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IMPLICIT HIGHLY DISCONTINUOUS BOUNDARY VALUE PROBLEMS INVOLVING THE *P*-LAPLACIAN

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ABSTRACT. Let $n \in \mathbf{N}$, with $n \geq 2$, and let $p \in]n, +\infty[$. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded connected open set, with smooth boundary $\partial\Omega$, and let $Y \subseteq \mathbf{R}$ be a closed interval. We study the existence of solutions $u \in W_0^{1,p}(\Omega)$ of the implicit equation $\psi(-\Delta_p u) = f(x,u)$, where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ and $\psi : Y \to \mathbf{R}$ are two given functions. We establish some existence results where f is allowed to be highly discontinuous in both variables. In particular, a function $f(x,z)$ satisfying the assumptions of our results can be discontinuous, with respect to the second variable, even at all points $z \in \mathbf{R}$. As regard ψ, we only require that it is continuous and locally nonconstant.

1. Introduction

Let $n \geq 2$, and let $\Omega \subseteq \mathbf{R}^n$ be a nonempty bounded and connected open set, with smooth boundary $\partial\Omega$. Let $p \in]1, +\infty[$, and let $Y \subseteq \mathbf{R}$ be a closed interval. In this paper we are concerned with the boundary value problem

$$\begin{cases} \psi(-\Delta_p u) = f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $\psi : Y \to \mathbf{R}$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ are given functions and Δ_p is the p-Laplace operator. More precisely, we deal with the existence of functions $u \in W_0^{1,p}(\Omega)$ such that $\Delta_p u \in L^r(\Omega)$ (for suitable $r \in [1, +\infty]$) and

$$-\Delta_p u(x) \in Y \quad \text{and} \quad \psi(-\Delta_p u(x)) = f(x, u(x)) \quad \text{for a.e.} \quad x \in \Omega.$$

Up to our knowledge, there is not much literature on this implicit problem (see Cabada and Heikkilä 2002; Heikkilä and Seikkala 2005; Shah *et al.* 2018; Ahmad, Zada, and Alzabut 2019, and the references therein). A typical assumption on f, even in the particular explicit case where $\psi(t) = t$, is that f is a Carathéodory function (see, for instance, Chabrowski 1997; Peral 1997; Dinca, Jebelean, and Mawhin 2001; Carl, Le, and Motreanu 2007; Bonanno and Molica Bisci 2010, and the references therein). When one goes out from this assumption, assuming that f can admit some points of discontinuity in the second variable, the study of the problem (1) becomes more difficult. Very recently, Marino and Paratore (2021) have obtained some existence results by considering separately the case where f is a Carathéodory function and the case where f may admit, with respect to the second variable, points of discontinuity. It is interesting to observe that, in this latter case, it is required that the set of the discontinuity points of f (as a function of two variables) is a zero-measure set in $\Omega \times \mathbf{R}$, with a suitable geometry. In particular, when f does not depend on $x \in \Omega$, it is required that the set of the points of discontinuity of f has null measure in $\mathbf{R}$.

Our aim in this paper is to prove an existence result for the above problem (Theorem 3.1 below), where f can be highly discontinuous in both variables, admitting a set of discontinuity points significantly larger than in the results obtained by Marino and Paratore (2021). In particular, a function f satisfying the assumptions of Theorem 3.1 below can be discontinuous, with respect to the second variable, even at each point $z \in \mathbf{R}$. As regards the function ψ, we only require that it is continuous and locally nonconstant.

As a matter of fact, the kind of discontinuity allowed for f is the main peculiarity of our results. Therefore, we now briefly illustrate in detail the difference between the results of Marino and Paratore (2021) and our results. Firstly, let π_0 and π_1 be the projections of $\Omega \times \mathbf{R}$ over Ω and $\mathbf{R}$, respectively. That is, for each $(x,z) \in \Omega \times \mathbf{R}$, we put $\pi_0(x,z) = x$ and $\pi_1(x,z) = z$. Moreover, put

$$\mathscr{F}_\Omega := \left\{ A \subseteq \Omega \times \mathbf{R} : \ m_n(\pi_0(A)) = 0 \text{ or } m_1(\pi_1(A)) = 0 \right\},$$

where m_n and m_1 denote the n-dimensional and the 1-dimensional Lebesgue measure in $\mathbf{R}^n$ and $\mathbf{R}$, respectively. The continuity assumption required in the main result of Marino and Paratore (2021, Theorem 4.1) is as follows:

(i_1) f is essentially bounded and the set

$$D_f := \{(x,z) \in \Omega \times \mathbf{R} : \ f \text{ is discontinuous at } (x,z) \}$$

belongs to the family $\mathscr{F}_\Omega$.

Hence, the set of discontinuity points of f can be quite large (Marino and Paratore 2021), but in any case it must be a null-measure set in $\mathbf{R}^{n+1}$, with a suitable geometry. In particular, when f does not depend explicitly on $x \in \Omega$ (that is, $f : \mathbf{R} \to \mathbf{R}$), assumption (i_1) implies that the set

$$D_f^* := \{z \in \mathbf{R} : \ f \text{ is discontinuous at } z \}$$

has null Lebesgue measure.

In our main result and in its consequences, conversely, the regularity assumption on f is the following:

(i_2) there exists a set $E \subseteq \mathbf{R}$, with $m_1(E) = 0$, such that for all $z \in \mathbf{R} \setminus E$, the function $f(\cdot, z)$ is measurable, and for a.e. $x \in \Omega$ the function $f(x, \cdot)|_{\mathbf{R} \setminus E}$ is continuous.

It is immediate to check that if a function $f : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies condition (i_1), then it satisfies condition (i_2), while the converse is not true in general. Moreover, a function f satisfying (i_2) can be discontinuous, with respect to the second variable, even at each point

$z \in \mathbf{R}$. To see this, it is enough to consider the function $f : \Omega \times \mathbf{R} \to \mathbf{R}$ defined by

$$f(x,z) = \begin{cases} 1 & \text{if } x \in C \text{ and } z \in \mathbf{Q}, \\ 2 & \text{if } x \in C \text{ and } z \in \mathbf{R} \setminus \mathbf{Q}, \\ 3 & \text{if } x \in \Omega \setminus C \text{ and } z \in \mathbf{Q}, \\ 4 & \text{if } x \in \Omega \setminus C \text{ and } z \in \mathbf{R} \setminus \mathbf{Q}, \end{cases} \tag{2}$$

where $C \subseteq \Omega$ is any nonempty measurable set, with $C \neq \Omega$, and $\mathbf{Q}$ denotes the set of rational real numbers. It is immediate to check that such a function f satisfies condition (i_2), by taking $E = \mathbf{Q}$. Moreover, for each $x \in \Omega$, the function $f(x,\cdot)$ is discontinuous at all points $z \in \mathbf{R}$. Finally, we observe that such a function f does not satisfy condition (i_1), since in this case we have $D_f = \Omega \times \mathbf{R}$.

Similarly, in the particular case where f does not depend on $x \in \Omega$, a function $f : \mathbf{R} \to \mathbf{R}$ satisfying condition (i_2) can be discontinuous even at all points $z \in \mathbf{R}$. To see this, it suffices to take

$$f(z) = \begin{cases} 1 & \text{if } z \in \mathbf{Q}, \\ 2 & \text{if } z \in \mathbf{R} \setminus \mathbf{Q}. \end{cases} \tag{3}$$

As before, condition (i_2) is satisfied by taking $E = \mathbf{Q}$, and, of course, f is discontinuous at all points $z \in \mathbf{R}$. Hence, condition (i_1) is not satisfied since in this case we have $D_f^* = \mathbf{R}$. Even if our results allow a higher discontinuity for f, we point out that our results and the ones obtained by Marino and Paratore (2021) are formally independent. For a more detailed comparison between these results, we refer the reader to Remark **3.12** below.

In the proof of our results, the framework is that of set-valued analysis, and the main tools are a recent selection theorem (Theorem 2.1 below) and a result on differential inclusions by Marano (2012, Theorem 2.2). Our main result will be stated and proved in Section 3, together with some consequences and corollaries, while in Section 2 we shall fix some notations and give some preliminaries.

2. Preliminaries

In what follows, $n \geq 2$ is a natural number, and $\Omega \subseteq \mathbf{R}^n$ is a nonempty bounded and connected open set, with smooth boundary $\partial\Omega$. Moreover, $p \in]n, +\infty[$ is a fixed real number, and $p' = p/(p-1)$ is the conjugate exponent of p. For every $q \in [1, +\infty]$, we denote by $\|\cdot\|_{L^q(\Omega)}$ the usual norm of the space $L^q(\Omega)$. As usual, we denote by $W^{1,p}(\Omega)$ the space of all functions $u \in L^p(\Omega)$ whose weak derivatives $\frac{\partial u}{\partial x_i}$, with $i = 1, \ldots, n$, belong to $L^p(\Omega)$ (see Adams and Fournier 2003). The space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{n} \|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)}.$$

We also denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The space $W_0^{1,p}(\Omega)$ will be considered with the norm

$$\|u\|_* := \| \, |\nabla u| \, \|_{L^p(\Omega)} = \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p},$$

which, by the Poincaré inequality, is equivalent (on $W_0^{1,p}(\Omega)$) to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$ (Brezis 2011, see Corollary 9.19). We shall denote be $W^{-1,p'}(\Omega)$ the topological dual of the space $(W_0^{1,p}(\Omega),\|\cdot\|_*)$, with corresponding norm $\|\cdot\|_{W^{-1,p'}(\Omega)}$. As known, the p-Laplacian is the operator

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

A weak formulation of this operator consists in regarding Δ_p as an operator acting from $W_0^{1,p}(\Omega)$ into its dual space $W^{-1,p'}(\Omega)$, by

$$\langle \Delta_p u, w\rangle := -\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla w\,dx \quad \text{for all} \quad u,w\in W_0^{1,p}(\Omega) \tag{4}$$

(see, for instance, Dinca, Jebelean, and Mawhin 2001). In what follows, we assume (4) as the definition of p-Laplacian. Of course, if for some $u\in W_0^{1,p}(\Omega)$ the operator $\Delta_p u\in W^{-1,p'}(\Omega)$ admits a representation of the type

$$\langle \Delta_p u, w\rangle = \int_\Omega g w\,dx \quad \text{for all} \quad w\in W_0^{1,p}(\Omega),$$

then, as usual, we can identify $\Delta_p u$ with g.

Now, we observe that, by the Rellich-Kondrachov theorem (see Adams and Fournier 2003, Theorem 6.3), or even Theorem 9.16 and Remark 20 at p. 290 of Brezis (2011), the space $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^p(\Omega)$. Hence, by Theorem 6.4 of Brezis (2011), it follows that $L^{p'}(\Omega)$ is compactly imbedded in $W^{-1,p'}(\Omega)$. Consequently, there exists a constant $\lambda>0$ such that

$$\|v\|_{W^{-1,p'}(\Omega)} := \sup_{u\in W_0^{1,p}(\Omega),\|u\|_*\le 1}\left|\int_\Omega v(x)\,u(x)\,dx\right| \le \lambda\,\|v\|_{L^{p'}(\Omega)} \quad \text{for all} \quad v\in L^{p'}(\Omega). \tag{5}$$

Moreover, since $p>n$, we have that $W_0^{1,p}(\Omega)$ is continuously imbedded in $L^\infty(\Omega)$ (again, see Adams and Fournier 2003; Brezis 2011); hence, there exists $\sigma>0$ such that

$$\|u\|_{L^\infty(\Omega)} \le \sigma\,\|u\|_* \quad \text{for all} \quad v\in W_0^{1,p}(\Omega). \tag{6}$$

Explicit estimates of the constants λ and σ have been obtained by Marano (2012) and Talenti (1987) (see Remark 3.4 below for more details). In the case where Ω is convex, the constant σ has been estimated in Theorem 1 of Burenkov and Gusakov (1987).

In the following, "measurable function" and "measurable set" will mean "Lebesgue measurable function" and "Lebesgue measurable set", respectively. Moreover, if $k\in\mathbf{N}$, we shall denote by m_k the k-dimensional Lebesgue measure in $\mathbf{R}^k$. If $A\subseteq\mathbf{R}^k$ is a Lebesgue measurable set, we shall denote by $\mathscr{L}(A)$ the family of all Lebesgue measurable subsets of A. Finally, for any set $A\subseteq\mathbf{R}^k$, we shall denote by $\overline{\operatorname{conv}}(A)$ the closed convex hull of the set A.

For what concerns the basic definitions and facts on multifunctions, we refer to Denkowski, Migórski, and Papageorgiou (2003) and to Klein and Thompson (1984). As regards measurable multifunctions, we also refer the reader to Himmelberg (1975). Here, we only recall that if X is a topological space and $(T,\mathscr{G})$ is a measurable space, then we say that a multifunction $F : T \to 2^X$ is $\mathscr{G}$-*measurable* (resp., $\mathscr{G}$-*weakly measurable*) in T if for any closed (resp., open) set $V \subseteq X$ one has

$$F^-(V) := \{x \in T :\ F(x) \cap V \neq \emptyset\} \in \mathscr{G}$$

(see Himmelberg 1975). If X is a metric space, then $\mathscr{G}$-measurability implies $\mathscr{G}$-weak measurability. If F has closed values and X is a σ-compact and separable metric space, then the two notions of measurability are equivalent (see Himmelberg 1975, Theorem 3.5). If X is a topological space, we denote by $\mathscr{B}(X)$ the Borel family of X. For what concerns the definition and the properties of Souslin sets, we refer to Chapter 6 of Bogachev (2007). For the reader's convenience, we now briefly recall some results that will be fundamental in the sequel. We begin with the following selection result, where $\mathscr{T}_\mu$ denotes the completion of $\mathscr{B}(T)$ with respect to the measure μ.

Theorem 2.1. (Cubiotti and Yao 2015, Theorem 2.1). *Let T and $X_1, X_2, \ldots X_k$ be complete separable metric spaces, with $k \in \mathbf{N}$, and let $X := \prod_{j=1}^k X_j$ (endowed with the product topology). Let $\mu, \psi_1, \ldots, \psi_k$ be positive regular Borel measures over $T, X_1, X_2, \ldots X_k$, respectively, with μ finite and $\psi_1, \ldots, \psi_k$ σ-finite.*

Let S be a separable metric space, $W \subseteq X$ a Souslin set, and let $F : T \times W \to 2^S$ be a multifunction with nonempty complete values. Let $E \subseteq W$ be a given set. Finally, for all $i \in \{1, \ldots, k\}$, let $P_{*,i} : X \to X_i$ be the projection over X_i. Assume that:

(i) *the multifunction F is $\mathscr{T}_\mu \otimes \mathscr{B}(W)$-weakly measurable;*
(ii) *for a.e. $t \in T$, one has*

$$\big\{x = (x_1, \ldots, x_k) \in W : F(t,\cdot) \text{ is not lower semicontinuous at } x\big\} \subseteq E.$$

Then, there exist sets $Q_1, \ldots, Q_k$, with $Q_i \in \mathscr{B}(X_i)$ and $\psi_i(Q_i) = 0$ for all $i = 1, \ldots, k$, and a function $\phi : T \times W \to S$ such that:

(a) *$\phi(t,x) \in F(t,x)$ for all $(t,x) \in T \times W$;*
(b) *for all $x := (x_1, x_2, \ldots, x_k) \in W \setminus \Big[\big(\bigcup_{i=1}^k P_{*,i}^{-1}(Q_i)\big) \cup E\Big]$, the function $\phi(\cdot, x)$ is $\mathscr{T}_\mu$-measurable over T;*
(c) *for a.e. $t \in T$, one has*

$$\big\{x = (x_1, x_2, \ldots, x_k) \in W : \phi(t,\cdot) \text{ is discontinuous at } x\big\} \subseteq$$
$$\subseteq E \cup \Big[W \cap \Big(\bigcup_{i=1}^k P_{*,i}^{-1}(Q_i)\Big)\Big].$$

The following result will be also a key tool in the sequel.

Theorem 2.2. (Marano 2012, Theorem 2.2). *Let U be a nonempty set, and let $\Phi : U \to W_0^{1,p}(\Omega)$ and $\Psi : U \to L^{p'}(\Omega)$ be two operators. Let $F : \Omega \times \mathbf{R} \to 2^{\mathbf{R}}$ be a multifunction, with nonempty closed convex values, and let $\varphi : [0, +\infty[\to [0, +\infty]$ be a non-decreasing function. Assume that:*

(i) *Ψ is bijective and for any sequence $\{v_h\}$, weakly convergent to v in $L^{p'}(\Omega)$, there is a subsequence of $\{\Phi(\Psi^{-1}(v_h))\}$ which converges to $\Phi(\Psi^{-1}(v))$ almost everywhere in Ω;*

(ii) *one has*

$$\|\Phi(u)\|_{L^\infty(\Omega)} \leq \varphi(\|\Psi(u)\|_{L^{p'}(\Omega)}) \quad \textit{for all} \quad u \in U;$$

(iii) *for all $z \in \mathbf{R}$, the multifunction $F(\cdot, z)$ is $\mathscr{L}(\Omega)$-measurable;*
(iv) *for a.e. $x \in \Omega$, the multifunction $F(x, \cdot)$ has closed graph;*
(v) *there exists $r > 0$ such that the function $\gamma(x) := \sup_{|z| \leq \varphi(r)} \inf\{|y| : y \in F(x,z)\}$, with $x \in \Omega$, belongs to $L^{p'}(\Omega)$ and $\|\gamma\|_{L^{p'}(\Omega)} \leq r$.*

Then, there exists $u \in U$ such that $\Psi(u)(x) \in F(x, \Phi(u)(x))$ and $|\Psi(u)(x)| \leq \gamma(x)$ for almost every $x \in \Omega$.

The next proposition follows by exactly the same proof of Proposition 2.6 of Cubiotti and Yao (2015).

Proposition 2.3. *Let $h, k \in \mathbf{N}$ be two natural numbers, $A \subseteq \mathbf{R}^n$ a measurable set, $\varphi : A \times \mathbf{R}^h \to \mathbf{R}^k$ a given function, $S \subseteq \mathbf{R}^h$ a Lebesgue measurable set, with $m_h(S) = 0$, and let D_0 be a countable dense subset of $\mathbf{R}^h$, with $D_0 \cap S = \emptyset$. Assume that:*
(i) *for all $x \in A$, the function $\varphi(x, \cdot)$ is bounded;*
(ii) *for all $z \in D_0$, the function $\varphi(\cdot, z)$ is measurable.*

Let $\Phi : A \times \mathbf{R}^h \to 2^{\mathbf{R}^k}$ be the multifunction defined by setting, for each $(x,z) \in A \times \mathbf{R}^h$,

$$\Phi(x,z) := \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}} \overline{\Bigg(\bigcup_{\substack{v \in D_0 \\ |v - z| \leq \frac{1}{m}}} \{\varphi(x,y)\} \Bigg)}.$$

Then, one has:
(a) *Φ has nonempty closed convex values;*
(b) *for all $z \in \mathbf{R}^h$, the multifunction $\Phi(\cdot, z)$ is $\mathscr{L}(A)$-measurable;*
(c) *for all $x \in A$, the multifunction $\Phi(x, \cdot)$ has closed graph;*
(d) *if $x \in A$, and $\varphi(x, \cdot)|_{\mathbf{R}^h \setminus S}$ is continuous at $z \in \mathbf{R}^h \setminus S$, then one has*

$$\Phi(x,z) = \{\varphi(x,z)\}.$$

3. Existence results

The following is our main result.

Theorem 3.1. *Let $Y \subseteq \mathbf{R}$ be a closed interval, with $0 \notin Y$, and let $\psi : Y \to \mathbf{R}$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be two given functions. Moreover, assume that there exist $\xi \in L^2(\Omega)$ and a measurable set $E \subseteq \mathbf{R}$, with $m_1(E) = 0$, such that:*
(i) *the function ψ is continuous in Y, and $\operatorname{int}(\psi^{-1}(r)) = \emptyset$ for every $r \in \operatorname{int}(\psi(Y))$;*
(ii) *for a.e. $x \in \Omega$, the function $f(x, \cdot)|_{\mathbf{R} \setminus E}$ is continuous;*
(iii) *for all $z \in \mathbf{R} \setminus E$, the function $f(\cdot, z)$ is measurable;*
(iv) *for a.e. $x \in \Omega$, one has $f(x, \mathbf{R} \setminus E) \subseteq \psi(Y)$;*
(v) *for a.e. $x \in \Omega$ and for all $z \in \mathbf{R} \setminus E$, one has*

$$\sup\{|y| : \ y \in Y \textit{ and } \psi(y) = f(x,z)\} \leq \xi(x).$$

Then, there exists $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^2(\Omega)$, and one also has:

(a) $-\Delta_p u(x) \in Y$ *and* $\psi(-\Delta_p u(x)) = f(x, u(x))$ *for a.e.* $x \in \Omega$;
(b) $u(x) \in \mathbf{R} \setminus E$ *for a.e.* $x \in \Omega$;
(c) $|\Delta_p u(x)| \leq \xi(x)$ *for a.e.* $x \in \Omega$.

Proof. Without loss of generality, we can assume that assumptions (ii), (iv) and (v) are satisfied for all $x \in \Omega$. Since $m_1(E) = 0$, there exist a set $U_0 \in \mathscr{B}(\mathbf{R})$ such that $E \subseteq U_0$ and $m_1(U_0) = 0$. By assumption (i) and Theorem 2.4 of Ricceri (1982), there exists a set $Y_0 \subseteq Y$ such that $\psi(Y_0) = \psi(Y)$ and the function $\psi|_{Y_0} : Y_0 \to \psi(Y)$ is open (it maps open subsets of Y_0 onto open subsets of $\psi(Y) = \psi(Y_0)$).

Let $G : \psi(Y) \to 2^{Y_0}$ be the multifunction defined by putting, for each $t \in \psi(Y)$,

$$G(t) := \psi^{-1}(t) \cap Y_0.$$

It is a routine matter to check that the openness of the function $\psi|_{Y_0} : Y_0 \to \psi(Y)$ implies that G is lower semicontinuous in $\psi(Y)$, with nonempty values. Now, let $\Lambda : \Omega \times (\mathbf{R} \setminus U_0) \to 2^{Y_0}$ be the multifunction defined by setting, for each $(x,z) \in \Omega \times (\mathbf{R} \setminus U_0)$,

$$\Lambda(x,z) := G(f(x,z)).$$

By assumption (iv), we get that the multifunction Λ is well-defined, with nonempty values. By the lower semicontinuity of G, by assumption (ii) and by Theorem 7.3.11 of Klein and Thompson (1984), we have that for each $x \in \Omega$ the multifunction $\Lambda(x,\cdot)$ is lower semicontinuous in $\mathbf{R} \setminus U_0$. Let $\overline{\Lambda} : \Omega \times (\mathbf{R} \setminus U_0) \to 2^Y$ be defined by putting, for each $(x,z) \in \Omega \times (\mathbf{R} \setminus U_0)$,

$$\overline{\Lambda}(x,z) := \overline{\Lambda(x,z)} = \overline{(G(f(x,z)))} = \overline{(\psi^{-1}(f(x,z)) \cap Y_0)}$$

(here and in the sequel, the closures of subsets of $\mathbf{R}$ are taken with respect to the space $\mathbf{R}$). By Proposition 7.3.3 of Klein and Thompson (1984), we have that for every $x \in \Omega$ the multifunction $\overline{\Lambda}(x,\cdot)$ is lower semicontinuous in $\mathbf{R} \setminus U_0$, with nonempty and closed (in $\mathbf{R}$) values.

Now, observe that by the Lemma at p.198 of Kucia (1991), taking into account assumptions (ii) and (iii), the function $f|_{\Omega \times (\mathbf{R} \setminus U_0)}$ is $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R} \setminus U_0)$-measurable. Therefore, by the lower semicontinuity of G and by Proposition 2.6 of Himmelberg (1975), the multifunctions Λ and $\overline{\Lambda}$ are $\mathscr{L}(\Omega) \otimes \mathscr{B}(\mathbf{R} \setminus U_0)$-weakly measurable. Moreover, by assumption (v) we have that

$$\overline{\Lambda}(x,z) \subseteq [-\xi(x), \xi(x)] \cap Y \quad \text{for all} \quad (x,z) \in \Omega \times (\mathbf{R} \setminus U_0). \tag{7}$$

Let $V : \overline{\Omega} \times (\mathbf{R} \setminus U_0) \to 2^Y$ be defined by setting, for each $(x,z) \in \overline{\Omega} \times (\mathbf{R} \setminus U_0)$,

$$V(x,z) = \begin{cases} \overline{\Lambda}(x,z) & \text{if } x \in \Omega \text{ and } z \in (\mathbf{R} \setminus U_0), \\ Y & \text{if } x \in \partial\Omega \text{ and } z \in (\mathbf{R} \setminus U_0). \end{cases}$$

Of course, by the above construction, the multifunction V is $\mathscr{L}(\overline{\Omega}) \otimes \mathscr{B}(\mathbf{R} \setminus U_0)$-weakly measurable. We now observe that by Corollary 6.6.7 of Bogachev (2007) the set $\mathbf{R} \setminus U_0$ is a Souslin set. Consequently, by Theorem 2.1, there exist two sets $\Omega_0 \in \mathscr{L}(\overline{\Omega})$ and $U_1 \in \mathscr{B}(\mathbf{R})$, with $m_n(\Omega_0) = 0$ and $m_1(U_1) = 0$, and a function $g_0 : \overline{\Omega} \times (\mathbf{R} \setminus U_0) \to Y$ such that:

(a_1) $g_0(x,z) \in V(x,z)$ for all $(x,z) \in \overline{\Omega} \times (\mathbf{R} \setminus U_0)$;
(a_2) for every $z \in \mathbf{R} \setminus (U_0 \cup U_1)$, the function $g_0(\cdot, z)$ is measurable on $\overline{\Omega}$;

(a_3) for every $x \in \overline{\Omega} \setminus \Omega_0$, one has

$$\{z \in (\mathbf{R} \setminus U_0) : \ g_0(x,\cdot) \text{ is discontinuous at } z\} \subseteq U_1 \setminus U_0.$$

Let $\Omega_1 := \Omega_0 \cap \Omega$ and $g_1 := g_0|_{\Omega \times (\mathbf{R} \setminus U_0)}$. Of course, we have that $g_1 : \Omega \times (\mathbf{R} \setminus U_0) \to Y$ satisfies the following conditions, that we explictly state for a clearer further exposition:

(b_1) $g_1(x,z) \in \overline{\Lambda}(x,z)$ for all $(x,z) \in \Omega \times (\mathbf{R} \setminus U_0)$;
(b_2) for every $z \in \mathbf{R} \setminus (U_0 \cup U_1)$, the function $g_1(\cdot,z)$ is measurable in Ω;
(b_3) for every $x \in \Omega \setminus \Omega_1$, one has 5

$$\{z \in \mathbf{R} \setminus U_0 : \ g_1(x,\cdot) \text{ is discontinuous at } z\} \subseteq U_1 \setminus U_0.$$

By (7) we immediately get

$$|g_1(x,z)| \leq \xi(x) \quad \text{for all} \quad (x,z) \in \Omega \times (\mathbf{R} \setminus U_0). \tag{8}$$

Let $g^* : \Omega \times \mathbf{R} \to \mathbf{R}$ by putting

$$g^*(x,z) = \begin{cases} g_1(x,z) & \text{if } x \in \Omega \text{ and } z \in (\mathbf{R} \setminus U_0) \\ 0 & \text{if } x \in \Omega \text{ and } z \in U_0. \end{cases}$$

By (8) we get

$$|g^*(x,z)| \leq \xi(x) \quad \text{for all} \quad (x,z) \in \Omega \times \mathbf{R}. \tag{9}$$

Since $m_1(U_0 \cup U_1) = 0$, there exists a countable set $D_1 \subseteq \mathbf{R} \setminus (U_0 \cup U_1)$ such that D_1 is dense in $\mathbf{R}$. Now, let $F : \Omega \times \mathbf{R} \to 2^{\mathbf{R}}$ be the multifunction defined by setting, for each $(x,z) \in \Omega \times \mathbf{R}$,

$$F(x,z) := \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}} \overline{\Bigg(\bigcup_{\substack{v \in D_1 \\ |v-z| \leq \frac{1}{m}}} \{g^*(x,v)\} \Bigg)} = \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}} \overline{\Bigg(\bigcup_{\substack{v \in D_1 \\ |v-z| \leq \frac{1}{m}}} \{g_1(x,v)\} \Bigg)}.$$

Again by (8), we get

$$F(x,z) \subseteq Y \cap [-\xi(x), \xi(x)] \quad \text{for every} \quad (x,z) \in \Omega \times \mathbf{R}. \tag{10}$$

By (b_2), for every $z \in D_1$ the function $g^*(\cdot,z)$ is measurable in Ω. Moreover, by (9), for every $x \in \Omega$ the function $g^*(x,\cdot)$ is bounded. Applying Proposition **2.3**, with $S = U_0 \cup U_1$, we get:

(c_1) F has nonempty closed convex values;
(c_2) for all $z \in \mathbf{R}$, the multifunction $F(\cdot,z)$ is $\mathscr{L}(\Omega)$-measurable;
(c_3) for all $x \in \Omega$, the multifunction $F(x,\cdot)$ has closed graph;
(c_4) if $x \in \Omega$, and the function $g^*(x,\cdot)|_{\mathbf{R} \setminus (U_0 \cup U_1)} = g_1(x,\cdot)|_{\mathbf{R} \setminus (U_0 \cup U_1)}$ is continuous at $z \in \mathbf{R} \setminus (U_0 \cup U_1)$, then one has $F(x,z) = \{g^*(x,z)\} = \{g_1(x,z)\}$.

In particular, by properties (c_4) and (b_3), we get

$$F(x,z) = \{g^*(x,z)\} = \{g_1(x,z)\} \quad \text{for all} \quad (x,z) \in \big(\Omega \setminus \Omega_1\big) \times \big[\mathbf{R} \setminus (U_0 \cup U_1)\big]. \tag{11}$$

By (10), we also get

$$\sup_{z \in \mathbf{R}} \Big(\inf\{|y| : \ y \in F(x,z)\} \Big) \leq \xi(x) \quad \text{for all} \quad x \in \Omega. \tag{12}$$

Now we want to apply Theorem **2.2** above to the multifunction F, choosing $U := (-\Delta_p)^{-1}(L^{p'}(\Omega))$, $\Phi(u) = u$, $\Psi(u) = -\Delta_p u$, $\varphi(t) = \sigma(\lambda t)^{1/(p-1)}$ and $r = \|\xi\|_{L^{p'}(\Omega)}$, where λ and σ are as in (5) and (6). To this aim, we observe what follows:

(d_1) the operator

$$\Psi = -\Delta_p|_U : U \to L^{p'}(\Omega)$$

is bijective. This follows at once from the fact that the operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bijective (see Dinca, Jebelean, and Mawhin 2001, Theorem 8);

(d_2) for every sequence $\{v_h\}$ in $L^{p'}(\Omega)$, weakly converging to v in $L^{p'}(\Omega)$, there is a subsequence $\{v_{h_k}\}$ such that $\{(-\Delta_p)^{-1}(v_{h_k})\} \to (-\Delta_p)^{-1}(v)$ a.e. in Ω. Indeed, assume that $\{v_h\}$ converges weakly to v in $L^{p'}(\Omega)$. Since $L^{p'}(\Omega)$ is compactly embedded in $W^{-1,p'}(\Omega)$, there exists a subsequence $\{v_{h_l}\}$ of $\{v_h\}$ such that $\{v_{h_l}\} \to v$ strongly in $W^{-1,p'}(\Omega)$. Since the operator

$$(-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$$

is strongly continuous (see Dinca, Jebelean, and Mawhin 2001, Theorem 8), the sequence $\{(-\Delta_p)^{-1}(v_{h_l})\} \to (-\Delta_p)^{-1}(v)$ strongly in $W_0^{1,p}(\Omega)$, hence strongly in $L^p(\Omega)$. Consequently, the sequence $\{(-\Delta_p)^{-1}(v_{h_l})\}$ has a subsequence which converges to $(-\Delta_p)^{-1}(v)$ almost everywhere in Ω, and this proves our claim.

(d_3) we have $\|u\|_{L^\infty(\Omega)} \leq \varphi(\|-\Delta_p u\|_{L^{p'}(\Omega)})$ for all $u \in U$. Indeed, by (5) and (6), taking into account Remark 2 at p. 349 of Dinca, Jebelean, and Mawhin (2001), for each $u \in U$ we have

$$\|u\|_{L^\infty(\Omega)} \leq \sigma\|u\|_* = \sigma\,\|-\Delta_p u\|_{W^{-1,p'}(\Omega)}^{1/(p-1)} \leq \sigma\Big(\lambda\,\|-\Delta_p u\|_{L^{p'}(\Omega)}\Big)^{1/(p-1)},$$

that is our claim;

(d_4) the function

$$\gamma(x) := \sup_{|z|\leq\varphi(\|\xi\|_{L^{p'}(\Omega)})} \big(\inf\{|y| : y \in F(x,z)\}\big), \qquad \text{with } x \in \Omega$$

belongs to $L^{p'}(\Omega)$ and $\|\gamma\|_{L^{p'}(\Omega)} \leq \|\xi\|_{L^{p'}(\Omega)}$. This follows at once by (12), since $p' < 2$; for what concerns the measurability of γ, we refer to p. 262 of Naselli Ricceri and Ricceri (1990).

Thus, all the assumptions of Theorem **2.2** are satisfied. Consequently, there exists a function $u \in U$ (that is, $u \in W_0^{1,p}(\Omega)$ and $-\Delta_p u \in L^{p'}(\Omega)$) such that $-\Delta_p u(x) \in F(x,u(x))$ for a.e. $x \in \Omega$. We now prove that the function u satisfies our conclusion. To this aim, let $\Omega_2 \subseteq \Omega$, with $m_n(\Omega_2) = 0$, be such that

$$-\Delta_p u(x) \in F(x,u(x)) \quad \text{for all} \quad x \in \Omega \setminus \Omega_2. \tag{13}$$

By (10) we have

$$-\Delta_p u(x) \in Y \cap [-\xi(x), \xi(x)] \quad \text{for all} \quad x \in \Omega \setminus \Omega_2, \tag{14}$$

hence, in particular, $-\Delta_p u \in L^2(\Omega)$ and $|\Delta_p u(x)| \leq \xi(x)$ for a.e. $x \in \Omega$. Since $0 \notin Y$, by (14) we get that the function $-\Delta_p u$ has constant sign in $\Omega \setminus \Omega_2$. Let us assume that

$$-\Delta_p u(x) > 0 \quad \text{for all} \quad x \in \Omega \setminus \Omega_2 \tag{15}$$

(if, conversely, $-\Delta_p u(x) < 0$ for all $x \in \Omega \setminus \Omega_2$, then the argument is analogous). By Lemma 1 of De Giorgi, Buttazzo, and Dal Maso (1983), we have

$$\nabla u(x) = 0_{\mathbf{R}^n} \quad \text{for a.e.} \quad x \in u^{-1}(U_0 \cup U_1).$$

On the other side, by (15) and by Corollary 1.1 of Lou (2008), the set

$$\{x \in \Omega : \nabla u(x) = 0_{\mathbf{R}^n}\}$$

has null Lebesgue measure, hence we easily get that $m_n(u^{-1}(U_0 \cup U_1)) = 0$. Let

$$\Omega^* := \Omega_1 \cup \Omega_2 \cup [u^{-1}(U_0 \cup U_1)].$$

By the above construction, it follows that $m_n(\Omega^*) = 0$. Fix any $x \in \Omega \setminus \Omega^*$. By (13), we have that $-\Delta_p u(x) \in F(x, u(x))$. Moreover, $u(x) \notin U_0 \cup U_1$, hence, in particular, $u(x) \notin E$. Since $x \notin \Omega_1$, by (11) we have that

$$F(x, u(x)) = \{g_1(x, u(x))\},$$

hence

$$-\Delta_p u(x) = g_1(x, u(x)) \in \overline{\Lambda}(x, u(x)) \subseteq Y. \tag{16}$$

By (16), by the continuity of ψ and by the closedness of Y, we get

$$-\Delta_p u(x) \in \overline{(\psi^{-1}(f(x, u(x))) \cap Y_0)} \subseteq \overline{\psi^{-1}(f(x, u(x)))} = \psi^{-1}(f(x, u(x))),$$

and thus

$$\psi(-\Delta_p u(x)) = f(x, u(x)).$$

This completes the proof. □

Now we state explicitly some special cases and some corollaries of Theorem **3.1**. Firstly, we consider the case where f does not depend on $x \in \Omega$. In this case, Theorem **3.1** immediately gives the following result.

Corollary 3.2. *Let $Y \subseteq \mathbf{R}$ be a closed interval, with $\inf Y > 0$, $\psi : Y \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ two given functions, and let $E \subseteq \mathbf{R}$, with $m_1(E) = 0$. Assume that:*

(i) *ψ is continuous and $\operatorname{int}(\psi^{-1}(t)) = \emptyset$ for all $t \in \operatorname{int}(\psi(Y))$;*
(ii) *$f|_{\mathbf{R} \setminus E}$ is continuous;*
(iii) *one has $f(\mathbf{R} \setminus E) \subseteq \psi(Y)$ and $\sup \psi^{-1}(f(\mathbf{R} \setminus E)) < +\infty$.*

Then, there exists $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^\infty(\Omega)$, and:

(a) *$-\Delta_p u(x) \in Y$ and $\psi(-\Delta_p u(x)) = f(u(x))$ for a.e. $x \in \Omega$;*
(b) *$u(x) \in \mathbf{R} \setminus E$ for a.e. $x \in \Omega$.*

Proof. It follows at once from Theorem 3.1. □

As an application of Corollary **3.2**, we can now prove the following result, where the constants λ and σ are as in (5) and (6).

Theorem 3.3. *Let Let $a > 0$, $\psi : [a, +\infty[\to \mathbf{R}$ a continuous function, and let $f : \mathbf{R} \to \mathbf{R}$ be a given function. Let $E \subseteq \mathbf{R}$, with $m_1(E) = 0$. Assume that:*

(i) *the function* $f|_{\mathbf{R}\setminus E}$ *is continuous;*
(ii) $\operatorname{int}(\psi^{-1}(t)) = \emptyset$ *for all* $t \in \operatorname{int}(\psi([a,+\infty[))$*;*
(iii) *there exists* $\rho > a$ *such that*

$$f(]-\sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}, \sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}[\setminus E) \subseteq \psi([a,\rho]).$$

Then, there exists a function $u \in W_0^{1,p}(\Omega)$ *such that* $-\Delta_p u \in L^\infty(\Omega)$*, and also one has:*
(a) $-\Delta_p u(x) \in [a,\rho]$ *and* $\psi(-\Delta_p u(x)) = f(u(x))$ *for a.e.* $x \in \Omega$*;*
(b) $u(x) \in]-\sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p},\ \sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}[\setminus E$ *for a.e.* $x \in \Omega$*.*

Proof. Choose any

$$\widetilde{z} \in]-\sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p},\ \sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}[\setminus E.$$

Let $Y := [a,\rho]$, $\hat{\psi} := \psi|_Y$, and let $\hat{f} : \mathbf{R} \to \mathbf{R}$ be defined by

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \in [-\sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p},\ \sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}], \\ f(\widetilde{z}) & \text{otherwise.} \end{cases}$$

Put

$$\widetilde{E} := E \cup \{-\sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p},\ \sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}\}.$$

Then, we have that $m_1(\widetilde{E}) = 0$, the function $\hat{f}|_{\mathbf{R}\setminus\widetilde{E}}$ is continuous and

$$\hat{f}(\mathbf{R}\setminus\widetilde{E}) = f(]-\sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}, \sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}[\setminus E) \subseteq \hat{\psi}([a,\rho]).$$

Moreover, $\operatorname{int}(\hat{\psi}^{-1}(t)) = \emptyset$ for all $t \in \operatorname{int}(\hat{\psi}(Y))$. Finally, we have $\sup \hat{\psi}^{-1}(\hat{f}(\mathbf{R}\setminus\widetilde{E})) \leq \rho$. By Corollary **3.2**, there exists a function $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^\infty(\Omega)$ and one has

$$-\Delta_p u(x) \in [a,\rho], \qquad \hat{\psi}(-\Delta_p u(x)) = \hat{f}(u(x)) \quad \text{and} \quad u(x) \in \mathbf{R}\setminus\widetilde{E} \tag{17}$$

for a.e. $x \in \Omega$. By (5) and (6), taking into account Remark 2 at p. 349 of Dinca, Jebelean, and Mawhin (2001), we have that

$$\|u\|_{L^\infty(\Omega)} \leq \sigma\|u\|_* = \sigma\,\|-\Delta_p u\|_{W^{-1,p'}(\Omega)}^{1/(p-1)} \leq \sigma\Big(\lambda\,\|-\Delta_p u\|_{L^{p'}(\Omega)}\Big)^{1/(p-1)}.$$

Since by (17) we have

$$\|\Delta_p u\|_{L^{p'}(\Omega)} \leq \rho\, m_n(\Omega)^{1/p'},$$

we get

$$\|u\|_{L^\infty(\Omega)} \leq \sigma\Big(\lambda\,\rho\, m_n(\Omega)^{(p-1)/p}\Big)^{1/(p-1)} = \sigma(\lambda\,\rho)^{1/(p-1)}m_n(\Omega)^{1/p}.$$

Taking into account (17), this implies that

$$u(x) \in]-\sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p},\ \sigma(\lambda\rho)^{1/(p-1)}m_n(\Omega)^{1/p}[\setminus E$$

for a.e. $x \in \Omega$. Thus, for a.e. $x \in \Omega$ we get

$$\psi(-\Delta_p u(x)) = \hat{\psi}(-\Delta_p u(x)) = \hat{f}(u(x)) = f(u(x)),$$

and this completes the proof. □

Remark 3.4. In order to apply concretely Theorem **3.3**, one needs to have explicit estimates of the constants λ and σ. To this aim, we observe that by Formula (6b) of Talenti (1987) we have

$$\sigma \leq \frac{n^{-1/p}}{\sqrt{\pi}} \left(\frac{p-1}{p-n}\right)^{1-1/p} \left(\Gamma\left(1+\frac{n}{2}\right)\right)^{1/n} m_n(\Omega)^{1/n-1/p},$$

where Γ is the Gamma function. Moreover, we have (see Marano 2012, Remark 3.4):

$$\lambda \leq \frac{n^{-1/p}}{\sqrt{\pi}} \left(\frac{p-1}{p-n}\right)^{1-1/p} \left(m_n(\Omega)\cdot\Gamma\left(1+\frac{n}{2}\right)\right)^{1/n}.$$

We now consider the explicit case where $\psi(t) = t$. In this case, Theorem 3.1 immediately gives the following result.

Theorem 3.5. *Let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be a given function. Let $\alpha > 0$, $E \subseteq \mathbf{R}$, with $m_1(E) = 0$, and $\xi \in L^2(\Omega)$ be such that:*

(i) *for a.e. $x \in \Omega$, the function $f(x,\cdot)|_{\mathbf{R}\setminus E}$ is continuous;*
(ii) *for all $z \in \mathbf{R}\setminus E$, the function $f(\cdot,z)$ is measurable;*
(iii) *for a.e. $x \in \Omega$, one has* $\inf f(x,\mathbf{R}\setminus E) \geq \alpha$ *and* $\sup f(x,\mathbf{R}\setminus E) \leq \xi(x)$.

Then, there exists a function $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^2(\Omega)$, and one also has:

(a) $-\Delta_p u(x) \in [\alpha, \xi(x)]$ *and* $u(x) \in \mathbf{R}\setminus E$ *for a.e.* $x \in \Omega$;
(b) $-\Delta_p u(x) = f(x,u(x))$ *for a.e.* $x \in \Omega$.

Proof. Choose $Y := [\alpha, +\infty[$ and $\psi(y) = y$. Then, the conclusion follows by at once by Theorem **3.1**. □

By Theorem **3.5** we immediately get the following result, which concerns the problem $-\Delta u = g(x,u) + h(x)$.

Corollary 3.6. *Let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be a given function. Let $h, \eta \in L^2(\Omega)$, $\alpha > 0$ and $E \subseteq \mathbf{R}$, with $m_1(E) = 0$, be such that:*

(i) *for a.e. $x \in \Omega$, the function $g(x,\cdot)|_{\mathbf{R}\setminus E}$ is continuous;*
(ii) *for all $z \in \mathbf{R}\setminus E$, the function $g(\cdot,z)$ is measurable;*
(iii) *for a.e. $x \in \Omega$, one has*

$$\alpha - h(x) \leq \inf g(x,\mathbf{R}\setminus E), \qquad \sup g(x,\mathbf{R}\setminus E) \leq \eta(x).$$

Then, there exists a function $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^2(\Omega)$, and one also has $u(x) \in \mathbf{R}\setminus E$, $-\Delta_p u(x) \in [\alpha, \eta(x) + h(x)]$, and $-\Delta_p u(x) = g(x,u(x)) + h(x)$ for almost every $x \in \Omega$.

Proof. It follows at once by Theorem **3.5**, by choosing $f(x,z) = g(x,z) + h(x)$ and $\xi = \eta + h$. □

Finally, when the function g does not depend on $x \in \Omega$, by Corollary **3.6** we get the following existence result for the problem $-\Delta u = \varphi(u) + h(x)$, where the function $\varphi : \mathbf{R} \to \mathbf{R}$ can be discontinuous even at all points $z \in \mathbf{R}$.

Corollary 3.7. *Let $\varphi : \mathbf{R} \to \mathbf{R}$ be a given function. Let $h \in L^2(\Omega)$, $\alpha > 0$ and $E \subseteq \mathbf{R}$, with $m_1(E) = 0$, be such that:*

(i) *the function $\varphi|_{\mathbf{R}\setminus E}$ is continuous and bounded;*
(ii) *for a.e. $x \in \Omega$, one has*

$$h(x) \geq \alpha - \inf \varphi(\mathbf{R} \setminus E).$$

Then, there exists a function $u \in W_0^{1,p}(\Omega)$, with $-\Delta_p u \in L^2(\Omega)$, such that $u(x) \in \mathbf{R} \setminus E$ and $-\Delta_p u(x) = \varphi(u(x)) + h(x)$ for a.e. $x \in \Omega$.

We now give some remarks about possible improvements and applications of our results.

Remark 3.8. Let us consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(z) = \begin{cases} 0 & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

The Example 4.3 of Marino and Paratore (2021) shows that in this case, for any $p > n$, the equation $-\Delta_p u = f(u)$ has no solutions in $W_0^{1,p}(\Omega)$. It is immediate to check that, in this case, all the assumptions of our Theorem **3.1** (except for the assumption $0 \notin Y$) are satisfied by taking $Y = [0,1]$, $E = \{0\}$, $\psi(y) = y$, $\xi(x) \equiv 0$. Consequently, the Example 4.3 of Marino and Paratore (2021) shows that the assumption $0 \notin Y$ cannot be dropped from the statement of Theorem **3.1**. Moreover, it shows that the assumptions $\inf Y > 0$ and $\inf f(x, \mathbf{R} \setminus E) \geq \alpha > 0$ in Corollary **3.2** and Theorem **3.5**, respectively, are essential. Finally, the same example shows that the assumptions $\alpha - h(x) \leq \inf g(x, \mathbf{R} \setminus E)$ and $h(x) \geq \alpha - \inf \varphi(\mathbf{R} \setminus E)$ in Corollary **3.6** and in Corollary **3.7**, respectively, cannot be dropped from the statements.

Remark 3.9. It is quite simple to provide examples of application of our results, where for every $x \in \Omega$ the function $f(x, \cdot)$ is discontinuous at all points $z \in \mathbf{R}$. To this aim, it suffices to take f as defined in (2). In this case, applying Theorem **3.5** with $E = \mathbf{Q}$, $\alpha = 2$ and $\xi(x) \equiv 4$, it follows that there exists $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^\infty(\Omega)$ and for a.e. $x \in \Omega$ one has $-\Delta_p u(x) \in [2,4]$, $u(x) \in \mathbf{R} \setminus \mathbf{Q}$ and $-\Delta_p u(x) = f(x, u(x))$. In a similar way, as regards the case where f does not depend on $x \in \Omega$, one can take f as defined by (3), and then apply Theorem **3.5**. As we have already observed, such a function f is discontinuous at all points $z \in \mathbf{R}$. We now want to present, stated as corollaries, two more articulated examples of application of our results. Such examples are simply intended to illustrate possible uses of our results.

Corollary 3.10. *Let $h \in L^\infty(\Omega)$, with $a := \operatorname{essinf}_{x \in \Omega} h(x) > 0$. Let $\beta > 0$, $\alpha \geq 0$, $b \geq 1$, $\gamma \in [0,1]$ be four constants, and let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be defined by putting*

$$f(x,z) = \begin{cases} h(x) + \alpha(b + \cos z)^\beta & \text{if } x \in \Omega,\ z < 0 \text{ and } z \in \mathbf{R} \setminus \mathbf{Q}, \\ h(x) + \alpha(b + \cos z)^\beta + 1 & \text{if } x \in \Omega,\ z > 0 \text{ and } z \in \mathbf{R} \setminus \mathbf{Q}, \\ h(x) + \alpha(b + \cos z)^\beta + 2 & \text{if } x \in \Omega \text{ and } z \in \mathbf{Q}. \end{cases}$$

Finally, let $V \subseteq \mathbf{R}$ be a measurable set, with $m_1(V) = 0$.

Then, there exists a function $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^\infty(\Omega)$ and also one has:

(a) $-\Delta_p u(x) = f(x,u(x)) + \gamma \sin(-\Delta_p u(x))$ *for almost every* $x \in \Omega$*;*
(b) $u(x) \in \mathbf{R} \setminus (\mathbf{Q} \cup V)$ *for almost every* $x \in \Omega$.

Proof. Put $E := \mathbf{Q} \cup V$. It is immediate to check that for all $z \in \mathbf{R} \setminus E$ the function $f(\cdot, z)$ is measurable, and for all $x \in \Omega$ the function $f(x,\cdot)|_{\mathbf{R}\setminus E}$ is continuous. Now, let $\Omega^* \subseteq \Omega$, with $m_n(\Omega^*) = 0$, be such that $a \le h(x) \le \|h\|_{L^\infty(\Omega)}$ for all $x \in \Omega \setminus \Omega^*$. Hence, we get

$$a \le f(x,z) \le \|h\|_{L^\infty(\Omega)} + \alpha(b+1)^\beta + 1 \quad \text{for all} \quad x \in \Omega \setminus \Omega^*,\ z \in \mathbf{R} \setminus \mathbf{Q}. \tag{18}$$

Since $\lim_{y\to 0^+}(y - \gamma \sin y) = 0$, there exists $y_0 > 0$ such that $y_0 - \gamma \sin y_0 < a$. Moreover, since $\lim_{y\to+\infty}(y - \gamma \sin y) = +\infty$, there exists $y_1 > y_0$ such that

$$y_1 - \gamma \sin y_1 > \|h\|_{L^\infty(\Omega)} + \alpha(b+1)^\beta + 1.$$

Now, we want to apply Theorem **3.1**, by choosing $Y = [y_0, y_1]$, $\xi(x) \equiv y_1$ and $\psi(y) = y - \gamma \sin y$, and with E and f defined as above. To this aim, we firstly observe that assumption (i) is satisfied since ψ' never vanishes identically on a interval. Moreover, we have already observed that f satisfies assumptions (ii) and (iii). Assumption (iv) is satisfied since, by (18) and by the above construction, we have that

$$f(x, \mathbf{R}\setminus E) \subseteq [a, \|h\|_{L^\infty(\Omega)} + \alpha(b+1)^\beta + 1] \subseteq [\psi(y_0), \psi(y_1)] \subseteq \psi(Y) \quad \text{for all} \quad x \in \Omega \setminus \Omega^*.$$

Finally, assumptions (v) is obviously satisfied. Hence, by Theorem **3.1** there exists a function $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^\infty(\Omega)$, and for almost every $x \in \Omega$ one has $u(x) \in \mathbf{R} \setminus E$ and $-\Delta_p u(x) - \gamma \sin(-\Delta_p u(x)) = f(x, u(x))$, that is our conclusion. □

Corollary 3.11. *Let* $h \in L^\infty(\Omega)$*, with* $a := \operatorname{essinf}_{x\in\Omega} h(x) > 0$*. Let* $\beta > 0$*,* $\alpha \ge 0$*,* $b \ge 1$ *and* $\gamma \in [0, a[$ *be four constants, and let* $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *be defined by*

$$f(x,z) = \begin{cases} h(x) + \alpha(b+\cos z)^\beta & \text{if } x \in \Omega,\ z < 0 \text{ and } z \in \mathbf{R} \setminus \mathbf{Q}, \\ h(x) + \alpha(b+\cos z)^\beta + 1 & \text{if } x \in \Omega,\ z > 0 \text{ and } z \in \mathbf{R} \setminus \mathbf{Q}, \\ h(x) + \alpha(b+\cos z)^\beta + 2 & \text{if } x \in \Omega \text{ and } z \in \mathbf{Q}. \end{cases}$$

Finally, let $V \subseteq \mathbf{R}$ *be a measurable set, with* $m_1(V) = 0$*. Then, there exists a function* $u \in W_0^{1,p}(\Omega)$ *such that* $-\Delta_p u \in L^\infty(\Omega)$*, and one also has:*

(a) $-\Delta_p u(x) = f(x,u(x)) - \gamma e^{-\Delta_p u(x)}$ *for almost every* $x \in \Omega$*;*
(b) $u(x) \in \mathbf{R} \setminus (\mathbf{Q} \cup V)$ *for almost every* $x \in \Omega$.

Proof. We essentially argue as in Corollary **3.10**. Let us put $E := \mathbf{Q} \cup V$. As above, it is immediate to check that for all $z \in \mathbf{R} \setminus E$ the function $f(\cdot, z)$ is measurable, and for all $x \in \Omega$ the function $f(x,\cdot)|_{\mathbf{R}\setminus E}$ is continuous. Let $\Omega^* \subseteq \Omega$, with $m_n(\Omega^*) = 0$, be such that $a \le h(x) \le \|h\|_{L^\infty(\Omega)}$ for all $x \in \Omega \setminus \Omega^*$. Hence, we have

$$a \le f(x,z) \le \|h\|_{L^\infty(\Omega)} + \alpha(b+1)^\beta + 1 \quad \text{for all} \quad x \in \Omega \setminus \Omega^*,\ z \in \mathbf{R} \setminus \mathbf{Q}. \tag{19}$$

Since $\lim_{y\to 0^+}(y + \gamma e^y) = \gamma < a$, there exists $y_0 > 0$ in such a way that $y_0 + \gamma e^{y_0} < a$. Since $\lim_{y\to+\infty}(y + \gamma e^y) = +\infty$, there exists $y_1 > y_0$ in such a way that

$$y_1 + \gamma e^{y_1} > \|h\|_{L^\infty(\Omega)} + \alpha(b+1)^\beta + 1.$$

We now apply Theorem **3.1**, with $Y = [y_0, y_1]$, $\xi(x) \equiv y_1$, $\psi(y) = y + \gamma e^y$, and E and f as defined above. To this aim, we observe that assumption (i) is satisfied since $\psi'(y) \ge 1$ for all $y \in \mathbf{R}$. Moreover, we have already observed that f satisfies assmptions (ii) and (iii). For what concerns assumption (iv), we observe that by (19) and by the above construction we get

$$f(x, \mathbf{R} \setminus E) \subseteq [a, \|h\|_{L^\infty(\Omega)} + \alpha(b+1)^\beta + 1] \subseteq [\psi(y_0), \psi(y_1)] = \psi(Y) \quad \text{for all} \quad x \in \Omega \setminus \Omega^*,$$

hence assumption (iv) is satisfied. Since assumption (v) is obviously satisfied, by Theorem **3.1** we have that there exists a function $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in L^\infty(\Omega)$, and for almost every $x \in \Omega$ one has $u(x) \in \mathbf{R} \setminus E$ and $-\Delta_p u(x) + \gamma e^{-\Delta_p u(x)} = f(x, u(x))$, that is our conclusion. □

Remark 3.12. We now briefly compare our main result (Theorem **3.1**) with Theorem 4.1 of Marino and Paratore (2021), which is the main result of Marino and Paratore (2021) for what concerns the discontinuous framework. For the reader's convenience, we now state explicitly this latter result (Ω, n and p are assumed as in Section 2).

Theorem 3.13. (Marino and Paratore 2021, Theorem 4.1) *Let $(\alpha, \beta) \subseteq \mathbf{R}$ be such that $0 \notin (\alpha, \beta)$, let $\psi : (\alpha, \beta) \to \mathbf{R}$ be continuous, and let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be given. Assume that:*

(i) *f is $\mathscr{L}(\Omega \times \mathbf{R})$-measurable and essentially bounded;*
(ii) *the set $D_f := \{(x,z) \in \Omega \times \mathbf{R} : f \text{ is discontinuous at } (x,z)\}$ belongs to the family $\mathscr{F}_\Omega$;*
(iii) *$f^{-1}(r) \setminus \operatorname{int}(f^{-1}(r)) \in \mathscr{F}_\Omega$ for every $r \in \psi((\alpha, \beta))$;*
(iv) *$\overline{f((\Omega \times \mathbf{R}) \setminus D_f)} \subseteq \psi((\alpha, \beta))$.*

Then, there exists $u \in W_0^{1,p}(\Omega)$ such that $\Delta_p u \in L^{p'}(\Omega)$, and also $-\Delta_p u(x) \in Y$ and $\psi(-\Delta_p u(x)) = f(x, u(x))$ for a.e. $x \in \Omega$.

As we have already observed, Theorem **3.1** and Theorem **3.13** are formally independent. However, we observe that in Theorem 3.1, as well as in its consequences below, it is assumed that $\operatorname{int}(\psi^{-1}(t)) = \emptyset$ for all $t \in \operatorname{int}(\psi(Y))$. This is not a restrictive condition, and it is satisfied, for instance, if ψ is nonconstant over the intervals. It is worth noticing that, if such an assumption is satisfied, then Theorem **3.13** is a consequence of our Theorem **3.1**. To see this, assume that all the assumptions of Theorem **3.13** are satisfied, and that $\operatorname{int}(\psi^{-1}(t)) = \emptyset$ for all $t \in \operatorname{int}(\psi(Y))$. It is routine matter to check that assumption (i) of Theorem **3.13** implies that the restriction $f|_{(\Omega \times \mathbf{R}) \setminus D_f}$ is bounded. Put

$$\gamma := \inf f((\Omega \times \mathbf{R}) \setminus D_f), \qquad \delta := \sup f((\Omega \times \mathbf{R}) \setminus D_f).$$

By assumption (iv) of Theorem **3.13**, there exist $t_1, t_2 \in (\alpha, \beta)$ such that $\psi(t_1) = \gamma$, $\psi(t_2) = \delta$. Let $[y_1, y_2]$ be any compact interval, with $y_1 < y_2$, such that

$$t_1 \in [y_1, y_2], \quad t_2 \in [y_1, y_2], \quad \text{and} \quad [y_1, y_2] \subseteq (\alpha, \beta).$$

By assumption (ii) of Theorem **3.13**, we have that at least one of the two sets $\pi_0(D_f)$ and $\pi_1(D_f)$ has null Lebesgue measure. Firstly, assume that $m_n(\pi_0(D_f)) = 0$. We have that for every $x \in \Omega \setminus \pi_0(D_f)$ the function $f(x, \cdot)$ is continuous in $\mathbf{R}$. Moreover, for each fixed $z \in \mathbf{R}$, the function $f(\cdot, z)|_{\Omega \setminus \pi_0(D_f)}$ is continuous, hence measurable. Since $m_n(\pi_0(D_f)) = 0$, this implies that the function $f(\cdot, z)$ is measurable in Ω. Finally, for every $x \in \Omega \setminus \pi_0(D_f)$, and every $z \in \mathbf{R}$, we have $(x, z) \notin D_f$, hence by the above construction it follows easily that

$f(x,z) \subseteq \psi([y_1,y_2])$. Consequently, all the assumptions of Theorem **3.1** are satisfied by taking $E = \emptyset$, $Y = [y_1,y_2]$ and $\xi(x) \equiv \max\{|y_1|,|y_2|\}$.

Conversely, assume that $m_1(\pi_1(D_f)) = 0$. Then, arguing as above, it is easy to check that all the assumptions of Theorem **3.1** are satisfied by taking $E = \pi_1(D_f)$, $Y = [y_1,y_2]$ and $\xi(x) \equiv \max\{|y_1|,|y_2|\}$. Hence, in both cases Theorem **3.1** gives the existence of a function $u \in W_0^{1,p}(\Omega)$ such that $\Delta_p u \in L^\infty(\Omega)$, and also $-\Delta_p u(x) \in Y$ and $\psi(-\Delta_p u(x)) = f(x,u(x))$ for a.e. $x \in \Omega$, and thus our claim is proved.

At this point, it is natural to ask if the assumption "$\mathrm{int}\,(\psi^{-1}(t)) = \emptyset$ for all $t \in \mathrm{int}\,(\psi(Y))$" in the statement of Theorem **3.1** is essential. Since we did not succeed in finding a counterexample, we leave this as an open problem.

References

Adams, R. A. and Fournier, J. J. F. (2003). *Sobolev Spaces*. 2nd ed. Vol. 140. Amsterdam: Elsevier. URL: https://www.elsevier.com/books/sobolev-spaces/adams/978-0-12-044143-3.

Ahmad, M., Zada, A., and Alzabut, J. (2019). "Stability analysis of a nonlinear coupled implicit switched singular fractional differential system with p-Laplacian". *Advances in Difference Equations* **2019**, 436. DOI: 10.1186/s13662-019-2367-y.

Bogachev, V. I. (2007). *Measure Theory*. Berlin, Heidelberg: Springer. DOI: 10.1007/978-3-540-34514-5.

Bonanno, G. and Molica Bisci, G. (2010). "Infinitely many solutions for a Dirichlet problem involving the p-Laplacian". *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* **140**, 737–752. DOI: https://doi.org/10.1017/S0308210509000845.

Brezis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York, NY: Springer. DOI: 10.1007/978-0-387-70914-7.

Burenkov, V. I. and Gusakov, V. A. (1987). "On exact constants in Sobolev embedding theorems". *Doklady Akademii Nauk SSSR* **294**(6), 1293–1297. URL: http://mi.mathnet.ru/dan8079.

Cabada, A. and Heikkilä, S. (2002). "Implicit nonlinear discontinuous functional boundary value φ-Laplacian problems: extremality results". *Applied Mathematics and Computation* **129** (2-3), 537–549. DOI: 10.1016/S0096-3003(01)00061-3.

Carl, S., Le, V. K., and Motreanu, D. (2007). *Nonsmooth Variational Problems and Their Inequalities*. Springer Monographs in Mathematics. New York, NY: Springer. DOI: 10.1007/978-0-387-46252-3.

Chabrowski, J. H. (1997). *Variational Methods for Potential Operator Equations: With Applications to Nonlinear Elliptic Equations*. Vol. 24. De Gruyter Series in Mathematics. Berlin: De Gruyter. DOI: 10.1515/9783110809374.

Cubiotti, P. and Yao, J.-C. (2015). "On the two-point problem for implicit second-order ordinary differential equations". *Boundary Value Problems* **2015**, 211. DOI: 10.1186/s13661-015-0475-5.

De Giorgi, E., Buttazzo, G., and Dal Maso, G. (1983). "On the lower semicontinuity of certain integral functionals". *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni*. 8th ser. **74**(5), 274–282. URL: https://eudml.org/doc/287314.

Denkowski, Z., Migórski, S., and Papageorgiou, N. S. (2003). *An Introduction to Nonlinear Analysis: Theory*. New York, NY: Springer. DOI: 10.1007/978-1-4419-9158-4.

Dinca, G., Jebelean, P., and Mawhin, J. (2001). "Variational and topological methods for Dirichlet problems with p-Laplacian". *Portugaliae Mathematica. Nova Série* **58**(3), 339–378. URL: http://eudml.org/doc/49321.

Heikkilä, S. and Seikkala, S. (2005). "On singular, functional, nonsmooth and implicit phi-Laplacian initial and boundary value problems". *Journal of Mathematical Analysis and Applications* **308**(2), 513–531. DOI: 10.1016/j.jmaa.2004.11.033.

Himmelberg, C. J. (1975). "Measurable relations". *Fundamenta Mathematicae* **87**(1), 53–72. URL: https://eudml.org/doc/214809.

Klein, E. and Thompson, A. C. (1984). *Theory of Correspondences: Including Applications to Mathematical Economics*. Canadian Mathematical Society Series of Monographs and Advanced Texts. New York, NY: John Wiley and Sons Inc.

Kucia, A. (1991). "Scorza Dragoni type theorems". *Fundamenta Mathematicae* **138**, 197–203. DOI: 10.4064/fm-138-3-197-203.

Lou, H. (2008). "On singular sets of local solutions to p-Laplace equations". *Chinese Annals of Mathematics, Series B* **29**, 521–530. DOI: 10.1007/s11401-007-0312-y.

Marano, S. A. (2012). "On a Dirichlet problem with p-Laplacian and set-valued nonlinearity". *Bulletin of the Australian Mathematical Society* **86**(1), 83–89. DOI: 10.1017/S0004972711002905.

Marino, G. and Paratore, A. (2021). "Implicit equations involving the p-Laplace operator". *Mediterranean Journal of Mathematics* **18**, 74. DOI: 10.1007/s00009-021-01713-9.

Naselli Ricceri, O. and Ricceri, B. (1990). "An existence theorem for inclusions of the type $\Psi(u)(t) \in F(t, \Phi(u)(t))$ and application to a multivalued boundary value problem". *Applicable Analysis* **38**(4), 259–270. DOI: 10.1080/00036819008839966.

Peral, I. (1997). *Multiplicity of solutions for the p-Laplacian*. Second School of Nonlinear Functional Analysis and Applications to Differential Equations. Trieste, Italy: International Center for Theoretical Physics. URL: https://matematicas.uam.es/~ireneo.peral/ICTP.pdf.

Ricceri, B. (1982). "Sur la semi-continuité inférieure de certaines multifonctions". *Comptes Rendus des Seancés de l'Académie des Sciences*. 1st ser. **294**(7), 265–267. URL: https://gallica.bnf.fr/ark:/12148/bpt6k5532920h/f25.item.

Shah, K., Hussain, W., Thounthong, P., Borisut, P., Kumam, P., and Arif, M. (2018). "On nonlinear implicit fractional differential equations with integral boundary condition involving p-Laplacian operator without compactness". *Thai Journal of Mathematics* **20**, 301–321. URL: https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/744.

Talenti, G. (1987). "Some inequalities of Sobolev type on two-dimensional spheres". In: *General Inequalities 5*. Ed. by W. Walter. Vol. 80. International Series of Numerical Mathematics. Basel: Birkhäuser, pp. 401–408. DOI: 10.1007/978-3-0348-7192-1_32.

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