bigcup_{k \in G} A_k$ introduce the hyperproduct $\circ : H \times H \mapsto \mathcal{P}^*(H)$, defined as follows:

$$x \circ y = \begin{cases} A_{ij} & \text{if } x \in A_i, y \in A_j \text{ and } (i, j) \notin I \\ \varphi_{i,j}(x, y) & \text{if } x \in A_i, y \in A_j \text{ and } (i, j) \in I \end{cases} \tag{4}$$

for all $x, y \in H$. This hyperproduct is well defined because the sets in the family $\mathfrak{A} = \{A_k\}_{k \in G}$ are nonempty and pairwise disjoint. The hyperproduct is naturally extended to nonempty subsets of H as usual: For $X, Y \in \mathcal{P}^*(H)$ let

$$x \circ Y = \bigcup_{y \in Y} x \circ y, \quad X \circ y = \bigcup_{x \in X} x \circ y, \quad X \circ Y = \bigcup_{x \in X, y \in Y} x \circ y.$$

In particular, for every $i, j \in G$ and $x \in A_j$, we have

$$A_i \circ x = A_{ij}, \quad x \circ A_i = A_{ji}. \tag{5}$$

Indeed, if $(i, j) \notin I$ then $A_i \circ x = \bigcup_{y \in A_i} y \circ x = A_{ij}$. Otherwise, if $(i, j) \in I$, then from (3) we obtain $A_i \circ x = \bigcup_{y \in A_i} \varphi_{i,j}(y, x) = A_{ij}$. Analogously we can deduce that $x \circ A_i = A_{ji}$. From this observation, it is not difficult to derive that if $I = \emptyset$ or all functions $\varphi_{i,j}$ are trivial; for every $(i, j) \in I$, then (H, \circ) is a complete hypergroup. The following result shows that (H, \circ) is always a hypergroup under the sole condition that $I \in \text{PF}_G$.

Theorem 3. Let $I \in \text{PF}_G$. Then, in the previous notations,

(a) for every $i, j, k \in G, x \in A_i, y \in A_j$ and $z \in A_k$, we have

$$(x \circ y) \circ z = A_{ijk} = x \circ (y \circ z);$$

(b) for every integer $n \geq 3$ and for every $z_1, z_2, \dots, z_n \in H$ there exists $i \in G$ such that $z_1 \circ z_2 \circ \dots \circ z_n = A_i$;

(c) (H, \circ) is a hypergroup such that $\beta = \beta_2$;

Proof. (a) Let $i, j, k \in G, x \in A_i, y \in A_j$ and $z \in A_k$. If $(i, j) \notin I$ and $(j, k) \notin I$, then we have $x \circ y = A_{ij}, y \circ z = A_{jk}$. Consequently, by (5) we obtain

$$(x \circ y) \circ z = A_{ij} \circ z = A_{(ij)k} = A_{i(jk)} = x \circ A_{jk} = x \circ (y \circ z).$$

If $(i, j) \in I$ and $(j, k) \notin I$, we have $(ij, k) \notin I, x \circ y = \varphi_{i,j}(x, y) \subseteq A_{ij}$ and $y \circ z = A_{jk}$. Moreover, for every $a \in A_{ij}$ we have $a \circ z = A_{(ij)k}$. Hence,

$$(x \circ y) \circ z = \bigcup_{a \in \varphi_{i,j}(x,y)} a \circ z = A_{(ij)k}.$$

Moreover, by (5), we obtain $x \circ (y \circ z) = x \circ A_{jk} = A_{i(jk)}$. Therefore, $(x \circ y) \circ z = x \circ (y \circ z)$. We obtain same result also when $(i, j) \notin I$ and $(j, k) \in I$. Finally, if $(i, j) \in I$ and $(j, k) \in I$, we have $x \circ y = \varphi_{i,j}(x, y) \subseteq A_{ij}$ and $y \circ z = \varphi_{j,k}(y, z) \subseteq A_{jk}$. Since I is product free, we have $(ij, k) \notin I$ and $(i, jk) \notin I$. Thus,

$$\begin{aligned} (x \circ y) \circ z &= \bigcup_{a \in \varphi_{i,j}(x,y)} a \circ z = A_{(ij)k}, \\ x \circ (y \circ z) &= \bigcup_{b \in \varphi_{j,k}(y,z)} x \circ b = A_{i(jk)}. \end{aligned}$$

Hence, also in this case $(x \circ y) \circ z = x \circ (y \circ z) = A_{ijk}$.

(b) It suffices to apply (5) and the previous part a) and proceed by induction on n .

(c) To prove that (H, \circ) is a hypergroup, we only need to show that \circ is reproducible. Let $x \in H$ and $x \in A_i$. Clearly, $iG = G$ for all $i \in G$ and, by Equation (5), we obtain

$$x \circ H = x \circ \left(\bigcup_{j \in G} A_j \right) = \bigcup_{j \in G} x \circ A_j = \bigcup_{j \in G} A_{ij} = H.$$

The identity $H \circ x = H$ follows analogously for every $x \in H$, so (H, \circ) is a hypergroup. Finally, let $x\beta y$. By (2), there exists $n \geq 3$ such that $x\beta_n y$. By point b), there exists $i \in G$ such that $\{x, y\} \subseteq A_i$. Now, let $a \in A_{1_G}$. Since $(i, 1_G) \notin I$, by (4) we have $\{x, y\} \subseteq A_i = x \circ a$ and we deduce $x\beta_2 y$. \square

Example 5. Let G be a group and let $I \subset G \times G$ be a relation on G . Consider a family $\mathfrak{A} = \{A_k\}_{k \in G}$ of nonempty and pairwise disjoint sets such that $|A_k| \geq |G|$, for all $k \in G$. Moreover, let $\{f_k : A_k \rightarrow G\}_{k \in G}$ be a family of surjective functions. Proceeding as in Example 4, we obtain a family of bi-covering functions $\mathfrak{J} = \{\varphi_{i,j} : A_i \times A_j \rightarrow \mathcal{P}^*(A_{ij})\}_{(i,j) \in I}$. If $I \in PF_G$, then Theorem 3 provides a hypergroup (H, \circ) .

Remark 3. Product-free relations have a kind of optimality with respect to the rule (4). As shown in Theorem 3, every hyperproduct defined in terms of a PF-relation is associative and reproducible, independent of families $\mathfrak{A} = \{A_k\}_{k \in G}$ and $\mathfrak{J} = \{\varphi_{i,j}\}_{(i,j) \in I}$. The same property does not hold in general if the relation I is not a PF-relation. For example, consider the group $(\mathbb{Z}_3, +)$, the relation $I = \{(1, 1), (2, 2)\}$, the sets $A_0 = \{a, b\}, A_1 = \{c, d, e\}, A_2 = \{f, g, h\}$ and the bi-coverings $\varphi_{1,1} : A_1 \times A_1 \mapsto \mathcal{P}^*(A_2), \varphi_{2,2}, \varphi'_{2,2} : A_2 \times A_2 \mapsto \mathcal{P}^*(A_1)$ defined as follows:

$\varphi_{1,1}$	c	d	e	$\varphi_{2,2}$	f	g	h	$\varphi'_{2,2}$	f	g	h
c	A_2	A_2	f, g	f	A_1	A_1	c, d	f	A_1	A_1	c
d	A_2	f, g	A_2	g	d, e	A_1	A_1	g	d, e	A_1	A_1
e	A_2	A_2	f, g	h	d, e	A_1	d, e	h	d, e	A_1	d, e

Considering the functions $\varphi_{1,1}$ and $\varphi_{2,2}$, definition (4) returns the following hypergroup:

\circ_1	a	b	c	d	e	f	g	h
a	A_0	A_0	A_1	A_1	A_1	A_2	A_2	A_2
b	A_0	A_0	A_1	A_1	A_1	A_2	A_2	A_2
c	A_1	A_1	A_2	A_2	f, g	A_0	A_0	A_0
d	A_1	A_1	A_2	f, g	A_2	A_0	A_0	A_0
e	A_1	A_1	A_2	A_2	f, g	A_0	A_0	A_0
f	A_2	A_2	A_0	A_0	A_0	A_1	A_1	c, d
g	A_2	A_2	A_0	A_0	A_0	d, e	A_1	A_1
h	A_2	A_2	A_0	A_0	A_0	d, e	A_1	d, e

On the other hand, considering the functions $\varphi_{1,1}$ and $\varphi'_{2,2}$, we have the hyperproduct

\circ_1	a	b	c	d	e	f	g	h
a	A_0	A_0	A_1	A_1	A_1	A_2	A_2	A_2
b	A_0	A_0	A_1	A_1	A_1	A_2	A_2	A_2
c	A_1	A_1	A_2	A_2	f, g	A_0	A_0	A_0
d	A_1	A_1	A_2	f, g	A_2	A_0	A_0	A_0
e	A_1	A_1	A_2	A_2	f, g	A_0	A_0	A_0
f	A_2	A_2	A_0	A_0	A_0	A_1	A_1	c
g	A_2	A_2	A_0	A_0	A_0	d, e	A_1	A_1
h	A_2	A_2	A_0	A_0	A_0	d, e	A_1	d, e

which is not associative since $(f \circ_2 h) \circ_2 e \neq f \circ_2 (h \circ_2 e)$. This example reveals a specific quality of PF-relations: If a hyperproduct defined as in (4) is associative and reducible, independent of families $\mathfrak{F} = \{A_k\}_{k \in G}$ and $\mathfrak{I} = \{\varphi_{i,j}\}_{(i,j) \in I}$, then the relation I is product free. This fact is formalized in the following result.

Theorem 4. Let G be a group and suppose that $I \subset G \times G$ is not product free. Then, there exists a family $\mathfrak{F} = \{A_k\}_{k \in G}$ of nonempty and pairwise disjoint sets and there exists a family of bi-coverings $\mathfrak{I} = \{\varphi_{i,j}\}_{(i,j) \in I}$ such that the hyperproduct defined in (4) is not associative.

Proof. Firstly, note that we have $I \neq \emptyset$ as $I \notin \text{PF}_G$. The proof can be reduced to the analysis of two cases: (a) there exists $(i, j) \in I$ such that $1_G \in \{i, j\}$; and (b) there exists $i, j, k \in G^*$ such that $(i, j) \in I$ and $(ij, k) \in I$ (or, equivalently, $(k, ij) \in I$).

(a) If $i = j = 1_G$ then it suffices to consider arbitrary families \mathfrak{F} and \mathfrak{I} where $A_{1_G} = \{a, b\}$ and the function $\varphi_{1_G, 1_G}$ is described by the following table:

$\varphi_{1_G, 1_G}$	a	b
a	b	a
b	a, b	a, b

Then, associativity fails because $(a \circ a) \circ a = \{a, b\} \neq \{a\} = a \circ (a \circ a)$. Otherwise, without loss of generality, assume $j = 1_G$ and $(1_G, 1_G) \notin I$. Let \mathfrak{F} and \mathfrak{I} verify the following conditions: $|A_\ell| = 2$ for every $\ell \in G$ and $|\varphi_{p,q}(x, y)| = 1$ for all $(p, q) \in I$. Let $x \in A_i$ and $y, z \in A_{1_G}$. Then,

$$(x \circ y) \circ z = \varphi_{i, 1_G}(x, y) \circ z = \varphi_{i, 1_G}(\varphi_{i, 1_G}(x, y), z).$$

Hence, $|(x \circ y) \circ z| = 1$. On the other hand, $x \circ (y \circ z) = x \circ A_{1_G} = A_i$; hence $(x \circ y) \circ z \neq x \circ (y \circ z)$.

(b) Let $\mathfrak{F} = \{A_\ell : \ell \in G\}$ and $\mathfrak{I} = \{\varphi_{p,q} : (p, q) \in I\}$ be arbitrary families verifying the following conditions: (b1) $|A_\ell| = 2$ for every $\ell \in G$; (b2) if $(i, jk) \in I$ then $\varphi_{i, jk}(x, y) = A_{ijk}$ for every $x \in A_i$ and $y \in A_{jk}$; $|\varphi_{p,q}(x, y)| = 1$ in all remaining cases. Let $x \in A_i$, $y \in A_j$, and $z \in A_k$. Then,

$$(x \circ y) \circ z = \varphi_{i, j}(x, y) \circ z = \varphi_{ij, k}(\varphi_{i, j}(x, y), z).$$

Since $1_G \notin \{i, j, k\}$, then $(i, j) \neq (i, jk)$ and $(ij, k) \neq (i, jk)$. Hence, $|(x \circ y) \circ z| = 1$ by (b2). On the other hand, for some $w \in y \circ z \subseteq A_{jk}$, we have

$$A_{ijk} = x \circ w \subseteq x \circ (y \circ z).$$

By (b1) we can conclude that $(x \circ y) \circ z \neq x \circ (y \circ z)$. (The proof proceeds in a similar way if $(k, ij) \in I$). \square

Definition 3. The hypergroups (H, \circ) defined as in (4) with a PF-relation I are called weakly complete. A weakly complete hypergroup is n -uniform if $|A_i| = n$ for all $i \in G$; if the size n is not relevant, then we simply call it uniform.

The term "weakly complete" originates from the following observations: Let (H, \circ) be a weakly complete hypergroup built from families $\mathfrak{F} = \{A_k\}_{k \in G}$ and $\mathfrak{J} = \{\varphi_{i,j}\}_{(i,j) \in I}$, and let \diamond be the hyperproduct obtained from the same set family \mathfrak{F} using only trivial bi-coverings. Then, (H, \diamond) is a complete hypergroup and $x \circ y \subseteq x \diamond y$ for all $x, y \in H$. We also obtain the same conclusion by replacing the given relation I with the empty relation. Furthermore, both in complete hypergroups and weakly complete hypergroups, the fundamental relation β coincides with β_2 , as shown in Theorem 3.

In the following, we use the notation $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ to indicate a weakly complete hypergroup whose hyperproduct \circ is defined as (4) from $I \in \text{PF}_G$ and the families $\mathfrak{F} = \{A_k\}_{k \in G}$ and $\mathfrak{J} = \{\varphi_{i,j}\}_{(i,j) \in I}$. We call $\mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ a representation of (H, \circ) . It is worth noting that a weakly complete hypergroup may have multiple representations. Indeed, let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ and let $(i, j) \notin I$. If the relation $\tilde{I} = I \cup \{(i, j)\}$ is product free, then the same hypergroup admits the representation $\mathcal{W}(G, \tilde{I}, \mathfrak{F}, \tilde{\mathfrak{J}})$ where $\tilde{\mathfrak{J}} = \mathfrak{J} \cup \{\varphi_{i,j}\}$ and $\varphi_{i,j}(x, y) = A_{ij}$ for every $x \in A_i$ and $y \in A_j$. However, all possible representations of a given weakly complete hypergroup share the same group G and family \mathfrak{F} . This fact should be evident from the following proposition, where we explain the algebraic role of the parameters of a representation of a weakly complete hypergroup.

Proposition 3. Let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$. Then, we have:

1. The sets $A_i \in \mathfrak{F}$ are the β -classes of H , i.e, for every $x \in H$, $x \in A_i \Leftrightarrow \beta(x) = A_i$.
2. $H/\beta \simeq G$ and $\omega_H = A_{1_G}$.
3. Every subhypergroup K of (H, \circ) is a complete part of H , that is, $\mathfrak{C}(K) = K$.
4. A subset $K \subseteq H$ is a subhypergroup of (H, \circ) if and only if there exists a subgroup G' of G such that $K = \bigcup_{i \in G'} A_i$.

Proof. 1. Let $x \in A_k$ and $a \in A_{1_G}$. Then, $A_k = x \circ a$, and so $y \in A_k$ implies $y\beta_2x$. Conversely, if $y\beta_2x$, then there exist $a, b \in H$ such that $\{x, y\} \subseteq a \circ b$. By construction, there exists $r \in G$ such that $a \circ b \subseteq A_r$. Therefore, since $x \in A_k \cap A_r$ and the sets of the family \mathfrak{F} are pairwise disjoint, we obtain $y \in a \circ b \subseteq A_r = A_k$. Hence, $y \in A_k$ if and only if $y\beta_2x$. By Theorem 3, we conclude $A_k = \beta(x)$.

2. The map $f : G \mapsto H/\beta$ such that $f(k) = A_k$, for every $k \in G$, is a group isomorphism. Moreover, we have $\omega_H = A_{1_G}$ since $1_{H/\beta} = f(1_G) = A_{1_G}$.

3. We must prove that $\beta(x) \subseteq K$, for all $x \in K$. By reproducibility of K , if $x \in K$ then there exists $u \in K$ such that $x \in x \circ u$. Considering the canonical epimorphism $\pi : H \mapsto H/\beta$, we obtain $\pi(x) = \pi(x) \otimes \pi(u)$ and so $\pi(u) = 1_{H/\beta}$. Hence, from point 2., we have $u \in \pi^{-1}(1_{H/\beta}) = \omega_H = A_{1_G}$. Consequently, $\omega_H = A_{1_G} = u \circ u \subseteq K$ and $\beta(x) = x \circ \omega_H \subseteq K \circ K = K$, for all $x \in K$.

4. Since $iG' = G' = G'i$, for all $i \in G'$, the proof of the implication \Leftarrow is similar to the one used in point 3. of Theorem 3 to prove that (H, \circ) is a hypergroup. Now, we prove the implication \Rightarrow . By point 1., the β -classes of (H, \circ) are the sets A_i , for all $i \in G$. Let $\pi : H \mapsto H/\beta$ be the canonical epimorphism and $f : H/\beta \mapsto G$ be the isomorphism such that $f(A_i) = i$, for all $i \in G$. If K is a subhypergroup of (H, \circ) , then $G' = (f \circ \pi)(K)$ is a subgroup of G . Moreover, if $x \in K$ then there exists $i \in G$ such that $x \in A_i$. By point 1.,

we have $A_i = \beta(x)$ and $i = f(A_i) = f(\pi(x)) = (f \circ \pi)(x) \in G'$. Hence, $K \subseteq \bigcup_{i \in G'} A_i$. On the other hand, if $x \in \bigcup_{i \in G'} A_i$, there exists $i \in G'$ such that $x \in A_i$. Clearly, there exists $y \in K$ such that $(f \circ \pi)(y) = i$. If we suppose that $y \in A_j$, then we have $A_j = \beta(y)$ and $i = (f \circ \pi)(y) = f(\pi(y)) = f(A_j) = j$. Finally, by point 3., $x \in A_i = A_j = \beta(y) \subseteq K$. Therefore, $\bigcup_{i \in G'} A_i \subseteq K$. \square

The following result, which follows from the definition of hyperproduct in (4) and point 1 in Proposition 3, describes all cases where a weakly complete hypergroup is complete.

Corollary 1. *Let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$.*

1. *If $I = \emptyset$, then (H, \circ) is complete;*
2. *if $I \neq \emptyset$, then (H, \circ) is complete $\iff \varphi_{i,j}$ is trivial, for every $(i, j) \in I$.*

Example 6. *Let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ such that $|A_k| > 1$ for some $k \in G$ and $|A_i| = 1$ for $i \neq k$. Then, (H, \circ) is complete, as a consequence of the previous corollary. Indeed, if $(i, j) \in I$ and $ij \neq k$, then $|A_{ij}| = 1$ and $\varphi_{i,j}$ is trivial. On the other hand, if $k = ij$ then $k \notin \{i, j\}$ because I is product free. Thus, $|A_i| = |A_j| = 1$ and $\varphi_{i,j}$ are trivial since it is a bi-covering.*

The next example shows a weakly complete hypergroup that contains both complete and noncomplete subhypergroups.

Example 7. *Let $G = \{1, 2, 3, 4\}$ be a group isomorphic to the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ where $1 = 1_G$. Consider $I = \{(2, 2), (3, 3)\}$, $A_1 = \{a, b\}$, $A_2 = \{c, d\}$, $A_3 = \{e, f\}$ and $A_4 = \{g\}$. In the set $H = \{a, b, c, d, e, f, g, h\}$, define the hyperproduct represented in the following table:*

\circ	a	b	c	d	e	f	g
a	A_1	A_1	A_2	A_2	A_3	A_3	A_4
b	A_1	A_1	A_2	A_2	A_3	A_3	A_4
c	A_2	A_2	a	A_1	A_4	A_4	A_3
d	A_2	A_2	b	a	A_4	A_4	A_3
e	A_3	A_3	A_4	A_4	b	a	A_2
f	A_3	A_3	A_4	A_4	a	b	A_2
g	A_4	A_4	A_3	A_3	A_2	A_2	A_1

Then, (H, \circ) is a weakly complete hypergroup. The subsets $K_1 = A_1 \cup A_2$, $K_2 = A_1 \cup A_3$, $K_3 = A_1 \cup A_4$ are a subhypergroup of (H, \circ) . Moreover, K_3 is complete and K_1 and K_2 are not complete.

The next theorem characterizes weakly complete hypergroups, in that it yields a necessary and sufficient condition for a given hypergroup to be weakly complete, based on the structure of its quotient group.

Theorem 5. *Let (H, \circ) be a hypergroup, and let $\pi : H \mapsto H/\beta$ be the canonical projection. Consider the following relation $J \subseteq H/\beta \times H/\beta$:*

$$J = \{(i, j) : \exists x \in \pi^{-1}(i), \exists y \in \pi^{-1}(j) : x \circ y \neq \mathfrak{C}(x \circ y)\}.$$

The following conditions are equivalent:

1. *J is product free;*
2. *(H, \circ) is a weakly complete hypergroup.*

Proof. Suppose that J is product free. For every $i \in H/\beta$, let $A_i = \pi^{-1}(i)$, and note that $\bigcup_i A_i = H$. For every $(i, j) \in J$

introduce the function $f_{i,j} : A_i \times A_j \mapsto A_{ij}$ such that $f_{i,j}(x, y) = x \circ y$. It is not difficult to see that $f_{i,j}$ is a bi-covering. Indeed, for any fixed $x \in A_i$ we have by construction

$$\begin{aligned} \bigcup_{y \in A_j} f_{i,j}(x, y) &= \bigcup_{y \in A_j} x \circ y = x \circ A_j \\ &= x \circ (y \circ \omega_H) \\ &= (x \circ y) \circ \omega_H = \mathfrak{C}(x \circ y) = A_{ij}. \end{aligned}$$

The identity $\bigcup_{x \in A_i} f_{i,j}(x, y) = A_{ij}$ can be derived analogously, so $f_{i,j}$ is a bi-covering. It remains to observe that $(H, \circ) = \mathcal{W}(H/\beta, J, \{A_i\}, \{f_{i,j}\})$, and we have the first part of the claim.

Conversely, suppose that (H, \circ) is a weakly complete hypergroup, $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$. Identifying G with H/β modulo an isomorphism, we have $J \subseteq I$. Indeed, let $(i, j) \in J$. By hypothesis, there exist $x, y \in H$ such that $\pi(x) = i, \pi(y) = j$ and $x \circ y \neq \mathfrak{C}(x \circ y)$. Hence, $(i, j) \in I$ by (4). This conclusion follows immediately from the fact that a subset of a PF-relation is a PF-relation. \square

5. Completeness Degree of Finite Hypergroups

In this section, we introduce the notion of completeness degree of finite hypergroups and analyze the completeness degree of finite weakly complete hypergroups.

Definition 4. Let (H, \circ) be a finite hypergroup. Define the set $\mathcal{C}_H \subseteq H \times H$,

$$\mathcal{C}_H = \{(x, y) \in H \times H \mid \mathfrak{C}(x \circ y) = x \circ y\}.$$

The rational number

$$\Delta(H) = \frac{|\mathcal{C}_H|}{|H|^2}$$

is the completeness degree of (H, \circ) .

Thus, the completeness degree of a hypergroup is the probability that the hyperproduct of two randomly chosen elements is a β -class. Clearly, $\Delta(H) \in [0, 1]$ and $\Delta(H) = 1$ if and only if (H, \circ) is complete. In the next lemma, we deduce an explicit formula for the completeness degree of finite weakly complete hypergroups. For this purpose, we make use of the following auxiliary notation. Let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$. For every $i, j \in G$, let

$$\mathcal{C}_{i,j} = \{(x, y) \in A_i \times A_j \mid x \circ y = A_{ij}\}.$$

Lemma 2. Let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$. Then,

$$\Delta(H) = \frac{\sum_{(i,j) \notin I} |A_i||A_j| + \sum_{(i,j) \in I} |\mathcal{C}_{i,j}|}{|H|^2}. \tag{6}$$

Moreover, if (H, \circ) is uniform, then

$$\Delta(H) = 1 - \frac{|I|}{|G|^2} + \frac{\sum_{(i,j) \in I} |\mathcal{C}_{i,j}|}{|H|^2}. \tag{7}$$

Proof. Firstly, note that $\mathcal{C}_H = \bigcup_{i,j} \mathcal{C}_{i,j}$. From the definition of the hyperproduct \circ in (4), we deduce the alternative formula

$$\mathcal{C}_{i,j} = \begin{cases} A_i \times A_j & \text{if } (i, j) \notin I \\ \{(x, y) \in A_i \times A_j \mid \varphi_{i,j}(x, y) = A_{ij}\} & \text{if } (i, j) \in I \end{cases}$$

hence,

$$\mathcal{C}_H = \left(\bigcup_{(i,j) \notin I} A_i \times A_j \right) \cup \left(\bigcup_{(i,j) \in I} \mathcal{C}_{i,j} \right).$$

Recalling that the sets of the family \mathfrak{F} are pairwise disjoint, we obtain

$$\begin{aligned} |\mathcal{C}_H| &= \left| \bigcup_{(i,j) \notin I} A_i \times A_j \right| + \left| \bigcup_{(i,j) \in I} \mathcal{C}_{i,j} \right| \\ &= \sum_{(i,j) \notin I} |A_i| |A_j| + \sum_{(i,j) \in I} |\mathcal{C}_{i,j}|, \end{aligned}$$

and Equation (6) follows. Moreover, if (H, \circ) is n -uniform, then $|H| = n|G|$ and

$$\frac{\sum_{(i,j) \notin I} |A_i| |A_j|}{|H|^2} = \frac{\sum_{(i,j) \notin I} n^2}{|H|^2} = \frac{(|G|^2 - |I|)n^2}{|G|^2 n^2} = 1 - \frac{|I|}{|G|^2},$$

and we also obtain (7). \square

Our next result provides two lower bounds on $\Delta(H)$ that depend only on the size of the β -classes of H .

Theorem 6. *Let (H, \circ) be a finite weakly complete hypergroup. Then,*

$$\Delta(H) \geq \frac{|\omega_H|}{|H|} \left(2 - \frac{|\omega_H|}{|H|} \right),$$

where ω_H is the heart of H . Moreover, if (H, \circ) is uniform, then $\Delta(H) \geq \frac{3}{4}$.

Proof. Let $\{A_i\}_{i \in G}$ be the family of disjoint sets in the representation of (H, \circ) . Then,

$$\begin{aligned} |\mathcal{C}_H| &\geq \sum_{(i,j) \notin I} |A_i| |A_j| \\ &= \sum_{i,j=1}^{|G|} |A_i| |A_j| - \sum_{(i,j) \in I} |A_i| |A_j| \\ &= |H|^2 - \sum_{(i,j) \in I} |A_i| |A_j| \\ &\geq |H|^2 - (\sum_{i \in G^*} |A_i|)^2 \\ &= |H|^2 - (|H| - |A_{1_G}|)^2 = |A_{1_G}|(2|H| - |A_{1_G}|). \end{aligned}$$

Recalling that $A_{1_G} = \omega_H$ and using (6), we obtain the first inequality. Moreover, from (7) we have $\Delta(H) \geq 1 - |I|/|G|^2$; hence, the second part of the claim is an immediate consequence of Theorem 1. \square

The next example shows that the inequalities in Theorem 6 can hold as equalities.

Example 8. *Let $m \geq 2$ be an even number, and let G and I be the same as in Example 3. Let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ be any uniform weakly complete hypergroup such that $\mathcal{C}_{i,j} = \emptyset$ for all $(i, j) \in I$; i.e., all bi-coverings are proper. A straightforward application of Lemma 2 proves that $\Delta(H) = 3/4$. Moreover, if $m = 2$, then $|H| = 2n$ and $|\omega_H| = n$. Thus, also the first inequality in Theorem 6 holds as an equality.*

In the forthcoming example, we construct uniform weakly complete hypergroups where all bi-coverings are proper, that is, $\mathcal{C}_{i,j} = \emptyset$, for all $(i, j) \in I$. According to Lemma 2, these hypergroups achieve the smallest $\Delta(H)$ possible for a given PF-relation.

Example 9. Let G be a group and $I \in \text{PF}_G$. Let $\mathfrak{F} = \{A_k\}_{k \in G}$ be a family of finite, pairwise disjoint sets such that $|A_k| = n \geq 2$ for all $k \in G$. We assume $A_k = B_k \cup C_k$, with B_k, C_k nonempty disjoint sets. For every $(i, j) \in I$, let $\varphi_{i,j} : A_i \times A_j \mapsto A_{ij}$ be defined as follows:

$$\varphi_{i,j}(x, y) = \begin{cases} B_{ij} & \text{if } (x \in B_i \text{ and } y \in B_j) \text{ or } (x \in C_i \text{ and } y \in C_j) \\ C_{ij} & \text{else.} \end{cases}$$

It is not difficult to verify that $\varphi_{i,j}$ is a proper bi-covering. Moreover, the hypergroup $(H, \circ) = \mathcal{W}(G, I, \{A_i\}, \{\varphi_{i,j}\})$ is n -uniform. Owing to (7) and the finiteness of G , the completeness degree of (H, \circ) is

$$\Delta(H) = 1 - \frac{|I|}{|G|^2},$$

i.e., the smallest possible value for the given relation I .

6. Commutativity Degree of Weakly Complete Hypergroups

In a nonabelian group and, more generally, in any nonabelian algebraic structure, it makes sense to compute the probability that two randomly chosen elements commute. This problem was popularized by Gustafson in [14], who defined the commutativity degree $d(G)$ of a group G as the probability that two arbitrary elements commute,

$$d(G) = \frac{|\{(x, y) \in G^2 : xy = yx\}|}{|G|^2}, \tag{8}$$

and proved that if $d(G) > \frac{5}{8}$ then G is abelian. Moreover, we have $d(G) = \frac{5}{8}$ if and only if $G/Z(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, where $Z(G)$ is the center of G . The basic technique adopted for the proof relies on the relationship between $d(G)$ and the number of conjugacy classes of G , and can be traced back to a paper by Erdős and Turán [17]. Later on, there has been considerable interest in the use of probabilistic techniques in group theory, and this concept has had significant developments.

Recently, the concept of commutativity degree has been introduced also in hypergroup theory [13,18]. In particular, in [13] the authors defined the commutativity degree of a finite hypergroup (H, \circ) as

$$d(H) = \frac{|\{(x, y) \in H^2 : x \circ y = y \circ x\}|}{|H|^2} \tag{9}$$

and characterized this index when (H, \circ) is complete by considering a partitioning of H into suitably defined conjugacy classes. In this section, we study the commutativity degree of weakly complete hypergroups. Our main tool is the partitioning of H into β -classes. To begin with, we point out an important observation. For any $i, j \in G$ and for any $x \in A_i$ and $y \in A_j$, a necessary condition for the identity $x \circ y = y \circ x$ to be valid is $ij = ji$, because $x \circ y \subseteq A_{ij}$, $y \circ x \subseteq A_{ji}$ and $A_{ij} \cap A_{ji} = \emptyset$ if $ij \neq ji$. Hence, we can restrict our attention to pairs (i, j) belonging to the set

$$c(G) = \{(i, j) \in G \times G : ij = ji\}.$$

This set is directly related to the commutativity degree of G , since $d(G) = |c(G)|/|G|^2$.

Definition 5. We say that a relation $I \in \text{PF}_G$ is G -symmetric if its restriction to $c(G)$ is symmetric; that is, for every $i, j \in G$, if $ij = ji$ then $(i, j) \in I \iff (j, i) \in I$.

Equivalently, $I \in \text{PF}_G$ is G -symmetric if and only if $I \cap c(G) = I^T \cap c(G)$. It can be observed that if G is abelian then a relation in PF_G is G -symmetric if and only if it is symmetric. The relevance of the previous definition lies in the fact that every weakly

complete hypergroup admits a representation with a G -symmetric relation, as shown in the following lemma.

Lemma 3. *Let (H, \circ) be a weakly complete hypergroup. Then, there exists a representation $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ where I is G -symmetric.*

Proof. Let $(H, \circ) = \mathcal{W}(G, \widehat{I}, \mathfrak{F}, \widehat{\mathfrak{J}})$ be any representation of (H, \circ) . If \widehat{I} is G -symmetric, then it is complete. Otherwise, $\widehat{I} \cap c(G) \neq \widehat{I}^T \cap c(G)$ and we define the relation

$$I = \widehat{I} \cup (\widehat{I}^T \cap c(G)).$$

We have $I \cap c(G) = \widehat{I}^T \cap c(G)$, so I is G -symmetric, and I properly extends \widehat{I} . Moreover, from Lemma 1 we can deduce that $I \in \text{PF}_G$, because both the support and the span of I coincide with those of \widehat{I} .

For every $(i, j) \in I \setminus \widehat{I}$ let $\psi_{i,j} : A_i \times A_j \mapsto A_{ij}$ be the trivial bi-covering, and define $\mathfrak{J} = \widehat{\mathfrak{J}} \cup \{\psi_{i,j}\}$. To conclude the proof, it suffices to show that the hypergroup $(H, \diamond) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ coincides with (H, \circ) . Indeed, for arbitrary $x \in A_i$ and $y \in A_j$, if $(i, j) \in I \setminus \widehat{I}$, then

$$x \diamond y = \psi_{i,j}(x, y) = A_{ij} = x \circ y.$$

Otherwise, if either $(i, j) \in \widehat{I}$ or $(i, j) \notin I$ then the identity $x \diamond y = x \circ y$ follows trivially from the construction (4). We can conclude that $(H, \diamond) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$. \square

In what follows, we obtain different characterizations of the commutativity degree of a weakly complete hypergroup $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ in terms of the parameters of its representation. By virtue of Lemma 3, we can safely assume that I is G -symmetric. In this case, for every pair $(i, j) \in c(G) \cap I$ the sets

$$\mathcal{D}_{i,j} = \{(x, y) \in A_i \times A_j : \varphi_{i,j}(x, y) = \varphi_{j,i}(y, x)\} \tag{10}$$

$$\mathcal{E}_{i,j} = \{(x, y) \in A_i \times A_j : \varphi_{i,j}(x, y) \neq \varphi_{j,i}(y, x)\} \tag{11}$$

are well defined.

Theorem 7. *Let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$ where I is G -symmetric. Then,*

$$d(H) = \frac{\sum_{(i,j) \in c(G) \setminus I} |A_i| |A_j| + \sum_{(i,j) \in c(G) \cap I} |\mathcal{D}_{i,j}|}{|H|^2}. \tag{12}$$

Moreover, if (H, \circ) is uniform then

$$d(H) = d(G) - \frac{|c(G) \cap I|}{|G|^2} + \frac{\sum_{(i,j) \in c(G) \cap I} |\mathcal{D}_{i,j}|}{|H|^2} \tag{13}$$

$$= d(G) - \frac{\sum_{(i,j) \in c(G) \cap I} |\mathcal{E}_{i,j}|}{|H|^2}. \tag{14}$$

Proof. Let $i, j \in c(G)$, $x \in A_i$ and $y \in A_j$. Two cases are possible:

(a) $(i, j) \in c(G) \setminus I$. In this case, $x \circ y = A_{ij} = A_{ji} = y \circ x$; hence

$$\{(x, y) \in A_i \times A_j : x \circ y = y \circ x\} = A_i \times A_j.$$

(b) $(i, j) \in I \cap c(G)$. Owing to the G -symmetry of I , we have both $x \circ y = \varphi_{ij}(x, y)$ and $y \circ x = \varphi_{ji}(y, x)$. By (10),

$$\{(x, y) \in A_i \times A_j : x \circ y = y \circ x\} = \mathcal{D}_{i,j}.$$

The first claim follows from the fact that the set $c(G)$ is the disjoint union of $c(G) \setminus I$ and $I \cap c(G)$. Moreover, if $|A_i| = n$ for all $i \in G$, then

$$\sum_{(i,j) \in c(G) \setminus I} |A_i||A_j| = n^2|c(G) \setminus I| = n^2(|c(G)| - |c(G) \cap I|).$$

Since $|H| = n|G|$, we also have

$$\frac{\sum_{(h,k) \in c(G) \setminus I} |A_h||A_k|}{|H|^2} = \frac{|c(G)| - |c(G) \cap I|}{|G|^2} = d(G) - \frac{|c(G) \cap I|}{|G|^2},$$

and (13) follows. Finally, using (11) we obtain

$$\begin{aligned} \frac{|c(G) \cap I|}{|G|^2} - \frac{\sum_{(i,j) \in c(G) \cap I} |\mathcal{D}_{i,j}|}{|H|^2} &= \frac{\sum_{(i,j) \in c(G) \cap I} (n^2 - |\mathcal{D}_{i,j}|)}{|H|^2} \\ &= \frac{\sum_{(i,j) \in c(G) \cap I} |\mathcal{E}_{i,j}|}{|H|^2}, \end{aligned}$$

which yields (14), and the proof is complete. \square

The previous theorem yields a few notable consequences. For example, taking $I = \emptyset$ we conclude that if (H, \circ) is complete and

$$d(H) = \frac{\sum_{(i,j) \in c(G)} |A_i||A_j|}{|H|^2}.$$

In particular, if (H, \circ) is also uniform, then $d(H) = d(G)$. More generally, $d(H) \leq d(G)$ for any uniform weakly complete hypergroup, and the equality holds if and only if $\varphi_{i,j}(x, y) = \varphi_{j,i}(y, x)$ for every $(i, j) \in c(G) \cap I$.

Finally, the similarity between formulas (6) and (12) suggests that we should study the relationship between the degrees of commutativity and completeness, at least in the commutative case. We propose our result below. Before doing so, we recall that if G is abelian, then G -symmetric relations are symmetric. Hence, by Lemma 3, every weakly complete hypergroup built from an abelian group admits a representation whose PF-relation is symmetric.

Theorem 8. *Let G be abelian and let $(H, \circ) = \mathcal{W}(G, I, \mathfrak{F}, \mathfrak{J})$, where I is symmetric. Then,*

$$d(H) = \Delta(H) + \frac{\sum_{(i,j) \in I} (|\mathcal{D}_{i,j}| - |\mathcal{C}_{i,j}|)}{|H|^2}. \tag{15}$$

Moreover, if (H, \circ) is uniform then $|d(H) - \Delta(H)| \leq |I|/|G|^2 \leq \frac{1}{4}$.

Proof. Since G is abelian, we have $c(G) \cap I = I$ and the condition $(i, j) \in c(G) \setminus I$ reduces to $(i, j) \notin I$. Therefore, subtracting (13) from (6) we obtain (15). Furthermore, for every $(i, j) \in I$ we have $\mathcal{D}_{i,j} \cup \mathcal{C}_{i,j} \subseteq A_i \times A_j$. If H is n -uniform, then $|H| = n|G|$ and $|A_i \times A_j| = n^2$. Hence,

$$-n^2 \leq |\mathcal{D}_{i,j}| - |\mathcal{C}_{i,j}| \leq n^2.$$

Thus, $|d(H) - \Delta(H)| \leq n^2|I|/|H|^2 = |I|/|G|^2$. The rightmost inequality in the claim comes from Theorem 1. \square

7. Conclusions

The class of complete hypergroups is among the best known in hypergroup theory. Complete hypergroups have a variety of group-like properties and are characterized by the fact that the composition of two elements is a β -class [9–12]. In this paper, we introduce

a class of hypergroups (H, \circ) that includes complete hypergroups as a particular case. The construction of these hypergroups, called weakly complete, is crucially based on particular binary relations defined on the quotient group H/β . We call these relations product free because no group element is in relation with the product of two elements that are related to each other. Product-free relations are interesting by themselves, and we show a number of their main properties on generic groups in Section 2. For example, we prove an attainable upper bound on the cardinality of product-free relations in finite groups.

The main motivation of introducing weakly complete hypergroups lies in the possibility of measuring their “closeness” to complete hypergroups. Indeed, to every finite hypergroup, we can associate a completeness degree, which quantifies how close to completion the hypergroup is. We introduce and analyze this concept in Section 5. More precisely, the completeness degree of a hypergroup is the probability that the composition of two randomly chosen elements is a β -class. For a weakly complete hypergroup whose β -classes have the same cardinality, this probability is bounded from below by $\frac{3}{4}$. Indeed, the completeness degree of weakly complete hypergroups admits simple closed formulas. Furthermore, it can be related to the commutativity degree, which has been recently brought into hypercompositional algebra from group theory [13,18].

Completeness concepts and probabilistic methods are relevant topics nowadays not only in classical algebra but also in hypercompositional algebra, and this discipline is continually expanding with the introduction of structures with distinctive properties [19]. It would be interesting to discover more hypergroup classes, and more general hypercompositional structures, for which useful results can be found along these directions.

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