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Opportunity-based other-regarding preferences in general equilibrium: existence

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Abstract
We consider a pure exchange economy with a finite number of goods and households. Following the path-breaking paper by Kranich, we introduce two differences with respect to the standard model: (i) each household utility function depends not only on her own consumption but also on other households' welfare, measured by wealth; then, other-regarding preferences are based on other households opportunities; (ii) households are allowed to promise transfers to other households and the promises are bound to be honored. We provide a proof of existence of equilibria, solving several problems which are not addressed by the paper by Kranich. We present a robust example of non-existence of equilibria if the crucial assumption of an upper bound on transfers is violated. As a mathematical by-product of our analysis, we give easy-to-check conditions that ensure that commonly used constraint set-valued functions satisfy properties that are needed to apply standard existence theorems.

Keywords General equilibrium · Exchange economies · Other-regarding preferences · Existence · Non-existence of equilibria

Mathematics Subject Classification 91A80 · 91B50

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1 Introduction

Inspired by Kranich (1988), our paper studies the existence and non-existence problem in a general equilibrium model with a specific form of other-regarding preferences and the possibility of transfers of goods among individuals. To better understand the role of that model in the literature, we very briefly review the main available contributions on existence of equilibria in the general equilibrium model with so-called "imperfections". Then, we move to the analysis of the available contributions on the specific imperfection which is the object of our paper, i.e., other-regarding, or interdependent preferences. Among those papers, we distinguish between those that ignore and those that include a natural consequence of the altruistic concerns about other individuals: the possibility of providing transfers. Then, we describe the main features of our contribution.

The standard general equilibrium model analyzes an abstract economic environment in which individuals (or households) own goods or commodities and exchange them to maximize the satisfaction (or utility) they get from consuming the acquired goods. Production activities can also be easily introduced in the model. A fundamental goal of the analysis is to show existence of the so-called equilibrium prices, i.e., prices which are such that agents exchange goods following their own interest, and available resources are large enough to satisfy the total demand of goods, with no waste of desirable commodities. Even though the origin of general equilibrium dates back to Walras' work at the end of the nineteenth century, it was only in 1954 that Arrow and Debreu (1954) and McKenzie (1954), analyzed the model in a mathematical rigorous manner. They showed that under some conditions, for any economy, equilibria exist and associated distributions of resources are efficient.

The general equilibrium model is a highly simplified version of the real economic systems. Starting from the sixties, several models have been proposed to incorporate some "imperfection" in the "frictionless" Arrow–Debreu–McKenzie model to make it richer from an economic viewpoint. Indeed, in the original version of the model, there was no analysis of phenomena such as the incompleteness of financial markets or restricted participation in such markets, moral hazard and adverse selection, asymmetric and/or incomplete information, nonconvexities in consumption or production, absence of perfect foresight and temporary equilibrium, other-regarding-preferences and possibly related, altruistic or malevolent, behavior with respect to other households. A great effort has been made in the economic literature to present an analysis of the existence problem in the general equilibrium model enriched with one of the above-described features (see e.g. Bisin and Gottardi 2006; Bonnisseau and Cornet 1990; Prescott and Townsend 1984; Radner 1979; Rotschild and Stiglitz 1976; Rustichini and Siconolfi 2005; Villanacci et al. 2002 and the references therein). In any of the consistently obtained models, existence and non-existence results have been provided under different assumptions.

Different theoretical approaches have been used to show existence of equilibria in general equilibrium models. The most important are theorems of fixed points, variational inequalities, homotopy and degree, generalized games and quasi-equilibria. Indeed, those approaches are logically related one to the other and they show that some basic features of the model under analysis are necessary conditions to get existence. Those conditions are first of all related to the basic definition of the model and they can be summarized by the need for a substantial agreement on the values of some crucial exogenously given variables among individuals acting in the economy and the competitive behavior of utility-maximizing and profit-maximizing subjects. Moreover, from a more formal viewpoint, the needed conditions require some form of convexity and compactness of the actions available to the individuals and some form of continuity of the relevant set-valued functions. Models of asymmetric or private information and of temporary equilibrium may fail to satisfy the first type of conditions. Models with incomplete markets or with production and increasing return to scale and indivisibility fail to meet the second set of conditions: continuity of the demand function and convexity, respectively.

The model with other-regarding preferences and transfers displays non-existence results due to the non-compactness of the set of feasible promises of transfers. Moreover, all the available existence proofs assume some form of quasi-concavity on household choices, but this assumption is meaningless in the case of malevolent preferences. In our paper, we provide a solution to that problem—see Sect. 2.2.3.

Any contribution briefly described below presents a general equilibrium model with other-regarding preferences and, possibly, some other features and, studies existence and, in some cases, efficiency, production and regularity of equilibria. The main goal of our contribution is provide information on the existence.

In the well-known general equilibrium textbook by Arrow and Hahn (1971), a proof of existence is presented for a model with production and preferences described by utility functions. A similar independent result is provided by Laffont and Laroque (1972). Borglin (1973) presents a pure exchange economy where consumers have utility functions that depend upon the selfish utility of any individual in the economy (i.e., a Walrasian utility of the individual vector of consumption). He shows that a Pareto optimum is an equilibrium relative to a price system, and that competitive equilibria exist, under a somehow strong selfish assumption. Shafer and Sonnenschein (1975) provide a proof of existence in an exchange economy model with tastes described by a preference correspondence or set-valued function, and indeed preferences are other-regarding (or interdependent), price dependent and not necessarily transitive or complete. Similar results are obtained in the book by Florenzano (2003) in a model with consumption and production, using the so-called quasi-equilibrium approach. Gersbach and Haller (1999) present a model with a finite set of individuals who coordinate their behavior as households, i.e., the goal of each household is to find a Pareto optimal allocation among its members. Then, using sufficiently strong concavity assumptions, they easily provide an existence result. Del Mercato (2006) studies a model in which the consumption choices of each household affect both the utility function and the constraint set of herself and any other household in the economy. Under some relatively general smooth assumptions, equilibria are proven to exist.

Dufwenberg et al. (2011) show that if preferences are the sum of a selfish utility function and of a function that represents the impact of externalities, equilibrium prices and allocations are those of the economy without externalities. Moreover, the two theorems of welfare economics do not hold under wealth concerns even when consumers' preferences are separable in the selfish and other-regarding concerns. Balasko

(2015) analyzes a model in which a household's utility depends upon her consumption and other households' wealth. Under smooth assumptions, he proves that some crucial results in the standard exchange economies do survive: indeed equilibria exist for any economy. On the other hand, we show that some properties as the regularity of equilibrium allocations, the uniqueness of equilibrium at equilibrium allocations, and the stability of no-trade equilibria, do not resist the introduction of sufficiently large wealth concerns.

Another much smaller strand of literature recognizes the importance of other-regarding preferences as done in the papers presented above, but it also acknowledges an obvious consequence of those preferences: the possibility of transferring resources to individuals we care about.

There is ample motivation for an analysis of other-regarding behavior. In 2023, in the U.S., charitable giving from individuals amounted to 374.4 billion dollars; in Europe, more than 186,000 philanthropic organizations provided a giving of 54,4 billion euros and the non-profit sector contributed between 4.5 and 5.5 of GDP.

From a theoretical viewpoint, the role of transfers on optimal properties of equilibria deserves a careful analysis. If a household cares a lot about another household, then a transfer from the former one to the latter one may lead to a Pareto superior outcome. On the other hand, with many individuals, free-riding behavior may lead to an underprovision of transfers. Therefore, the efficiency properties of equilibria are both interesting and, at the time being, far away from being thoroughly analyzed. Indeed, in a companion paper, we do give some partial contributions on the topic—see Donato et al. (2024).

The first paper analyzing a general equilibrium model with other-regarding preferences and transfers is what we consider the absolute best contribution to other-regarding preferences in general equilibrium: the paper by Kranich (1988). He presents a model of a pure exchange economy with a finite number of goods and households and he introduces two differences concerning the standard model: i. each household utility function depends not only on her own consumption but also on other households' welfare, measured by wealth (then, other-regarding preferences are based on other households opportunities); ii. households are allowed to promise transfers to other households and the promises are bound to be honored. Several years later, Mercier Ythier (2000) analyzes a model with transfer in which other-regarding preferences are based on other people's utility of consumption. The main contribution of the paper is to discuss a quite strong assumption adopted by Kranich (1988) about an artificial upper bound on the level of transfers. He substitutes it with a relatively strong assumption on an endogenous object (the value function of the standard Walrasian consumer maximization problem, see Mercier Ythier 2000, page 52). Some crucial observations, about the model introduced by Kranich (1988), we present in our paper, apply to the model of Mercier Ythier (2000) as well—see Sects. 2.2.3 and 2.2.4.

The main contributions of our paper are the following ones. First of all, we discuss some problems contained in the set-up of the model and the proof of existence of equilibria presented in Kranich (1988). Indeed, the very definition of the maximization problem under analysis requires several observations concerning the role of the price normalizations, the quasi-concavity of the households' utility functions, the compactness of the set of admissible transfers and the possible emptiness of the constraint set.

Each of the above problems is discussed and some solutions are proposed. Those problems prevent to provide a proof under the general assumptions proposed by Kranich (1988). We then propose two different and logically independent sets of assumptions under which a proof of existence is presented. Such assumptions are less general than those provided by Kranich (1988). Following Kranich (1988), our existence result is proven imposing an artificial upper bound on the households' transfer. Therefore, we consider the example of a one-good two household Cobb–Douglas economy and we present a quite "robust" example of non-existence of equilibria, under the assumption of the absence of that bound.

With respect to the existence literature, the model we present contains a feature causing non-existence that is different from the ones presented in any other contributions described above. That feature is a sort of "insatiable altruism", which implies the loss of a crucial assumption in any proof of the existence of equilibria: compactness of the households' choice sets.

On the technical side and as a by-product of our analysis, we present a result that gives very easy-to-check conditions that are sufficient to guarantee the crucial properties of the constraint set-valued functions associated with commonly studied maximization problems in economics—see Proposition 18.

The present paper is organized as follows. In Sect. 2, we describe the set-up of the model, as introduced in Kranich (1988) and, we present, discuss and solve some of the above-mentioned problems on the basic set-up. In Sect. 3, we present the definition of equilibrium obtained as a consequence of the above discussion, and we analyze in detail how to overcome the possibility of emptiness of the constraint set defined in the households' maximization problems. Indeed, the issue can be addressed using some extension theorem for "nice" concave utility function from a subset of an Euclidean space to the whole space. Using those theorems, we present two existence results under different assumptions on the utility functions. As a consequence of our analysis, at the best of our knowledge, we can say that the existence result claimed in Kranich (1988) can be shown to be true only under assumptions that are stronger than those he proposes. In Sect. 4, we discuss in detail the non-existence problem in the model without an upper bound on transfers. We present a simple Cobb–Douglas, two households, one good version of the model and we show that there is indeed an open, nonempty, and "interesting" set of economies for which equilibria do not exist if upper bounds on transfers are not imposed. Intuition on the non-existence results is presented. Finally, Sect. 5 is dedicated to the conclusions.

2 Set-up of the model

2.1 A first version

We describe an economy in which households exchange goods (or commodities) in order to maximize their well-being. A commodity is denoted by $c \in \{1, \dots, C\} := \mathcal{C}$.

A household is denoted by $h \in \{1, \dots, H\} := \mathcal{H}$ and she is described by the following objects[1]:

A consumption set $X_h \subseteq \mathbb{R}^C$ with generic element $x_h = \left(x_h^c\right)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $x_h^c \in \mathbb{R}$ denotes the consumption of good c by household h;

An endowment set $E_h \subseteq \mathbb{R}^C$ with generic element $e_h = \left(e_h^c\right)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $e_h^c \in \mathbb{R}$ denotes the amount of good c owned by household h;

A transfer vector $t_h = (t_{hh'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{C(H-1)} := T_h$, where $t_{hh'} = \left(t_{hh'}^c\right)_{c \in \mathcal{C}}$ and $t_{hh'}^c$ denotes the transfer of good c from household h to household h'. We also define $t_{\setminus h} = (t_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{(H-1)C(H-1)} := T_{\setminus h}$ and $t = (t_h, t_{\setminus h}) \in R^{C(H-1)H} := T$.

Commodities can be exchanged with other commodities at exchange ratios described by a price vector p belonging to a price set $P \subseteq \mathbb{R}^C$.

To describe households' utility functions, we need some preliminary definitions. $\theta_h \in \Theta \subseteq \mathbb{R}$ is household h's wealth. We also define $\theta = (\theta_h)_{h \in \mathcal{H}} \in \Theta^H$ and $\theta_{\setminus h} = (\theta_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \Theta^{H-1}$.

Household h's wealth function depends upon the value of her initial endowment and net transfers and it is denoted and defined as follows.[2]

$$\widehat{w}_h : P \times \mathbb{R}^{C(H-1)H} \longrightarrow \Theta, \quad (p, t) \mapsto p e_h + p \sum_{h' \in \mathcal{H} \setminus \{h\}} t_{h'h} - p \sum_{h' \in \mathcal{H} \setminus \{h\}} t_{hh'},$$

i.e., household h's wealth is sum of the values of her endowment and of the transfers she receives minus the value of the transfers she delivers to other people.

Household h's utility function depends upon her own consumption and anyone else's wealth and it is denoted and defined as follows.

$$u_h : X_h \times \Theta^{H-1} \longrightarrow \mathbb{R}, \quad \left(x_h, \theta_{\setminus h}\right) \mapsto u_h\left(x_h, \theta_{\setminus h}\right).$$

For "physical/biological" reasons, we assume non-negativity of consumption. Using a standard "survival assumption", we endow households with a strictly positive vector of goods. For institutional reasons (households are not allowed to "steal" goods), we assume non-negativity of the transfer vectors. Consistently with the monotonicity assumption with respect the consumption of goods, we are going to impose on the utility function, we restrict prices to belong to $\mathbb{R}_+^C \setminus \{0\}$. Since wealth is going to be completely used to buy goods that are consumed, wealth as well is going to be nonnegative.

Summarizing, we assume that for any $h \in \mathcal{H}$, $X_h = \mathbb{R}_+^C$, $E_h = \mathbb{R}_{++}^C$, $T_h = \mathbb{R}_+^{C(H-1)}$ and $P = \mathbb{R}_+^C \setminus \{0\}$, $\Theta \subseteq \mathbb{R}_+$.[3] Then the utility function is specified as follows,

$$u_h : \mathbb{R}_+^C \times \Theta^{H-1} \longrightarrow \mathbb{R}, \quad \left(x_h, \theta_{\setminus h}\right) \mapsto u_h\left(x_h, \theta_{\setminus h}\right). \tag{1}$$

[1] Economically meaningful restrictions on the sets defined below will be presented in the remainder of the section.

[2] Conditions will be imposed below to get a well given definition of the wealth function.

[3] See (2) below for the a further specification of Θ.

A sort of naive version of household $h \in \mathcal{H}$ maximization problem is defined as follows.

Definition 1 For given $e_h \in \mathbb{R}^C_{++}$, $p \in P$, $t_{\backslash h} \in T_{\backslash h}$, u_h as defined in (1),

$$\max_{(x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{C(H-1)}} \quad u_h\left(x_h, \left(p\left(e_{h'} + \sum_{h''\in\mathcal{H}\backslash\{h'\}}(t_{h''h'} - t_{h'h''})\right)\right)_{h'\in\mathcal{H}\backslash\{h\}}\right)$$
s.t.
$$px_h \leq p\left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}}(t_{h'h} - t_{hh'})\right), \quad x_h \geq 0, \quad t_h \geq 0.$$

The Definition above requires a careful discussion. In each of the four subsections below, we present a problem related to that definition and a proposal on how to address that problem.

2.2 A discussion of the set-up of the model

2.2.1 Indeterminacy: price normalization matters

Following Kranich (1988), in Definition 1, we assumed that the utility function of household h depends upon other households' nominal wealth. Then, "normalizing price", i.e., multiplying prices by a strictly positive real number, affects the value of u_h (unless u_h is homogenous of degree zero in prices). In other words, different choices of normalization of prices give rise to different equilibria and there is no natural choice of normalization.

To avoid the fact that equilibrium allocations are normalization-dependent, we propose a simple change in the model: we substitute the wealth of other households in the utility function with the "relative wealth". There is indeed a vast literature in partial equilibrium, game theory, and behavioral economic analysis that follows this approach (see Dhami 2016, Chapter 6 and the references quoted there). Indeed, what is important in the analysis of other-regarding preferences that are opportunity-based is not (the value of) the amount of goods other households own, but those amounts compared with what is generally available in the economy, i.e., the value of the total resources $r := \sum_{h\in\mathcal{H}} e_h$.

We can then write the household h's utility function as

$$u_h\left(x_h, \left(\frac{p(e_{h'} + \sum_{h''\in\mathcal{H}\backslash\{h'\}} t_{h''h'} - \sum_{h''\in\mathcal{H}\backslash\{h'\}} t_{h'h''})}{pr}\right)_{h'\in\mathcal{H}\backslash\{h\}}\right),$$

Under the above specification of the utility function, it is immediate to see that the price normalization does not affect households' maximizing choices. Indeed, a convenient normalization[4] is to divide prices by the strictly positive amount pr. In other words, we assume $p \in S := \left\{p' \in \mathbb{R}^C_+ : p'r = 1\right\}$ and, with innocuous abuse of

[4] The normalization we choose is the one introduced in Kranich (1988).

notation for the symbols denoting the price and the wealth, which is now the relative wealth, we can rewrite the utility function and the objective function as

$$u_h : \mathbb{R}_+^C \times [0, 1]^{H-1} \longrightarrow \mathbb{R}, \quad \left(x_h, \theta_{\setminus h}\right) \mapsto u_h\left(x_h, \theta_{\setminus h}\right), \tag{2}$$

and

$$u_h\left(x_h, \left(p(e_{h'} + \sum_{h'' \in \mathcal{H}\setminus\{h'\}} t_{h''h'} - \sum_{h'' \in \mathcal{H}\setminus\{h'\}} t_{h'h''})\right)_{h' \in \mathcal{H}\setminus\{h\}}\right),$$

respectively.

2.2.2 Compactness of the choice set at the economy level

In many proofs of existence of equilibria, it is provisionally assumed that the consumption vectors of each household are bounded above by a vector bigger than total resources, which are indeed a "natural upper bound" on households' consumption vectors. Using a standard trick, it is then shown the upper bound is never reached in equilibrium. Then, "an equilibrium with an upper bound on consumption" is "an equilibrium without upper bound on consumption"—see Proposition 7. The procedure described above does not work in the case of transfers, for which no natural, physical bound exists. Following Kranich (1988), we assume that there exists an artificial bound on transfers. Indeed, without an upper bound on transfers, equilibria may not exist—see Proposition 13 and the discussion presented in Sect. 4. Formally, we assume that for any $h \in \mathcal{H}$, there exists $k'_h = \left(k'_{hh'}\right)_{h' \in \mathcal{H}\setminus\{h\}} \in \mathbb{R}_{++}^{C(H-1)}$ such that $t_h \leq k'_h$.

2.2.3 The role of quasi-concavity

In the existence theorem we are going to use, it is required the utility function has to be quasi-concave in each households'choice variables—see Theorem 3 below. That condition easily follows the quasi-concavity in the utility function arguments. Unfortunately, while quasi-concavity in the goods and "the wealth of people we like" is a widely accepted and reasonable assumption, it is not so in terms of the wealth of people we dislike, as briefly explained below.

Remark 1 Let $a \in \mathbb{R}_{++}$ and the utility function $u : \mathbb{R}_{++}^2 \longrightarrow \mathbb{R}$ such that $(x, y) \mapsto \ln(x) - a\ln(y)$ be given. Then u is a function of a good (whose quantity is x) and a bad (whose quantity is y). As a simple application of the Implicit Function Theorem, it is easy to verify that associated indifference curves are (increasing) and strictly concave if $a > 1$ and strictly convex if $a \in (0, 1)$.

Then, some work is required to deal with the above tension between "needed assumptions" and "realistic assumptions". The main idea of the analysis we present below is what follows. If you dislike someone, then you transfer nothing to her; therefore, there is no loss of generality in "ignoring" the part of the utility function related to households who are disliked.

To proceed in our argument, we need to define the set of people household h likes. That definition is based upon the following Assumption.

Assumption. For any $h \in \mathcal{H}$, there exists an exogenously given partition[5] of $\mathcal{H} \setminus \{h\}$ in the two sets $\mathcal{B}_h$ and $\mathcal{B}_h^{\setminus}$ such that u_h is strictly increasing in $\theta_{h'}$ for any $h' \in \mathcal{B}_h$ and decreasing in $\theta_{h'}$ for any $h' \in \mathcal{B}_h^{\setminus}$.

We are going to refer to $\mathcal{B}_h$ and $\mathcal{B}_h^{\setminus}$ as the sets of individuals that household h likes and dislikes, respectively. Then,

1. we define a concept of equilibria in which each household "ignores" people she dislikes, i.e. households who are disliked by household h, by construction, do not receive transfers from household h;
2. we show that the original and the newly introduced so-called $\mathcal{B}$-equilibria—see Definition 4—are equivalent.

"Appendix A" presents a detailed description of the above two steps.

Using what was said above, it is enough to reasonably assume (quasi-)concavity in the wealth of people we like in order to get (quasi-)concavity with respect to household h choice variables—as verified in Step 3 in Proposition 4. No (unreasonable) concavity assumption is needed on the wealth of people you dislike, simply because that wealth does not depend upon your choice variables.

Remark 2 In Kranich (1988), Kranich assumes quasi-concavity of the utility function in both consumption and any other household wealth.

2.2.4 Emptiness of the constraint set

The main point of this section can be easily made considering the case in which there are three households who like everybody else and one commodity. Assume that, for any $h \in \{1, 2, 3\}$, $k_h = (4, 4)$. Let's describe the constraint set of the maximization problem for household $h = 1$. For given $e_1, e_2, e_3 \in \mathbb{R}_{++}$, $t_2, t_3 \in \mathbb{R}^2_+$, $p \in S$, we have

$$\begin{aligned}&\Gamma_1(p, t_{21}, t_{23}, t_{31}, t_{32})\\&\quad = \{(x_1, t_{12}, t_{13}) \in \mathbb{R}^3 : \ \ p(x_1 + t_{12} + t_{13}) \leq p(e_1 + t_{21} + t_{31}),\\&\qquad t_{12} \in [0, 4]\,,\ t_{13} \in [0, 4]\,,\ \ x_1 \in \mathbb{R}_+,\\&\qquad p(e_2 + t_{12} + t_{32} - t_{21} - t_{23}) \in [0, 1]\,,\ \ p(e_3 + t_{13} + t_{23} - t_{31} - t_{32}) \in [0, 1]\}\end{aligned}$$

where the last two conditions follow from the very definition of the domain of the utility function presented in (2).[6]

It is easy to observe that Γ_1 may be empty for "many" values of the variables which are taken for given by household 1. Indeed, household 2's wealth may take negative

[5] We allow the possibility that either $\mathcal{B}_h$ or $\mathcal{B}_h^{\setminus}$ may be empty. We also use the standard convention that $\sum_{x \in \varnothing} x = 0$.

[6] Usually, conditions requiring the choice variables to belong to the domain of the objective functions are not "repeated" in the definition of the constraint set. In some of the literature on the model we are analyzing, that convention lead to ignoring that requirement. That is the reason why we decided to add it explicitly in the description of Γ_1.

values if, t_{21} and t_{32} are small, t_{23} is large and household 1's budget constraint does not allow to choose t_{12} to compensate that e_1 is small and household 1 receives no transfers. For example, if $e_1 = e_2 = 1, t_{21} = t_{31} = t_{32} = 0$ and $t_{23} = 3$, and since $p > 0$, the household 1's budget constraint and the nonnegativity constraint on household 2's wealth become

$$x_1 + t_{12} + t_{13} \leq 1 \text{ and } e_2 - t_{21} - t_{23} + t_{32} + t_{12} \in [0, 1],$$

from which it follows $t_{12} \leq 1$ and $t_{12} \in [2, 3]$.

To avoid the above-described problem, we introduce a fictitious utility function that extends the utility defined in (2), in order to allow negative wealth and consumption. We then construct a game associated with "the original economy with the extended utility function"; we show that the game has an equilibrium and finally that equilibrium is an equilibrium of the "true" economy. The above strategy of proof is presented in detail in the following Sect. 3.

3 Existence of equilibria

In this section, we want to write the definition of an economy and associated equilibrium, keeping into account the discussion presented in the previous section. To do that we need some preliminary definitions. For any $h \in \mathcal{H}$, we define

$$\mathcal{B}_{\to h} = \{h' \in \mathcal{H} : h \in \mathcal{B}_{h'}\}, \text{ the set of households who like } h,$$
$$\mathcal{B}^{\setminus}_{\to h} = \{h' \in \mathcal{H} : h \in \mathcal{B}^{\setminus}_{h'}\}, \text{ the set of households who dislike } h.$$
$$\mathcal{B} = (\mathcal{B}_h)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}} \mathcal{P}(\mathcal{H}\setminus\{h\}), \text{ where } \mathcal{P}(\mathcal{H}\setminus\{h\})$$
$$\text{is the family of all subsets of } \mathcal{H}\setminus\{h\},$$
$$B_h = \#\mathcal{B}_h, B = \sum_{h\in\mathcal{H}} B_h, B^{\setminus}_h = \#\mathcal{B}^{\setminus}_h, \ B_{\longrightarrow h} = \#\mathcal{B}_{\longrightarrow h}, B^{\setminus}_{\longrightarrow h} = \#\mathcal{B}^{\setminus}_{\longrightarrow h}.$$

We also define the transfers

$$\tau_h = (\tau_{hh'})_{h'\in\mathcal{B}_h} \in \mathbb{R}^{CB_h} = T_h(\mathcal{B}),$$
$$\tau_{\setminus h} = \left(\tau_{\mathcal{B}_{h'}}\right)_{h'\neq h} = ((\tau_{h'h''})_{h''\in\mathcal{B}_{h'}})_{h'\in\mathcal{H}\setminus\{h\}} \in \mathbb{R}^{C\sum_{h'\neq h} B_{h'}} = T_{\setminus h}(\mathcal{B}), \text{ and}$$
$$\tau = (\tau_h, \tau_{\setminus h}) \in T(\mathcal{B}) = T_h(\mathcal{B}) \times T_{\setminus h}(\mathcal{B}),$$

and the wealth vectors

$$\theta_{\mathcal{B}_h} = (\theta_{h'})_{h'\in\mathcal{B}_h} \in [0, 1]^{B_h}, \theta_{\mathcal{B}^{\setminus}_h} = (\theta_{h'})_{h'\in\mathcal{B}^{\setminus}_h} \in [0, 1]^{B^{\setminus}_h}$$

Remark 3 We have to introduce a new notation to denote transfers simply because $t_h \in \mathbb{R}^{C(H-1)}$ and $\tau_h \in \mathbb{R}^{CB_h}$.

Definition 2 For any $h \in \mathcal{H}$, the wealth function is $w_h : S \times T(\mathcal{B}) \to \mathbb{R}$,

$$w_h(p, \tau) = p(e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau_{h'h} - \sum_{h' \in \mathcal{B}_h} \tau_{h'h}).$$

As motivated in Sect. 2.2.3, we describe the vector of "wealth" of individuals household h does not like.

Definition 3 For any $h \in \mathcal{H}$ and any $h' \in \mathcal{B}_h^{\setminus}$, we define $q_h : S \times T_{\setminus h}(\mathcal{B}) \longrightarrow \mathbb{R}^{B_h^{\setminus}}$ as

$$q_h(p, \tau_{\setminus h}) = \left(p(e_{h'} + 0 + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \tau_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \tau_{h'h''}) \right)_{h' \in \mathcal{B}_h^{\setminus}}.$$

Remark 4 Observe that, consistently with the discussion contained in Sect. 2.2.4, we take the whole $\mathbb{R}$ and not the set [0, 1] to be the codomain of the components of the functions presented in the above Definitions 2 and 3. Of course, in equilibrium, we verify that for any $h \in \mathcal{H}$, $w_h(p, \tau), q_h(p, \tau_{\setminus h}) \in [0, 1]$.

As briefly described in Sect. 2.2.3 and formalized and proved in "Appendix A", to show existence of equilibria—see Definition 17 there—is equivalent to show existence of $\mathcal{B}$-equilibria. For the reader's convenience, we repeat the definition of $\mathcal{B}$-equilibrium. To describe the related concept of $\mathcal{B}$-economy and we also state our assumptions on the exogenous data of the model.

Assumption (i) For any $h \in \mathcal{H}$, there exists an exogenously given partition of $\mathcal{H} \setminus \{h\}$ in the two sets $\mathcal{B}_h$ and $\mathcal{B}_h^{\setminus}$ such that

$$u_h : \mathbb{R}_+^C \times [0, 1]^{B_h} \times [0, 1]^{B_h^{\setminus}} \longrightarrow \mathbb{R}, \left(x_h, \theta_{\mathcal{B}_h}, \theta_{\mathcal{B}_h^{\setminus}}\right) \mapsto u_h\left(x_h, \theta_{\mathcal{B}_h}, \theta_{\mathcal{B}_h^{\setminus}}\right) \quad (3)$$

is Lipschitz continuous; strictly increasing in $\theta_{h'}$ for any $h' \in \mathcal{B}_h$ and decreasing in $\theta_{h'}$ for any $h' \in \mathcal{B}_h^{\setminus}$; concave in $\left(x_h, \theta_{\mathcal{B}_h}\right)$. Let $\mathcal{U}$ be the set of function satisfying the above assumptions.
(ii) For any $h \in \mathcal{H}$, $e_h \in \mathbb{R}_{++}^C$;
(iii) For any $h \in \mathcal{H}$, $k_h \in \mathbb{R}_{++}^{CB_h}$.

Remark 5 The strong assumptions of concavity and continuity are needed to extended the utility function as said in Sect. 2.2.4 and analyzed in detail in Sect. 3.1 below. Some sufficient conditions for a function to be Lipschitz are presented in Corollary 1. Roughly speaking, we need to assume that the utility function has slopes in points close to zero which are bounded above.

Definition 4 A $\mathcal{B}$-economy is $\mathcal{E} := (e_h, u_h, k_h)_{h \in \mathcal{H}} \in \mathbb{R}_{++}^{CH} \times \mathcal{U} \times \mathbb{R}_{++}^{CB_h} := \mathbb{E}$.
$(\widetilde{x}, \widetilde{\tau}, \widetilde{p}) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^{CB} \times S$ is a $\mathcal{B}$- **equilibrium** for the $\mathcal{B}$-economy $\mathcal{E} \in \mathbb{E}$ if
a. for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $\widetilde{p} \in S$ and $\widetilde{\tau}_{\setminus h} \in \mathbb{R}^{C \sum_{h' \neq h} B_{h'}}$,

$(\widetilde{x}_h, \widetilde{\tau}_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ solves

$$\begin{array}{l} \max_{(x_h,\tau_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} \\ u_h\left(x_h, (\widetilde{p}(e_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \widetilde{\tau}_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \widetilde{\tau}_{h'h''}))_{h'\in\mathcal{B}_h}, q_h(\widetilde{p}, \widetilde{\tau}_{\setminus h})\right) \\ \text{s.t.} \\ (x_h, \tau_h) \in \Gamma_h\left(\widetilde{p}, \widetilde{\tau}_{\setminus h}\right), \end{array}$$

where

$$\begin{array}{l} \Gamma_h : S \times T_{\setminus h}(\mathcal{B}) \longrightarrow\longrightarrow \mathbb{R}^C \times \mathbb{R}^{CB_h}, \\ (\widetilde{p}, \widetilde{\tau}_{\setminus h}) \mapsto\mapsto \Big\{(x_h, \tau_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \widetilde{p}x_h \le \widetilde{p}\left(e_h + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} \widetilde{\tau}_{h'h} - \sum_{h'\in\mathcal{B}_h} \tau_{hh'}\right) \\ \qquad x_h \ge 0, \quad \tau_h \ge 0, \quad \tau_h \le k_h.\Big\} \end{array} \tag{4}$$

b. markets clear, i.e., $\sum_{h\in\mathcal{H}}(\widetilde{x}_h - e_h) = 0$.

The reason for which the above definition contains no "market clearing conditions for transfers" is explained in the simple result below.

Proposition 1 *For any $t \in \mathbb{R}^{HCB}$,*

$$\sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_h} t_{hh'} = \sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_{\longrightarrow h}} t_{h'h}. \tag{5}$$

Proof By definition of $\mathcal{B}_{\longrightarrow h'}$, we have

$$h \in \mathcal{B}_{\longrightarrow h'} \Leftrightarrow h' \in \mathcal{B}_h. \tag{6}$$

Defined $\mathcal{S} = \left\{(h, h') \in \mathcal{H}^2 : h' \in \mathcal{B}_h\right\}$ and $\mathcal{T} = \left\{(h', h) \in \mathcal{H}^2 : h' \in \mathcal{B}_{\longrightarrow h}\right\}$, we show that $\mathcal{S} = \mathcal{T}$. Indeed, $\mathcal{S} := \left\{(h, h') \in \mathcal{H}^2 : h' \in \mathcal{B}_h\right\} \overset{(6)}{=} \left\{(h, h') \in \mathcal{H}^2 : h \in \mathcal{B}_{\longrightarrow h'}\right\} = \left\{(h', h) \in \mathcal{H}^2 : h' \in \mathcal{B}_{\longrightarrow h}\right\} := \mathcal{T}$. □

To provide a proof of existence of equilibrium, we need to give the definition of equilibrium with an upper bound on consumption. To this aim, we introduce the upper bounds k_x on the consumption vectors.

Definition 5 $r^m = \min\{r^c : c \in \mathcal{C}\}$; $r^M = \max\{r^c : c \in \mathcal{C}\}$; $e_h^m = \min\left\{e_h^c : c \in \mathcal{C}\right\}$. $\widetilde{k}_x = \frac{1}{r^m} \cdot \sum_{\mathbf{c}\in\mathcal{C}} \sum_{h\in\mathcal{H}} \left(e_h^c + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} k_{h'h}^c\right) + 1 \in \mathbb{R}_{++}$; $\widetilde{k}_x^c = \max\left\{\widetilde{k}_x,\ r^c + 1, \widetilde{k}_x \cdot r^c\right\} \in \mathbb{R}_{++}$ and $k_x = \left(\widetilde{k}_x^c\right)_{c\in\mathcal{C}} \in \mathbb{R}_{++}^C$.

The following result can be easily shown.

Lemma 1 *For any $p \in S$, 1. for any $c \in \mathcal{C}$, $p^c \le \frac{1}{r^m}$; 2. there exists $c \in \mathcal{C}$ such that $p^c \ge \frac{1}{C \cdot r^M}$; 3. for any $e_h \in \mathbb{R}_{++}^C$, $pe_h \ge \frac{e_h^m}{C \cdot r^M}$.*

Remark 6 For any $p \in S$, $h \in \mathcal{H}$, $h' \in \mathcal{B}_h$ and $\tau = \left((\tau_{hh'})_{h'\in\mathcal{B}_h}\right)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}}\left(\times_{h'\in\mathcal{B}_h} [0, k_{hh'}]\right) \subseteq \mathbb{R}^{C\cdot\Sigma_{h\in\mathcal{H}} B_h}$, we get $w_h(p, \tau) < \widetilde{k}_x$.

Definition 6 An equilibrium with an upper bound on consumption is defined as in Definition 4 apart from the constraint set which is instead

$$\begin{aligned}&\widehat{\Gamma}_h : S \times T_{\setminus h}(\mathcal{B}) \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}^C \times \mathbb{R}^{CB_h},\\&\left(p, \tau_{\setminus h}\right) \mapsto\!\!\!\!\mapsto \Gamma_h\left(p, \tau_{\setminus h}\right) \cap \left\{(x_h, \tau_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : x_h \leq k_x\right\}.\end{aligned} \tag{7}$$

We show existence of equilibria using the definition of a so-called generalized game—see Definition 8 below—and a main associated result—see Theorem 3 below.

More precisely, following also what we said in Sect. 2.2, our strategy of proof goes through the following steps—each of which is the content of Sections from 3.1 to Sect. 3.4.

1. We present conditions that insure existence of the needed extension of the utility function.
2. We describe a generalized game associated with the economy under analysis and verify that game satisfies the sufficient conditions stated in Theorem 3 and therefore has a Nash equilibrium.
3. We show equilibria of the generalized game are such that the associated wealth is positive and therefore they are equilibria with an upper bound on consumption—as in Definition 6—of the economic model under analysis.
4. Using a standard trick, we show that an equilibrium with an upper bound on consumption is an equilibrium—as in Definition 4.

3.1 Extension of the utility function

It is usually assumed that for any $h \in \mathcal{H}$, the utility function u_h is continuous and quasi-concave on $\mathbb{R}^n_+$, for appropriately chosen $n \in \mathbb{N}$. We would like to extend u_h to a function $U_h : \mathbb{R}^n \longrightarrow \mathbb{R}$, which is still continuous and quasi-concave. Unfortunately, the following statement is false (see De Bernardi and Vesely 2023, page 7).

Conjecture 1 *Let the following objects be given. (1) A convex set $S \subseteq \mathbb{R}^n$; (2) a (Lipschitz) continuous, quasi-convex function $f : S \longrightarrow \mathbb{R}$. Then there exists a function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that 1. F is an extension of f (i.e., $F_{|S} = f$), 2. F is quasi-convex and 3. F is continuous.*

Indeed, stronger assumptions on u_h have to be added, as described in the result below.

Proposition 2 (see McShane 1934[7]) *Let A be a convex subset of a normed space X. If $g : A \to \mathbb{R}$ is an L-Lipschitz concave function then it admits an L-Lipschitz concave extension G to the whole X; moreover, such an extension G can be defined by the formula*

$$G(x) = \sup_{y \in A}\left[g(y) - L\|x - y\|\right], \qquad x \in X.$$

[7] A more recent reference is Borwein and Vanderwerff (2010), Exercise 8.3.4, p. 399.

Lemma 2 *The function* $G_h : \mathbb{R}^C \times \mathbb{R}^{B_h} \times \mathbb{R}^{B_h^\setminus} \longrightarrow \mathbb{R}$,

$$\left(\xi_h, \eta_{\mathcal{B}_h}, \eta_{\mathcal{B}_h^\setminus}\right) \mapsto \sup_{(x_h, \theta_{\mathcal{B}_h}) \in \mathbb{R}_+^C \times [0,1]^{B_h}}$$

$$\left[u_h\left(x_h, \theta_{\mathcal{B}_h}, \left(\max\{0, \eta_{h'}\}_{h' \in \mathcal{B}_h^\setminus}\right)\right) - L\left\|\left(\xi_h, \eta_{\mathcal{B}_h}\right) - \left(x_h, \theta_{\mathcal{B}_h}\right)\right\|\right]$$

is an extension of u_h*, defined in* (3)*. Moreover, it is concave in* $\left(\xi_h, \eta_{\mathcal{B}_h}\right)$ *and it is continuous on* $C_1 \times C_2$*, where* C_1 *is a compact subset of* $\mathbb{R}^C \times \mathbb{R}^{B_h}$ *and* C_2 *is a compact subset of* $\mathbb{R}^{B_h^\setminus}$*.*

Proof The concavity result follows immediately from Proposition 2, identifying X, A and g with $\mathbb{R}^C \times \mathbb{R}^{B_h}$, $\mathbb{R}_+^C \times [0,1]^{B_h}$ and $u_h(\cdot, \theta_{\mathcal{B}_h^\setminus})$, respectively. The continuity follows from the definition of G_h and from the fact that if a function $\omega : C \times K \longrightarrow \mathbb{R}$, $(X, Z) \mapsto \omega(x, z)$ defined on a compact subset $C \times K \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is continuous, then the function $\Omega : C \longrightarrow \mathbb{R}$, $z \mapsto \sup_{x \in K} \omega(x, z)$ is continuous. □

We can then give the following definition.

Definition 7 $(\widetilde{x}, \widetilde{\tau}, \widetilde{p}) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^{CB} \times S$ is a $\mathcal{B}$-**equilibrium** with extended utility function and upper bound on consumption for the $\mathcal{B}$-economy $\mathcal{E} \in \mathbb{E}$ if

(a) for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $\widetilde{p} \in S$ and $\widetilde{\tau}_{\setminus h} \in \mathbb{R}^{C \sum_{h' \neq h} B_{h'}}$, $(\widetilde{x}_h, \widetilde{\tau}_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ solves

$$\max_{(x_h, \tau_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}}$$
$$G_h\left(x_h, (\widetilde{p}(e_{h'} + \tau_{hh'} + \textstyle\sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{\tau}_{h'h''}))_{h' \in \mathcal{B}_h}, q_h(\widetilde{p}, \widetilde{\tau}_{\setminus h})\right)$$
s.t.
$$(x_h, \tau_h) \in \widehat{\Gamma}_h\left(\widetilde{p}, \widetilde{\tau}_{\setminus h}\right),$$

(b) markets clear.

3.2 The generalized game associated with the economy

In this section, we define a generalized game, the related concept of Nash equilibrium, and a standard result on existence. We then construct a generalized game associated with the model we are analyzing. We finally show that a Nash equilibrium in that generalized game does exist.

Definition 8 (Kreps 2013, page 339) Given $n \in \mathbb{N}$, an n-player generalized game is a triple $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$, where for any $i \in \{1, \ldots, n\}$,

1. A_i is a set of strategies or actions with generic element a_i;
2. $C_i : A_{\setminus i} := \times_{j \in \{1,\ldots,n\} \setminus \{i\}} A_j \longrightarrow\!\!\!\!\longrightarrow A_i$, $a_{\setminus i} = (a_j)_{j \in \{1,\ldots,n\} \setminus \{i\}} \mapsto\!\!\!\!\mapsto C_i\left(a_{\setminus i}\right)$ is a constraint set-valued function;
3. $\pi_i : A := \times_{i \in \{1,\ldots,n\}} A_i \longrightarrow \mathbb{R}$, $a \mapsto \pi_i(a)$ is a utility or payoff function.

Definition 9 A Nash equilibrium for the generalized game $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$ is an n-tuple of actions $a^* := \left(a_i^*\right)_{i=1}^n \in A$ such that for any $i \in \{1, \ldots, n\}$, a_i^* solves the following problem. For given $a_{\setminus i}^* := \left(a_j^*\right)_{j \in \{1,\ldots,n\} \setminus \{i\}} \in A_{\setminus i}$,

$$\max_{a_i \in A_i} \pi_i(a_i, a_{\setminus i}^*) \quad \text{s.t.} \quad a_i \in C_i(a_{\setminus i}^*).$$

Theorem 3 (Debreu 1952. See also Kreps 2013, p. 340) *Let a generalized game* $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$ *be given. If for any* $i \in \{1, \ldots, n\}$

1. *there exists* $n_i \in \mathbb{N}$ *such that* A_i *is a nonempty, compact, convex subset of* $\mathbb{R}^{n_i}$*;*
2. C_i *is a non-empty valued, convex valued, lower hemicontinuous and upper hemicontinuous set-valued function;*
3. π_i *is a continuous function and for any* $a_{\setminus i} \in A_{\setminus i}$*, the function* $\pi_i\left(\cdot, a_{\setminus i}\right) : A_i \longrightarrow \mathbb{R}$, $a_i \mapsto \pi_i\left(a_i, a_{\setminus i}\right)$ *is quasi-concave,*

then $\mathcal{G}$ *has a Nash equilibrium.*

We now define the generalized game associated with an economy $\mathcal{E}$ we are going to use.

Definition 10 There are $n = 1 + H$ players. For each player $h \in \{0, 1 \ldots, H\}$, we describe below *the* appropriate definition of the triple of 1. set of actions, 2. constraint set-valued functions and 3. utility functions.

1.

$$\begin{array}{l} A_0 = S \subseteq \mathbb{R}^C \\ A_h = \widehat{X}_h \times \widehat{T}_h(\mathcal{B}) \subseteq \mathbb{R}^{C + B_h \cdot C}, \quad \text{for any } h \in \mathcal{H} \end{array}$$

where for any $h \in \mathcal{H}$, $\widehat{X}_h = \left\{x_h \in \mathbb{R}^C : 0 \leq x_h \leq k_x\right\}$ and $\widehat{T}_h(\mathcal{B}) = \left\{\tau_h \in \mathbb{R}^{B_h \cdot C} : 0 \leq \tau_h \leq k_h\right\}$.

2.

$$\begin{array}{l} C_0 : \times_{h \in \mathcal{H}} A_h \longrightarrow\!\!\!\!\longrightarrow A_0 \\ C_0 : \left(\times_{h \in \mathcal{H}} \left(\widehat{X}_h \times \widehat{T}_h(\mathcal{B})\right)\right) \longrightarrow\!\!\!\!\longrightarrow S \\ (x, t) \mapsto\!\!\!\!\mapsto S; \\ C_h : A_0 \times \left(\times_{h' \in \mathcal{H} \setminus \{h\}} A_{h'}\right) \longrightarrow\!\!\!\!\longrightarrow A_h \\ \widehat{\widehat{\Gamma}}_h : S \times \left(\times_{h' \in \mathcal{H} \setminus \{h\}} \left(\widehat{X}_{h'} \times \widehat{T}_{h'}(\mathcal{B})\right)\right) \longrightarrow\!\!\!\!\longrightarrow \widehat{X}_h \times \widehat{T}_h(\mathcal{B}) \\ \left(p, (x_{h'}, \tau_{h'})_{h' \in \mathcal{H} \setminus \{h\}}\right) \mapsto\!\!\!\!\mapsto \widehat{\Gamma}_h\left(p, \tau_{\setminus h}\right) \end{array}$$

3.

$$\begin{array}{l} \pi_0 : A_0 \times (\times_{h \in \mathcal{H}} A_h) \longrightarrow \mathbb{R} \\ \pi_0 : S \times \left(\times_{h \in \mathcal{H}} \left(\widehat{X}_h \times \widehat{T}_h(\mathcal{B})\right)\right) \longrightarrow \mathbb{R} \\ \left(p, \left(x_h, \tau_h\right)_{h \in \mathcal{H}}\right) \longrightarrow p \cdot \sum_{h \in \mathcal{H}} \left(x_h - e_h\right); \\ \pi_h : A_0 \times \left(\times_{h \in \mathcal{H}} A_h\right) \longrightarrow \mathbb{R} \\ \widetilde{G}_h : S \times \left(\times_{h \in \mathcal{H}} \left(\widehat{X}_h \times \widehat{T}_h(\mathcal{B})\right)\right) \longrightarrow \mathbb{R} \\ \left(p, (x_h, \tau_h)_{h \in \mathcal{H}}\right) \mapsto G_h(x_h, (p(e_{h'} + \sum_{h'' \in \mathcal{B}_{\rightarrow h'}} \tau_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \tau_{h'h''}))_{h' \in \mathcal{B}_h}, q_h(p, \tau_{\setminus h})) \end{array}$$

Definition 11 A Nash equilibrium for the generalized game associated with a $\mathcal{B}$-economy $\mathcal{E} \in \mathbb{E}$ is a vector $(p^*, (x_h^*, \tau_h^*)_{h\in\mathcal{H}}) \in S \times (\times_{h\in\mathcal{H}}(\widehat{X}_h \times \widehat{T}_h(\mathcal{B})))$ such that

for given $(x_h^*, \tau_h^*)_{h\in\mathcal{H}}, \quad p^*$ solves

$$\max_{p\in S} \; p \cdot \sum_{h\in\mathcal{H}} \left(x_h^* - e_h\right),$$

and for any $h \in \mathcal{H}$, for given p^* and $\left(x_{h'}^*, \tau_{h'}^*\right)_{h'\in\mathcal{H}\setminus\{h\}}, \quad \left(x_h^*, \tau_h^*\right)$ solves

$$\max_{(x_h,\tau_h)\in\left(\widehat{X}_h\times\widehat{T}_h(\mathcal{B})\right)} \; G_h\left(x_h, \left(p^*\left(e_{h'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}} \tau^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}\right)\right)_{h'\in\mathcal{B}_h}, q_h\left(p^*, \tau^*_{\setminus h}\right)\right),$$

such that $(x_h, \tau_h) \in \widehat{\widehat{\Gamma}}_h(p^*, (x_{h'}^*, \tau_{h'}^*)_{h'\in\mathcal{H}\setminus\{h\}}) = \widehat{\Gamma}_h(p^*, \tau^*_{\setminus h})$.

To show our existence result, we need two Lemmas.

Lemma 3 *Let X_1, X_2 and Y be metric spaces and $\varphi : X_1 \rightarrow\rightarrow Y, x_1 \longmapsto\!\!\!\!\rightarrow \varphi(x_1)$ and $\psi : X_1 \times X_2 \rightarrow\rightarrow Y, (x_1, x_2) \longmapsto\!\!\!\!\rightarrow \varphi(x_1)$ be set valued functions.*

Then if φ satisfies any of the properties listed below, then ψ does as well: (1) non-empty valued; (2) convex valued; (3) closed; (4) compact valued; (5) lower hemi-continuous; (6) upper hemi-continuous.

Proof Obvious. □

Lemma 4 *For any $h \in \mathcal{H}$, the set-valued function $\widehat{\Gamma}_h$, defined in (7), is (1) non-empty valued; (2) convex valued; (3) closed; (4) compact valued and $Im(\widehat{\Gamma}_h) \subseteq \widehat{X}_h \times \widehat{T}_h(\mathcal{B})$; (5) lower hemi-continuous; (6) upper hemi-continuous.*

Proof Define

$$\begin{aligned}&\widetilde{\Gamma}_h : S \times \widehat{T}_{\setminus h}(\mathcal{B}) \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}^C \times \mathbb{R}^{C\cdot B_h}\\ &\widetilde{\Gamma}_h\left(p, \tau_{\setminus h}\right) = \left\{\left(x_h, \tau_h\right) \in \mathbb{R}^C \times \mathbb{R}^{C\cdot B_h} : -p\left(x_h + \textstyle\sum_{h'\in\mathcal{B}_h} \tau_{hh'}\right)\right.\\ &+p\left(e_h + \textstyle\sum_{h'\in\mathcal{B}_{\longrightarrow h}} \tau_{h'h}\right) > 0,\\ &\qquad\qquad \left. x_h >> 0,\;\; k_x - x_h >> 0,\;\; \tau_h >> 0,\;\; k_h - \tau_h >> 0\right\}\end{aligned} \tag{8}$$

and $f : S \times \widehat{X}_h \times \widehat{T}_h(\mathcal{B}) \times \widehat{T}_{\setminus h}(\mathcal{B}) \longrightarrow \mathbb{R} \times \mathbb{R}^C \times \mathbb{R}^C \times \mathbb{R}^{C\cdot B_h} \times \mathbb{R}^{C\cdot B_h}$ such that $f(p, x_h, \tau_h, \tau_{\setminus h}) = (-p(x_h + \sum_{h'\in\mathcal{B}_h} \tau_{hh'}) + p(e_h + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} \tau_{h'h}), x_h, k_x - x_h, \tau_h, k_h - \tau_h)$.

To get the desired result, we apply Proposition 18 in "Appendix C", identifying f, B and $\widetilde{B}$ with f, $\widehat{\Gamma}_h$ and $\widetilde{\Gamma}_h$, respectively.

More precisely, we have to check that

1. $\widetilde{\Gamma}_h$ is nonempty valued,
2. f is continuous and for any $j = 1, \ldots m$ and for any $\pi \in \Pi$, $f_{j|\{\pi\}}$ is Locally NonSatiated and quasi-concave,

3. $\widehat{\Gamma}_h$ is compact valued or b. $Im(\widehat{\Gamma}_h)$ is contained in a compact set. Indeed, both results hold true.

1. Take $\widetilde{x}_h = \frac{e_h}{2H} \gg 0$ and for any $c \in \mathcal{C}$ and for any $h' \in \mathcal{B}_h$, $\widetilde{\tau}^c_{hh'} = \min\left\{\frac{k^c_{hh'}}{2}, \frac{e^c_h}{2H}\right\} \in \left(0, k^c_{hh'}\right)$, the vector $(\widetilde{x}_h, \widetilde{\tau}_h)$ belongs to $\widetilde{\Gamma}_h$, indeed

$$-p\left(\widetilde{x}_h + \sum_{h'\in\mathcal{B}_h} \tau_{hh'}\right) + p\left(e_h + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} \widetilde{\tau}_{h'h}\right) \overset{(a)}{\geq} p\left(-\frac{e_h}{2H} - \frac{H-1}{2H}e_h + e_h\right) = \frac{pe_h}{2} > 0.$$

where (a) follows from the facts that for any $h' \in \mathcal{B}_h$, $\tau_{hh'} \leq \frac{e_h}{2H}$ and $B_h \leq H-1$, and also that for any $h' \in \mathcal{B}_{\longrightarrow h}$, $\widetilde{\tau}_{h'h} \geq 0$.
2. f is clearly continuous and any component function of f for fixed $\left(p, \tau_{\setminus h}\right)$ is affine and not constant a fact which implies the desired assumptions.
3. Since $\widehat{\Gamma}_h$ is defined in terms of weak inequalities via continuous function, it is closed valued. Moreover, $Im(\widehat{\Gamma}_h) \subseteq \widehat{X}_h \times \widehat{T}_h(\mathcal{B})$ and $\widehat{X}_h \times \widehat{T}_h(\mathcal{B})$ is a compact set. Since closed subsets of compact sets are compact, the desired result follows.

□

Proposition 4 *For any $\mathcal{B}$-economy $\mathcal{E} \in \mathbb{E}$, the generalized game*

$$\left(\left(S, \times_{h\in\mathcal{H}}\left(\widehat{X}_h \times \widehat{T}_h(\mathcal{B})\right)\right), \left(C_0, \left(\widehat{\widehat{\Gamma}}_h\right)_{h\in\mathcal{H}}\right), \left(\pi_0, \left(\widetilde{G}_h\right)_{h\in\mathcal{H}}\right)\right)$$

presented in Definition 10 has a Nash equilibrium (p^*, x^*, τ^*).

Proof We apply Theorem 3 and show that all its Assumptions are verified.

1. $A_0 = S$ and for any $h \in \mathcal{H}$, $A_h = \widehat{X}_h \times \widehat{T}_h(\mathcal{B})$ satisfy the needed assumptions by definition.
2. By definition of S and since $C_0 : (\times_{h\in\mathcal{H}}(\widehat{X}_h \times \widehat{T}_h(B))) \to\to S$, $(x, \tau) \mapsto\mapsto S$, the desired results follow because C_0 is the constant set-valued function and S is a compact, convex and nonempty set. Moreover, from Lemma 4, $\widehat{\Gamma}_h$ satisfies the desired properties. Then, Lemma 3, $\widehat{\widehat{\Gamma}}_h$ satisfies the desired properties as well.
3. For given $(x_h)_{h\in\mathcal{H}}$, π_0 is linear in p and therefore concave and quasi-concave.

From Lemma 2, G_h is continuous. We want to show that the function $\widetilde{G}_h : S \times (\times_{h\in\mathcal{H}}(\widehat{X}_h \times \widehat{T}_h(\mathcal{B}))) \longrightarrow \mathbb{R}$ such that

$$\left(p^*, x, \tau_h, \tau^*_{\setminus h}\right) \mapsto G_h\left(x_h, \left(p^*\left(e_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau^*_{h'h''}\right)\right)_{h'\in\mathcal{B}_h}, q_h\left(p^*, \tau^*_{\setminus h}\right)\right)$$

is continuous. To this aim, define the function which associates the argument of G_h with values of the endogenous variables as follows $\varphi_h : S \times (\times_{h\in\mathcal{H}}(\widehat{X}_h \times \widehat{T}_h(\mathcal{B}))) \longrightarrow \mathbb{R}^C \times \mathbb{R}^{B_h} \times \mathbb{R}^{B_h^\setminus}$,

$$\begin{aligned}&\left(p^*, x, \tau_h, \tau^*_{\setminus h}\right)\\ &\mapsto \left(x_h, \left(p^*\left(e_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau^*_{h'h''}\right)\right)_{h'\in\mathcal{B}_h}, q_h\left(p^*, \tau^*_{\setminus h}\right)\right). \quad (9)\end{aligned}$$

Then $\widetilde{G}_h = G_h \circ \varphi_h$ and it is clearly continuous because G_h is Lipschitz continuous and φ_h is continuous.

Finally, define the function $\widehat{G}_h : \widehat{X}_h \times \widehat{T}_h(\mathcal{B}) \longrightarrow \mathbb{R}$, $(x_h, \tau_h) \mapsto \widetilde{G}_h(p^*, x, \tau_h, \tau^*_{\setminus h})$, which is concave by construction. Then, $\widetilde{G}$ is quasi-concave in (x_h, τ_h), as desired. □

3.3 Equilibria of the game and equilibria with an upper bound on consumption

In this Section, we first show that a Nash equilibrium does involve economically meaningful, i.e., non-negative values of consumption and wealth and it implies that an appropriate version of Walras law holds true. Finally, using the above results, we show that any Nash equilibrium is a $\mathcal{B}$-equilibrium with upper bound.

Definition 12 For[8] any $(x^*_{\setminus h} := (x^*_{h'})_{h'\in\mathcal{H}\setminus\{h\}}, p^*, \tau^*_{\setminus h}) \in \mathbb{R}_+^{C(H-1)} \times S \times \widehat{T}_{\setminus h}(\mathcal{B})$,

$$\begin{aligned}&\varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})} := \varphi_h\left(\cdot; x^*_{\setminus h}, p^*, \tau^*_{\setminus h}\right) : \widehat{X}_h \times \widehat{T}_h(\mathcal{B}) \longrightarrow \mathbb{R}_+^C \times \mathbb{R}^B,\\ &\left(x_h, \tau_h\right) \mapsto \varphi_h\left(\left(x_h, \tau_h\right), x^*_{\setminus h}, p^*, \tau^*_{\setminus h}\right)\end{aligned}$$

Lemma 5 *If (x^*, p^*, τ^*) is a Nash equilibrium as presented in Definition 11, then for any $h \in \mathcal{H}$,*

1. *$w_h(p^*, \tau^*) \in [0, 1]$, and*
2.

$$\left(G_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})}\right)_{|\widehat{\Gamma}_h\left(p^*, \tau^*_{\setminus h}\right)} = \left(u_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})}\right)_{|\widehat{\Gamma}_h\left(p^*, \tau^*_{\setminus h}\right)}.$$

Proof 1. First of all, observe that for any $h \in \mathcal{H}$, by definition of $\widehat{\Gamma}_h$, we have $x^*_h \geq 0$; by definition of S, we have $p^* \geq 0$. Then, from the budget constraint,

$$\text{for any } h \in \mathcal{H},\ 0 \leq p^* x^*_h \leq p^* \left(e_h + \sum_{h'\in\mathcal{B}_{\to h}} \tau^*_{h'h} - \sum_{h'\in\mathcal{B}_h} \tau^*_{hh'}\right) = w_h\left(p^*, \tau^*\right). \quad (10)$$

[8] The definition of the function φ_h was presented in (9).

Now assume our claim is false, i.e.,

$$\text{there exists } h' \in \mathcal{H} \text{ such that } w_h\left(p^*, \tau^*\right) > 1. \tag{11}$$

Then,

$$1 \overset{p^* \in S}{=} \sum_{h \in \mathcal{H}} p^* e_h \overset{\text{Prop. 1}}{=} \sum_{h \in H} p^* \left(e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h'h} - \sum_{h' \in \mathcal{B}_h} \tau^*_{hh'} \right) \overset{(10),(11)}{>} 1,$$

which is the desired contradiction.

2. Observe that $G_{h|\mathbb{R}^C_+ \times [0,1]^{H-1}} = u_h$, and from 1. above, we have $\varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})}(\widehat{\Gamma}_h(p^*, \tau^*_{\setminus h})) \subseteq \mathbb{R}^C_+ \times [0,1]^{H-1}$, and then $(G_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})})_{|\widehat{\Gamma}_h(p^*, \tau^*_{\setminus h})} = (u_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})})_{|\widehat{\Gamma}_h(p^*, \tau^*_{\setminus h})}$

□

Proposition 5 *If (x^*, p^*, τ^*) is a Nash equilibrium for the generalized game presented above, then*

$$p^* \cdot \sum_{h \in \mathcal{H}} \left(x^*_h - e_h\right) = 0.$$

Proof The idea of the proof is as follows. Using the strict monotonicity of u_h with respect to x_h, Lemma 5 and the fact that $p^* \in S$, it is easy to show that budget constraints hold as equalities, i.e.,

$$p^* x^*_h = p^* \left(e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h'h} - \sum_{h' \in \mathcal{B}_h} \tau^*_{hh'} \right). \tag{12}$$

Then summing up with respect to households and using Proposition 1, we get the desired result.

Suppose our claim (12) does not hold, i.e.,

$$p^* x^*_h < p^* \left(e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h'h} - \sum_{h' \in \mathcal{B}_h} \tau^*_{hh'} \right) = w_h\left(p^*, \tau^*\right). \tag{13}$$

Since $p^* \in S$, then we can define $\mathcal{C}^+ = \{c \in \mathcal{C} : p^c > 0\} \neq \varnothing$. We then distinguish the following two cases.

Case a. There exists $\widetilde{c} \in \mathcal{C}^+$ such that $x^{*\widetilde{c}}_h < \widetilde{k}^{\widetilde{c}}_x$;

Case b. For any $c \in \mathcal{C}^+$, $x^{*c}_h = \widetilde{k}^c_x$.

Case a. Define $x_h^{**} = (x_h^{**c})_{c\in\mathcal{C}}$ such that

$$x_h^{**c} = \begin{cases} x_h^{*c} & \text{if } c \in \mathcal{C}^+ \setminus \{\widetilde{c}\} \\ x_h^{*\widetilde{c}} + \frac{1}{2}\min\left\{ \frac{w_h(p^*,\tau^*)-p^*x_h^*}{p^{*\widetilde{c}}}, \quad \widetilde{k}_x^{\widetilde{c}} - x_h^{*\widetilde{c}} \right\} > x_h^{*\widetilde{c}} & \text{if } c = \widetilde{c} \\ x_h^{*c} & \text{otherwise,} \end{cases}$$

where the strictly inequality follows from the fact that $w_h(p^*, \tau^*) - p^* x_h^* \overset{(13)}{>} 0$ and $\widetilde{k}_x^{\widetilde{c}} - x_h^{*\widetilde{c}} > 0$.

Since $(x_h^{**}, \tau_h^*), (x_h^*, \tau_h^*) \in \widehat{\Gamma}_h(p^*, \tau_{\setminus h}^*)$ and u_h is strictly increasing, from Lemma 5, we have

$$\begin{aligned} &\left(G_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})}\right)\left(x_h^{**}, \tau_h^*\right) = \left(u_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})}\right)\left(x_h^{**}, \tau_h^*\right) \\ &> \\ &\left(u_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})}\right)\left(x_h^{*}, \tau_h^*\right) = \left(G_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})}\right)\left(x_h^{*}, \tau_h^*\right), \end{aligned}$$

a fact that contradicts the assumption that (x_h^*, τ_h^*) is a solution to household h's maximization problem in Definition 7.

Case b. This case cannot hold. Assume it does. Then,

$$\begin{aligned} \widetilde{k}_x \overset{\text{Remark } 6}{>} w_h\left(p^*, \tau^*\right) \overset{(13)}{>} p^* x_h^* &= \sum_{c\in\mathcal{C}} p^{*c} \widetilde{k}_x^c \overset{\text{Def. } 5}{\geq} \sum_{c\in\mathcal{C}} p^{*c} \cdot \widetilde{k}_x \cdot r^c \\ &= \widetilde{k}_x \cdot (p^* r) \overset{p^* r = 1}{=} \widetilde{k}_x, \end{aligned}$$

which is the desired contradiction. □

Remark 7 If the upper bound on consumption is not big enough, then Walras' law does not hold, because the consumption vector hits the corner of the box $\left[0, \text{upper bound vector}\right]$. In Kranich (1988) (see page 377, last paragraph in the proof of Proposition 3.4.), Kranich uses the Walras law, but he does not seem to consider that possibility.

Proposition 6 *If (x^*, p^*, τ^*) is a Nash equilibrium for the generalized game presented in Definition 11, then it is a $\mathcal{B}$-equilibrium with an upper bound on consumption—as in Definition 6—and $p^* \gg 0$.*

Proof By definition of Nash equilibrium, each player is maximizing. Therefore, for player $h = 0$, we have that

$$\text{for any } p \in S, \quad p^* \cdot \sum_{h\in\mathcal{H}} \left(x_h^* - e_h\right) \geq p \cdot \sum_{h\in\mathcal{H}} \left(x_h^* - e_h\right). \tag{14}$$

We want to show that, for given $p^* \in S$ and $\tau^*_{\setminus h} \in \times_{h' \in \mathcal{H}\setminus\{h\}} [0, k_{h'}]$,

$$\begin{array}{l}\left(x^*_h, \tau^*_h\right) \ \text{ solves} \\ \max_{(x_h,\tau_h)\in(\widehat{X}_h \times \widehat{T}_h(\mathcal{B}))} \ (u_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})})(x_h, \tau_h) \ \text{ s.t. } \ (x_h, \tau_h) \in \widehat{\Gamma}_h(p^*, \tau^*_{\setminus h}).\end{array}$$

By assumption, for any $h \in \mathcal{H}$, we have that, for given $p^* \in S$ and $(x^*_{h'}, \tau^*_{h'}) \in \times_{h' \in \mathcal{H}\setminus\{h\}}(\widehat{X}_{h'} \times \widehat{T}_{h'}(\mathcal{B}))$,

$$\begin{array}{l}(x^*_h, \tau^*_h) \ \text{ solves} \\ \max_{(x_h,\tau_h)\in(\widehat{X}_h \times \widehat{T}_h(\mathcal{B}))} \ (G_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})})(x_h, \tau_h) \ \text{ s.t. } \ (x_h, \tau_h) \in \widehat{\Gamma}_h(p^*, \tau^*_h).\end{array} \tag{15}$$

From Lemma 5.2, we have $(G_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})})_{|\widehat{\Gamma}_h(p^*, \tau^*_{\setminus h})} = (u_h \circ \varphi_{h|(x^*_{\setminus h}, p^*, \tau^*_{\setminus h})})_{|\widehat{\Gamma}_h(p^*, \tau^*_{\setminus h})}$.

Then, the desired result holds true simply because constraint sets and objective functions restricted to the constraint sets of the two problems are the same. Then, households maximize. We are left with checking market clearing. From (15), we get that for any $h \in \mathcal{H}$

$$0 \geq p^* x^*_h - p^* \left(e_h + \sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h'h} - \sum_{h' \in \mathcal{B}_h} \tau^*_{hh'} \right).$$

Summing up with respect to $h \in \mathcal{H}$, we get

$$0 \geq \sum_{h \in \mathcal{H}} p^* \left(x^*_h - e_h\right) + \sum_{h \in \mathcal{H}} \left(\sum_{h' \in \mathcal{B}_{\to h}} \tau^*_{h'h} - \sum_{h' \in \mathcal{B}_h} \tau^*_{hh'} \right) = \sum_{h \in \mathcal{H}} p^* \left(x^*_h - e_h\right), \tag{16}$$

where last equality follows from Proposition 1. From (16 and (14), we then get

$$\text{for any } p \in S, \qquad 0 \geq p^* \cdot \sum_{h \in \mathcal{H}} \left(x^*_h - e_h\right) \geq p \cdot \sum_{h \in \mathcal{H}} \left(x^*_h - e_h\right). \tag{17}$$

For any $c \in \mathcal{C}$, define $p(c) = (p(c)^{c'})_{c' \in \mathcal{C}}$ such that

$$p(c)^{c'} = \begin{cases} \frac{1}{r^{c'}} & \text{if } c' = c \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $p(c) \in S$. Then, from (17), we get $0 \geq \frac{1}{r^{c'}} \sum_{h \in \mathcal{H}} (x^{*c'}_h - e^{c'}_h)$, and therefore,

$$\sum_{h \in \mathcal{H}} \left(x^*_h - e_h\right) \leq 0. \tag{18}$$

Let's now show that $p^* \gg 0$. Suppose our claim is false and without loss of generality, assume that $p^{*1} = 0$. Then, from strict monotonicity of u_h in x_h (and since $x_h^* \in \text{dom}u_h$), we would have for any $h \in \mathcal{H}$, $x_h^{*1} = \widetilde{k}_x^1$. Then, $\sum_{h\in\mathcal{H}} x_h^{*1} = H\widetilde{k}_x^1 \overset{\text{Def. 5}}{>} Hr^1 > r^1$, contradicting (18).

From Proposition 5.2, we have $p^* \sum_{h\in\mathcal{H}}(x_h^* - e_h) = 0$. Since $p^* >> 0$, from (18), we also have $\sum_{h\in\mathcal{H}}(x_h^* - e_h) = 0$. □

3.4 Equilibria with upper bound on consumption and existence of equilibria

This Section contains a main existence result of the paper.

Proposition 7 *For any economy, a $\mathcal{B}$-equilibrium with an upper bound on consumption is a $\mathcal{B}$-equilibrium (without the upper bound on consumption).*

Proof Since u_h is strictly increasing in $(x_h, \theta_{\mathcal{B}_h})$, then u_h is Locally NonSatiated; since u_h is concave in $(x_h, \theta_{\mathcal{B}_h})$, then u_h is quasi-concave. Therefore, u_h is semistrictly quasi-concave in $(x_h, \theta_{\mathcal{B}_h})$—see Definition 16 and Proposition 17 in "Appendix C". Using those results, then the proof becomes quite standard. See for example, Donato and Villanacci (2023), Theorem 49. □

We can then get the main result of the section.

Theorem 8 *For any economy $\mathcal{E} \in \mathbb{E}$, such that for any $h \in \mathcal{H}$, u_h is Lipschitz continuous and concave in $(x_h, \theta_{\mathcal{B}_{h,}})$, strictly increasing in $(x_h, \theta_{\mathcal{B}_{h,}})$, $e_h \in \mathbb{R}^C_{++}$, then a $\mathcal{B}$-equilibrium $(x^*, p^*, \tau^*) \in \mathbb{R}^{CH}_+ \times S \times T(\mathcal{B})$ exists and $p^* >> 0$.*[9]

Proof It follows from the results we proved in the previous sections and which are summarized below.

From Proposition 4, generalized Nash equilibria exist; then, from Propositions 6 and 7, equilibria exist. □

Remark 8 From Corollary 1 in the Appendix, the above result holds true if for any $h \in \mathcal{H}$, u_h is continuous and concave, strictly increasing in $(x_h, \theta_{\mathcal{B}_{h,}})$ and $\exists L \in \mathbb{R}_{++}$ such that, defined $y = (x_h, \theta_{\mathcal{B}_h})_{h\in\mathcal{H}} \in \mathbb{R}^{CH}_+ \times [0, 1]^{B_h}$, for any $h \in \mathcal{H}$, $y_{\setminus h} \in \mathbb{R}^{C(H-1)}_+ \times [0, 1]^{B_h(H-1)}$, $(u'_h)_{\{y_{\setminus h}\}}(0^+) \leq L$.

Remark 9 We conclude the section, listing some other, less important differences, between Kranich's contribution and ours.

a. Kranich uses a theorem which is not stated properly: a correct statement requires lower hemicontinuity of the constraint set-valued functions.
b. The statement that Walras law holds is not proved in the presence of bounds on the consumption levels—see our Proposition 5 and related Remark 7.
c. The boundedness of T has to be imposed to get existence, as we show in Sect. 4. Kranich requires boundedness of the consumption vector as well, a requirement which can be dispensed of using a standard trick.

[9] The existence theorem presented by Kranich (1988) is as follows. For any $h \in \mathcal{H}$, assume that: the consumption set is $\mathbb{R}^C_+$; the set of admissible transfer contains the origin and it is convex and compact; the utility function U_h is continuous, quasi-concave, strictly increasing in x_h; and $e_h \in \mathbb{R}^C_{++}$. Then an equilibrium exists for any economy.

Remark 10 As in any general equilibrium model, conditions under which uniqueness of equilibria is obtained are quite strong—see for example the recent survey on uniqueness by Toda and Walsh (2024). Using the example mentioned above and presented in Sect. 4, it can be shown that imposing of an upper bound on the level of transfers, an equilibrium exists and is unique. Under relatively general assumptions, the only hope is to get a generic local uniqueness result, i.e., the fact that typically in the space of economies, there exists a finite number of associated equilibria. We do study that problem in a companion paper—see Donato et al. (2024).

3.5 Existence under some other assumptions on the utility functions

In this section, we show existence of equilibria under different assumptions on the utility functions. The strategy of proof is the same as the one presented in the above sections. Below, we provide the proofs of the steps that are peculiar to the chosen specification of the utility function. We proceed as follows. 1. We first present the specific form of the utility function we want to analyze and we state the theorem we want to prove. Then, 2. we state and apply an appropriate extension theorem. Finally, 3. to get compactness of the constraint set, we add a fictitious constraint to those already presented in the previous section; following a standard strategy—see, for example, Section 9.2 in Villanacci et al. (2002)—we verify that the maximization with the "true constraint set" and the one "with the fictitious constraint set" have the same solution set; we prove the latter one has the standard needed properties.

It is easy to check that, by construction, the equilibrium values of the arguments of the extended utility function are in the domain of the function before the extension—as done in detail in Proposition 5 for the model presented in the previous section.

1. First of all, we assume that endowment and consumption vectors are strictly positive and therefore relative wealth are strictly positive and smaller than 1. Moreover, we assume that the utility function is separable with respect to own consumption vector and each other wealth, i.e., we assume that the utility function is of the form

$$\widehat{u}_h\left(x_h\right)+\sum_{h'\in\mathcal{B}_h} v_{hh'}\left(\theta_{h'}\right)+\sum_{h''\in\mathcal{B}_h^{\setminus}} v_{hh''}\left(\theta_{h''}\right),\quad \forall h\in\mathcal{H}, \tag{19}$$

where $\widehat{u}_h:\mathbb{R}^C_{++}\to\mathbb{R}$ and $v_{hh'}, v_{hh''}:\mathbb{R}_{++}\to\mathbb{R}$. Observe that due to the analysis presented in Sect. 2.2.3, there is no loss of generality in dropping the third term in (19).

Based on the above remarks, we can give the following definition.

Definition 13 An economy is $\mathcal{E}'':=\left(\widehat{u}_h,(v_{hh'})_{h'\in\mathcal{B}_h},k_h,e_h\right)_{h\in\mathcal{H}}\in\mathbb{E}''$, where $\mathbb{E}''$ is the set of the economies such that for any $h\in\mathcal{H}$ and $h'\in\mathcal{B}_h$, $\widehat{u}_h$ and $v_{hh'}$ satisfy properties 1–4 in the statement of Theorem 9 below, and for any $h\in\mathcal{H}$, $k_h\in\mathbb{R}^{CB_h}_{++}$ and $e_h\in\mathbb{R}^C_{++}$.

Definition 14 The vector $(x^*,\tau^*,p^*)\in\mathbb{R}^{CH}_+\times\mathbb{R}^{BC}_+\times S$ is a $\mathcal{B}$-**equilibrium** for the economy $\mathcal{E}''\in\mathbb{E}''$ if

i. for any $h\in\mathcal{H}$, household h maximizes, i.e., for given $\mathcal{E}''\in\mathbb{E}''$, $p^*\in S$, $\tau^*_{\setminus h}\in T_{\setminus h}(\mathcal{B})$,

$\left(x_h^*, \tau_h^*\right) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ solves

$$\begin{array}{l} \max_{(x_h,\tau_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} \\ \widehat{u}_h(x_h) + \sum_{h'\in\mathcal{B}_h} v_{hh'}\left(p^*\left(e_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}} \tau^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau^*_{h'h''}\right)\right) \\ \text{s.t.} \left(x_h, \tau_h\right) \in \Gamma_h^*\left(p^*, \tau^*_{\backslash h}\right), \end{array}$$

where

$$\begin{array}{l} \Gamma_h^* : S \times T_{\backslash h}(\mathcal{B}) \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}^C \times \mathbb{R}^{CB_h}, \\ (p^*, \tau^*_{\backslash h}) \mapsto\!\!\!\mapsto \left\{(x_h, \tau_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : p^* x_h \le p^*\left(e_h + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} \tau^*_{h'h} - \sum_{h'\in\mathcal{B}_h} \tau_{hh'}\right)\right. \\ \qquad\qquad \left. x_h >> 0,\ \ \tau_h \ge 0,\ \ \tau_h \le k_h.\right\} \end{array} \tag{20}$$

ii. Markets clear.

Theorem 9 *For for any economy $\mathcal{E}''$, if for any $h \in \mathcal{H}$,*

1. $\widehat{u}_h : \mathbb{R}^C_{++} \longrightarrow \mathbb{R}$ *is continuous, strictly increasing, concave,*
2. *for any* $\alpha \in \mathbb{R}$, $\mathrm{Cl}_{\mathbb{R}^C}\left\{x_h \in \mathbb{R}^C_{++} : \widehat{u}_h\left(x_h\right) \ge \alpha\right\} \subseteq \mathbb{R}^C_{++}$,
3. *for any* $h' \in \mathcal{B}_h$, $v_{hh'} : (0, 1) \longrightarrow \mathbb{R}$ *is (continuous,) increasing, concave and satisfies the condition*

$$\exists\varepsilon > 0 \text{ and } \gamma > 0 \text{ such that } \forall\sigma \in (0, \varepsilon), v_{hh'}\left(\sigma^-\right) < \gamma, \tag{21}$$

4. $e_h \in \mathbb{R}^C_{++}$, *then an equilibrium* $(x^*, p^*, \tau^*) \in \mathbb{R}^{CH}_{+} \times S \times T$ *exists and* $p^* >> 0$.

Remark 11 Under the assumptions listed in Theorem 9, a. $\widehat{u}_h$ is unbounded below; b. $v_{hh'}$ is bounded; c. Assumption 2 is equivalent to the following condition:
for any $\underline{x}_h \in \mathbb{R}^C_{++}$, $Cl_{\mathbb{R}^C}\left\{x_h \in \mathbb{R}^C_{++} : \widehat{u}_h\left(x_h\right) \ge \widehat{u}_h\left(\underline{x}_h\right)\right\} \subseteq \mathbb{R}^C_{++}$.

2.

Proposition 10 *If* $v : (0, 1) \to \mathbb{R}$, $t \longmapsto v(t)$ *is (continuous,) increasing, concave and satisfies Condition 21, then there exists a continuous, concave, increasing function* $V : \mathbb{R} \to \mathbb{R}$, $t \longmapsto V(t)$ *which is an extension of* v.

The proof of this result is presented in Proposition 23. From that Proposition, there exists a continuous, concave and increasing function, denoted by $V_{hh'} : \mathbb{R} \to \mathbb{R}$, which is an extension of $v_{hh'}$ on the whole Euclidean space.
3.
We first present two preliminary results.

Remark 12 If (x_h, τ_h) is a solution to household h maximization problem, then

$$\begin{array}{l} \widehat{u}_h\left(x_h\right) + \sum_{h'\in\mathcal{B}_h} V_{hh'}\left(p^*\left(e_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\backslash\{h\}} \tau^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau^*_{h'h''}\right)\right) \ge \\ \widehat{u}_h\left(\frac{e_h}{4}\right) + \sum_{h'\in\mathcal{B}_h} V_{hh'}\left(p^*\left(e_{h'} + 0 + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\backslash\{h\}} \tau^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau^*_{h'h''}\right)\right) \end{array} \tag{22}$$

where the inequality follows from the fact that $\left(\frac{e_h}{4}, 0\right) \in \Gamma_h^*\left(p^*, \tau_{\backslash h}^*\right)$ and from the monotonicity of $V_{hh'}$.

Remark 13 Defined

$$\underline{u}_h(e,k) :=$$
$$\min_{\left(p^*,\tau_{\backslash h}^*\right)\in S\times T_{\backslash h}(\mathcal{B})} \sum_{h'\in\mathcal{B}_h} V_{hh'}\left(p^*\left(e_{h'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\backslash\{h\}} \tau_{h''h'}^* - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}^*\right)\right) +$$
$$- \sum_{h'\in\mathcal{B}_h} V_{hh'}\left(p^*\left(e_{h'} + k_{hh'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\backslash\{h\}} \tau_{h''h'}^* - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}^*\right)\right),$$

if inequality (22) holds, from the Extreme Value Theorem and the fact that the involved functions are continuous and $S \times T_h(\mathcal{B})$ is a compact set, then we have

$$\widehat{u}_h(x_h) \geq \widehat{u}_h\left(\frac{e_h}{4}\right) + \underline{u}_h(e,k). \tag{23}$$

Based on the above Remarks, the new constraint to be added to the constraint set is the one presented in (22). Hence, we define the following fictitious constraint set-valued function.

$$\begin{array}{l}
\Gamma_h^{**} : S \times T_{\backslash h}(\mathcal{B}) \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}_{++}^C \times \mathbb{R}^{CB_h},\\
\left(p^*, \tau_{\backslash h}^*\right) \mapsto\!\!\!\!\mapsto \Big\{ (x_h, \tau_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \\
\qquad p^* x_h \leq p^*\left(e_h + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} \tau_{h'h}^* - \sum_{h'\in\mathcal{B}_h} \tau_{hh'}\right)\\
\qquad x_h >> 0,\ \ x_h \leq k_x,\ \ \tau_h \geq 0,\ \ \tau_h \leq k_h,\\
\qquad \widehat{u}_h(x_h) + \sum_{h'\in\mathcal{B}_h} V_{hh'}\left(p^*\left(e_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\backslash\{h\}} \tau_{h''h'}^* - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}^*\right)\right) \geq\\
\qquad \widehat{u}_h\left(\frac{e_h}{4}\right) + \sum_{h'\in\mathcal{B}_h} V_{hh'}\left(p^*\left(e_{h'} + 0 + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\backslash\{h\}} \tau_{h''h'}^* - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}^*\right)\right) \Big\}.
\end{array} \tag{24}$$

Moreover,

$$\widetilde{\Gamma}_h^{**} : S \times T_{\backslash h}(\mathcal{B}) \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}_{++}^C \times \mathbb{R}^{CB_h}$$

is defined as Γ_h^{**} with weak inequalities substituted by strict inequalities.

Therefore, from (22), the true maximization problem and the one with constraint as defined in (24) have the same solution sets.

Now, denote by f the function which is naturally defined using the left hand sides of the constraints used in the definition of Γ_h^*. We can then prove that the main conditions in Proposition 18 are indeed satisfied and therefore the constrained set-valued functions have the desired properties.

Proposition 11 *(i) $\widetilde{\Gamma}_h^{**}$ is nonempty valued; (ii) f is continuous; (iii) any component function of f for given values of the variables which are not chosen by the consumer is Locally NonSatiated and quasi-concave; (iv) Γ_h^{**} is compact valued (and $Im(\Gamma_h^{**})$ is contained in a compact set).*

Proof i. $\widetilde{\Gamma}_h^{**}$ is nonempty valued because the vector $\left(\frac{e_h}{3}, \frac{1}{2}\left(\min\left\{k_{hh'}^c, \frac{e_h^c}{3B_h}\right\}\right)_{c\in\mathcal{C},h'\in\mathcal{B}_h}\right)$ belongs to it.

ii. Obvious.

iii. All the constraints apart from the last are affine and not constant and the utility function is quasi-concave.

iv. First of all, observe that Γ_h^{**} is bounded below by 0_{C+CB_h} and above by $k := (k_x, k_h) \in \mathbb{R}_{++}^{C+CB_h}$. We want to show that Γ_h^{**} is sequentially compact. Since Γ_h^{**} is bounded, up to a subsequence, $\left(x_h^n, \tau_h^n\right) \stackrel{n}{\longrightarrow} (\overline{x}_h, \overline{\tau}_h) \in [0, k] \subseteq \mathbb{R}_{+}^{C+CB_h}$. We are then left with showing that $\overline{x}_h \in \mathbb{R}_{++}^C$. For any $n \in \mathbb{N}$, $\left(x_n^n, \tau_h^n\right) \in \Gamma_h^{**}$ satisfies the added constraint (22). Then, from Remark 13, $x_h^n \in \left\{x_h \in \mathbb{R}_{++}^C : \widehat{u}_h(x_h) \geq \widehat{u}_h(\frac{e_h}{4}) + \underline{u}_h(e, k)\right\}$. Since $x_h^n \stackrel{n}{\longrightarrow} \overline{x}_h$, then $\overline{x}_h \in \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}_{++}^C : \widehat{u}_h(x_h) \geq \widehat{u}_h(\frac{e_h}{4}) + \underline{u}_h(e, k)\right\}\right)$ which is contained in $\mathbb{R}_{++}^C$ by Assumption 2. in the statement of existence Theorem 9.

□

4 A robust example of non-existence

In this section, we want to address a quite reasonable question about equilibria. In the version of the model presented above, could we get existence without imposing an upper bound on transfers? The answer is negative, as the analysis of the following Cobb–Douglas economy shows. Consider the following informally described game. There are two players: each player chooses one real number, i.e., her strategy set is $\mathbb{R}$. The player who chooses a number bigger than one chosen by the other player wins 1 euro; a player who chooses a smaller number gets 0 euros. If both players choose the same number, they both get 0 euros. Since a best response against $x \in \mathbb{R}$ is $x + 1 \in \mathbb{R}$, then the game has no Nash equilibria—not even in mixed strategies. Indeed, if the level of altruism of households is sufficiently high, then each of them overbids the other one transfers. We will come back to this statement after the description of the main results in the Cobb–Douglas economy we analyze below.

We describe the problem of non-existence of equilibria in a 2 households—one good—Cobb–Douglas economy. For given, $(e_1, e_2, \beta_{12}) \in \mathbb{R}_{++}^3$ and $t_{21} \in \mathbb{R}_+$, the utility function of household 1 is

$$v_1 : \mathbb{R}_{++}^2 \times \mathbb{R}_{++} \longrightarrow \mathbb{R}, \quad (x_1, t_{12}) \mapsto \log x_1 + \beta_{12} \log\left(e_2 - t_{21} + t_{12}\right).$$

Symmetric definition applies to household 2.

Definition 15 A vector $(x^*, t^*) \in \mathbb{R}_{++}^2 \times \mathbb{R}^2$ is an equilibrium associated with the economy $(\beta, e) \in \mathbb{R}_{++}^2 \times \mathbb{R}_{++}^2$ if

1. $\left(x_1^*, t_{12}^*\right)$ solve the following problem:

$$\begin{array}{l} \text{for given } (e_1, e_2, \beta_{12}) \in \mathbb{R}_{++}^3 \text{ and } t_{21} \in \mathbb{R}_+, \\ \max_{(x_1, t_{12}) \in \mathbb{R}_{++} \times (-e_2 + t_{21}, +\infty)} \log x_1 + \beta_{12} \log\left(e_2 - t_{21} + t_{12}\right) \\ s.t. - x1 - t12 + e1 + t21 \geq 0, t12 \geq 0, \end{array} \tag{25}$$

and similar condition holds for $\left(x_2^*, t_{21}^*\right)$, and
2. markets clear, i.e.,

$$x_1^* + x_2^* = e_1 + e_2.$$

Proposition 12 *A vector* $(x^*, t^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^2$ *is an equilibrium associated with the economy* $(\beta, e) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ *iff there exists* $(\lambda^*, \gamma^*) \in \mathbb{R}^4$ *such that* $(x^*, t^*, \lambda^*, \gamma^*)$ *is a solution to the following system in the exogenous variables* (β, e)

$$\begin{array}{l}
\frac{1}{x_1} - \lambda_1 = 0 \\
\beta_{12}\frac{1}{e_2 - t_{21} + t_{12}} - \lambda_1 + \gamma_{12} = 0 \\
-x_1 - t_{12} + e_1 + t_{21} = 0 \\
\min\{\gamma_{12}, t_{12}\} = 0 \\
e_2 - t_{21} + t_{12} > 0 \\
\frac{1}{x_2} - \lambda_2 = 0 \\
\beta_{21}\frac{1}{e_1 + t_{21} - t_{12}} - \lambda_2 + \gamma_{21} = 0 \\
-x_2 - t_{21} + e_2 + t_{12} = 0 \\
\min\{\gamma_{21}, t_{21}\} = 0 \\
e_1 + t_{21} - t_{12} > 0 \\
\sum_{h\in\mathcal{H}}(x_h - e_h) = 0
\end{array} \tag{26}$$

Proof It is easy to verify that the set of maximizers to maximization problem (25) is characterized by the associated Kuhn–Tucker conditions. □

Proposition 13 *If* $\beta_{12} \cdot \beta_{21} > 1$, *then the system* (26) *has no solution, i.e., there is no equilibrium.*

Proof Observe that $x_2 = e_2 + t_{12} - t_{21}$. Then, $\beta_{12}\frac{1}{x_2} = \lambda_1 - \gamma_{12}$ and $\lambda_1 - \beta_{12}\lambda_2 = \gamma_{12}$. Then, using again the symmetry of the problems and observing that $\lambda_2 = \frac{1}{x_2} > 0$, we have

$$\begin{cases} \lambda_1 - \beta_{12}\lambda_2 = \gamma_{12} \\ -\beta_{21}\lambda_1 + \lambda_2 = \gamma_{21} \end{cases}$$

$$\begin{cases} \beta_{21}\lambda_1 - \beta_{12}\beta_{21}\lambda_2 = \beta_{21}\gamma_{12} \\ -\beta_{21}\lambda_1 + \lambda_2 = \gamma_{21} \end{cases}$$

$$0 > \overset{(<0)}{(1 - \beta_{12}\beta_{21})}\ \overset{(>0)}{\lambda_2} = \overset{(>0)}{\beta_{21}}\ \overset{(\geq 0)}{\gamma_{12}} + \overset{(\geq 0)}{\gamma_{21}} \geq 0,$$

which shows there is no solution to system (26). □

Remark 14 The above analysis says that there is no equilibrium if $\beta_{12}\beta_{21} > 1$. Since the objective function of household 1 is of the type $\log(x_1) + \beta_{12}\log(x_2)$, then 1 and β_{12} are coefficients that say how important household 1 and household 2 are for household 1, respectively. In other words, we can interpret β_{12} as "how much household 1 cares about household 2" and $\beta_{12} > 1$ means household 1 cares about 2 more than 1 cares about 1. Then, equilibria do not exist if "either $\beta_{12} > 1$ or $\beta_{21} > 1$, i.e., if there exists at least one household who cares “too much” about the other household. It is also possible to prove that if $\beta_{12}\beta_{21} \leq 1$, then an equilibrium exists without imposing the artificial bound on transfers.

5 Conclusions

We have presented a discussion of the existence problem in a model with other-regarding wealth-based preferences and possibilities of transfers among households, introduced by Kranich (1988), a paper we consider the best available one in the literature on general equilibrium and interdependent preferences. Our analysis pointed out several problems in the analysis by Kranich: all of them were discussed in Sect. 2.2 and Remark 9. Under some strong continuity and concavity assumptions and imposing an ad-hoc upper bound on transfers, we show existence of equilibria.

Several open problems we are working on in related working-in-progress papers require future analysis. First of all, some of the main difficulties presented by Kranich (1988) are still present in the consumption-based-other-regarding preference model presented by Mercier Ythier (2000). The proof of existence based on assumptions on exogenously given parameters still needs to be provided. An existence result without upper bounds on transfers and assuming some sort of more realistic “bound on altruism” in line with the discussion presented in Sect. 4 is an important point in the future work agenda. Finally, a general analysis of Pareto Optimality and regularity of equilibria and constrained sub-optimality seems feasible from an analytical viewpoint and desirable from an economic perspective.

Appendix A: Quasi-concavity assumption and $\mathcal{B}$-equilibria

As said in Sect. 2.2.3, in this Appendix,

1. We present the definitions of equilibrium and $\mathcal{B}$-equilibrium.

2. We show that the two concepts are equivalent in the sense formalized in Proposition 15.2.

1.

For any $h \in \mathcal{H}$, $t_{\mathcal{B}_h} = (t_{hh'})_{h' \in \mathcal{B}_h} \in \mathbb{R}^{CB_h}$, $t_{\mathcal{B}_h^\setminus} = (t_{hh'})_{h' \in \mathcal{B}_h^\setminus} \in \mathbb{R}^{CB_h^\setminus}$, $t_{\mathcal{B}_{\longrightarrow h}} = (t_{h'h})_{h' \in \mathcal{B}_{\longrightarrow h}} \in \mathbb{R}^{CB_{\longrightarrow h}}$, $t_{\mathcal{B}^\setminus_{\longrightarrow h}} = (t_{h'h})_{h' \in \mathcal{B}^\setminus_{\longrightarrow h}} \in \mathbb{R}^{CB^\setminus_{\longrightarrow h}}$.
$t_{\setminus h}(\mathcal{B}) = (t_{\mathcal{B}_{h'}})_{h' \in \mathcal{H}\setminus\{h\}} = ((t_{h'h''})_{h'' \in \mathcal{B}_{h'}})_{h' \in \mathcal{H}\setminus\{h\}} \in \mathbb{R}^{C\sum_{h' \neq h} B_{h'}} := T_{\setminus h}(\mathcal{B})$, $t_{\setminus h}(\mathcal{B}^\setminus) = (t_{\mathcal{B}^\setminus_{h'}})_{h' \in \mathcal{H}\setminus\{h\}} = ((t_{h'h''})_{h'' \in \mathcal{B}^\setminus_{h'}})_{h' \in \mathcal{H}\setminus\{h\}} \in \mathbb{R}^{C\sum_{h' \neq h} B^\setminus_{h'}} := T_{\setminus h}(\mathcal{B}^\setminus)$ and then $t_{\setminus h} = (t_{\setminus h}(\mathcal{B}), t_{\setminus h}(\mathcal{B}^\setminus)) \in \mathbb{R}^{C\sum_{h' \neq h} B_{h'}} \times \mathbb{R}^{C\sum_{h' \neq h} B^\setminus_{h'}} = \mathbb{R}^{C\sum_{h' \neq h}(B_{h'} + B^\setminus_{h'})} = \mathbb{R}^{C(H-1)(H-1)} = T_{\setminus h}$.

Remark 15 In the notation above, we use the following convention. Given a finite index set $\mathcal{A}$, we give the following definition

$$x_{\mathcal{A}} = \begin{cases} (x_i)_{i \in \mathcal{A}} & \text{if } \mathcal{A} \neq \varnothing, \\ ()_{\varnothing} & \text{if } \mathcal{A} = \varnothing, \end{cases}$$

where $()_{\varnothing}$ is the empty sequence, i.e., a sequence with no elements.

Then, for example, if $\mathcal{A}_1 \neq \varnothing$ and $\mathcal{A}_2 = \varnothing$, then $\left(x_{\mathcal{A}_1}, x_{\mathcal{A}_2}\right) = x_{\mathcal{A}_1}$. We also define $\mathbb{R}^0 = \varnothing$.

Remark 16 From the definition of $\mathcal{B}^{\setminus}_{\to h}$, we have that

$$\text{for any } h, h' \in \mathcal{H} \text{ such that } h \neq h', \quad h' \in \mathcal{B}^{\setminus}_{\to h} \quad \Leftrightarrow \quad h \in \mathcal{B}^{\setminus}_{h'}. \tag{A1}$$

Based on the above Remark, we formalize the following very intuitive and easy to prove result, which will be crucial in our analysis: if households give nothing to households they dislike, then households get nothing from people they dislike them.

Lemma 6 *If for any $h \in \mathcal{H}$, $t_{\mathcal{B}^{\setminus}_h} = 0$, then for any $h \in \mathcal{H}$, $t_{\mathcal{B}^{\setminus}_{\to h}} = 0$.*

Proof By assumption

$$\text{for any } h' \in \mathcal{H}, h \in \mathcal{B}^{\setminus}_{h'} \Rightarrow \quad t_{h'h} = 0. \tag{A2}$$

Then, $h' \in \mathcal{B}^{\setminus}_{\to h} \overset{(A1)}{\Rightarrow} h \in \mathcal{B}^{\setminus}_{h'} \overset{(A2)}{\Rightarrow} t_{h'h} = 0.$ □

We now present the definition of economy and equilibrium.

Definition 16 An economy is $\mathcal{E}' := (e_h, u_h, k'_h)_{h \in \mathcal{H}} \in \mathbb{R}^{CH}_{++} \times \mathcal{U} \times \mathbb{R}^{C(H-1)}_{++} := \mathbb{E}'$.

Definition 17 $(\widetilde{x}, \widetilde{t}, \widetilde{p}) \in \mathbb{R}^{CH}_{+} \times \mathbb{R}^{H(H-1)C}_{+} \times S$ is an **equilibrium** for the economy $\mathcal{E}' \in \mathbb{E}'$ if

a. for any $h \in \mathcal{H}$, for given $\mathcal{E}' \in \mathbb{E}'$, $\widetilde{p} \in S$, $\widetilde{t}_{\setminus h} \in T_{\setminus h}$, $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)}$ solves

$$\begin{aligned}
&\max_{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)}} \\
&u_h \left(\begin{array}{l} x_h, \\ (\widetilde{p}(e_{h'} + t_{hh'} + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}^{\setminus}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}^{\setminus}_{h'}} \widetilde{t}_{h'h''}))_{h' \in \mathcal{B}_h}, \\ (\widetilde{p}(e_{h'} + t_{hh'} + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}^{\setminus}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}^{\setminus}_{h'}} \widetilde{t}_{h'h''}))_{h' \in \mathcal{B}^{\setminus}_h} \end{array} \right) \\
&\text{s.t. } \widetilde{p} x_h \leq \widetilde{p}(e_h + \textstyle\sum_{h' \in \mathcal{B}_{\to h}} \widetilde{t}_{h'h} + \sum_{h' \in \mathcal{B}^{\setminus}_{\to h}} \widetilde{t}_{h'h} - \sum_{h' \in \mathcal{B}_h} t_{hh'} - \sum_{h' \in \mathcal{B}^{\setminus}_h} t_{hh'}) \\
&x_h \geq 0 \\
&0 \leq t_h \leq k'_h
\end{aligned} \tag{A3}$$

b. Markets clear, i.e., $\sum_{h \in \mathcal{H}} (\widetilde{x}_h - e_h) = 0$.

We now introduce a simpler definition of equilibrium in which, roughly speaking, we assume each household ignores the existence of people she dislikes, i.e., any transfer to households belonging to the set $\mathcal{B}_h^{\setminus}$ are zero (and therefore, from the above Lemma 6 also transfers to households in the set $\mathcal{B}_{\to h}^{\setminus}$ are zero).

Definition 18 A $\mathcal{B}$-economy is $\mathcal{E} := (e_h, u_h, k_h)_{h\in\mathcal{H}} \in \mathbb{R}_{++}^{CH} \times \mathcal{U} \times \mathbb{R}_{++}^{CB_h} := \mathbb{E}$.

Definition 19 $(\widetilde{x}, \widetilde{\tau}, \widetilde{p}) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^{C\sum_h B_h} \times S$ is a $\mathcal{B}$-**equilibrium** for the economy $\mathcal{E} \in \mathbb{E}$ if

a. for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $\widetilde{p} \in S$ and $\widetilde{\tau}_{\setminus h} \in \mathbb{R}^{C\sum_{h'\neq h} B_{h'}}$, $(\widetilde{x}_h, \widetilde{\tau}_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ solves

$$\begin{aligned}
&\max_{(x_h,\tau_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} \\
&u_h\left(\begin{array}{l} x_h, \\ (\widetilde{p}(e_{h'} + \tau_{hh'} + \sum\limits_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \widetilde{\tau}_{h''h'} - \sum\limits_{h''\in\mathcal{B}_{h'}} \widetilde{\tau}_{h'h''}))_{h'\in\mathcal{B}_h}, \\ (\widetilde{p}(e_{h'} + 0 + \sum\limits_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \widetilde{\tau}_{h''h'} - \sum\limits_{h''\in\mathcal{B}_{h'}} \widetilde{\tau}_{h'h''}))_{h'\in\mathcal{B}_h^{\setminus}} \end{array}\right) \\
&\text{s.t. } \widetilde{p}x_h \le \widetilde{p}\left(e_h + \textstyle\sum_{h'\in\mathcal{B}_{\to h}} \widetilde{\tau}_{h'h} - \sum_{h'\in\mathcal{B}_h} \tau_{hh'}\right) \\
&x_h \ge 0 \\
&0 \le \tau_h \le k_h = (k_{hh'})_{h'\in\mathcal{B}_h},
\end{aligned} \tag{A4}$$

and

b. markets clear, i.e., $\sum_{h\in\mathcal{H}}(\widetilde{x}_h - e_h) = 0$.

3. We now define the utility functions in terms of the choice variables in both above maximization problems, for any given $\widetilde{p} \in S$ and economies $\mathcal{E}' \in \mathbb{E}'$ and $\mathcal{E} \in \mathbb{E}$.

$$\begin{aligned}
&u_h^{\mathcal{H}} : \mathbb{R}_+^C \times \mathbb{R}_+^{C(H-1)} \times \mathbb{R}_+^{C(H-1)(H-1)} \longrightarrow \mathbb{R} \\
&(x_h, (t_{\mathcal{B}_h}, t_{\mathcal{B}_h^{\setminus}}), (t_{\mathcal{B}_{h'}}, t_{\mathcal{B}_{h'}^{\setminus}})_{h'\in\mathcal{H}\setminus\{h\}}) \mapsto \\
&u_h\left(\begin{array}{l} (x_h, \\ (\widetilde{p}(e_{h'} + t_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} t_{h''h'} + \sum_{h''\in\mathcal{B}_{\to h'}^{\setminus}\setminus\{h\}} t_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} t_{h'h''} - \sum_{h''\in\mathcal{B}_{h'}^{\setminus}} t_{h'h''}))_{h'\in\mathcal{B}_h}, \\ (\widetilde{p}(e_{h'} + t_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} t_{h''h'} + \sum_{h''\in\mathcal{B}_{\to h'}^{\setminus}\setminus\{h\}} t_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} t_{h'h''} - \sum_{h''\in\mathcal{B}_{h'}^{\setminus}} t_{h'h''}))_{h'\in\mathcal{B}_h^{\setminus}} \end{array}\right) \\
&u_h^{\mathcal{B}} : \mathbb{R}_+^C \times \mathbb{R}_+^{CB_h} \times \mathbb{R}_+^{C\sum_{h'\neq h} B_{h'}} \longrightarrow \mathbb{R} \\
&(x_h, \tau_{\mathcal{B}_h}, (\tau_{\mathcal{B}_{h'}})_{h'\in\mathcal{H}\setminus\{h\}}) \mapsto \\
&u_h\left(\begin{array}{l} x_h, \\ (\widetilde{p}(e_{h'} + \tau_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}))_{h'\in\mathcal{B}_h}, \\ (\widetilde{p}(e_{h'} + 0 + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \tau_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \tau_{h'h''}))_{h'\in\mathcal{B}_h^{\setminus}} \end{array}\right)
\end{aligned}$$

Given the above definitions, then the results below follow.

Lemma 7 *If for any* $h \in \mathcal{H}$, $t_{\mathcal{B}_h}^{\setminus} = 0 \in \mathbb{R}^{B_h^{\setminus}}$, *then for any* $(x_h, t_{\mathcal{B}_h}, \left(t_{\mathcal{B}_{h'}}\right)_{h' \in \mathcal{H}\setminus\{h\}}) \in \mathbb{R}_+^C \times \mathbb{R}_+^{CB_h} \times \mathbb{R}^{C\sum_{h' \neq h} B_{h'}}$, *we have*

$$u_h^{\mathcal{H}}(x_h, (t_{\mathcal{B}_h}, (t_{\mathcal{B}_h^{\setminus}} = 0)), (t_{\mathcal{B}_{h'}}, (t_{\mathcal{B}_{h'}^{\setminus}} = 0))_{h' \in \mathcal{H}\setminus\{h\}}) = u_h^{\mathcal{B}}(x_h, t_{\mathcal{B}_h}, (t_{\mathcal{B}_{h'}})_{h' \in \mathcal{H}\setminus\{h\}}).$$

Proposition 14 *Let* $\mathcal{E} \in \mathbb{E}$ *and* $\widetilde{p} \in S$ *given.*

1. *If* $(\widetilde{x}_h, \widetilde{t}_h = (\widetilde{t}_{\mathcal{B}_h}, \widetilde{t}_{\mathcal{B}_h^{\setminus}})) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)}$ *solves problem* (A3) *at* $\widetilde{t}_{\setminus h} \in T_{\setminus h}$, *then* $\widetilde{t}_{\mathcal{B}_h^{\setminus}} = 0$.

2. *If* $(\widetilde{x}_h, \widetilde{t}_h = (\widetilde{t}_{\mathcal{B}_h}, \widetilde{t}_{\mathcal{B}_h^{\setminus}})) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)}$ *solves problem* (A3) *at*

$$\widetilde{t}_{\setminus h} := ((\widetilde{t}_{\mathcal{B}_{h'}})_{h' \in \mathcal{H}\setminus\{h\}}, (\widetilde{t}_{\mathcal{B}_{h'}^{\setminus}} = 0)_{h' \in \mathcal{H}\setminus\{h\}}) \in T_{\setminus h} \tag{A5}$$

then $(\widetilde{x}_h, \widetilde{\tau}_{\mathcal{B}_h} = \widetilde{t}_{\mathcal{B}_h}) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ *solves problem* (A4) *at* $\widetilde{\tau}_{\setminus h} := (\widetilde{t}_{\mathcal{B}_{h'}})_{h' \neq h} \in T_{\setminus h}(\mathcal{B})$.

3. *If* $(\widetilde{x}_h, \widetilde{\tau}_{\mathcal{B}_h}) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ *solves problem* (A4) *at* $(\widetilde{\tau}_{\mathcal{B}_{h'}})_{h' \neq h} \in T_{\setminus h}(\mathcal{B})$, *then*

$$(\widetilde{x}_h, \widetilde{t}_h) := (\widetilde{x}_h, \widetilde{\tau}_{\mathcal{B}_h}, \widetilde{t}_{\mathcal{B}_h^{\setminus}} = 0) \in \mathbb{R}^C \times \mathbb{R}^{H-1}, \tag{A6}$$

solves problem (A3) *at*

$$\widetilde{t}_{\setminus h} := ((\widetilde{\tau}_{\mathcal{B}_{h'}})_{h' \neq h}, (\widetilde{t}_{\mathcal{B}_{h'}^{\setminus}} = 0)_{h' \neq h}). \tag{A7}$$

Proof Let $\Psi_h(\widetilde{p}, \widetilde{t}_{\setminus h})$ be the constraint set presented in Definition 17, and $\Gamma_h(\widetilde{p}, \widetilde{t}_{\setminus h}(\mathcal{B}))$ be the constraint sets presented in Definition 19 of $\mathcal{B}$-equilibrium.

1. We want to show that for any $h' \in \mathcal{B}_h^{\setminus}$, we have $\widetilde{t}_{hh'} = 0$. If $\mathcal{B}_h^{\setminus} = \varnothing$, then we are done. Suppose now that $\mathcal{B}_h^{\setminus} \neq \varnothing$ and that our claim is false, i.e., $\widetilde{t}_{\mathcal{B}_h^{\setminus}} > 0$. Then, the simple idea of the proof is to use those transfers as consumption of household h to get the desired contradiction.
Indeed, we prove that, take $x_h^* = \widetilde{x}_h + \sum_{h' \in \mathcal{B}_h^{\setminus}} \widetilde{t}_{hh'}$ and $t_h^* = (\widetilde{t}_{\mathcal{B}_h}, t^*_{\mathcal{B}_h^{\setminus}} := 0)$, the vector
$(x_h^*, (t^*_{\mathcal{B}_h}, t^*_{\mathcal{B}_h^{\setminus}}), (\widetilde{t}_{\mathcal{B}_{h'}}, \widetilde{t}_{\mathcal{B}_{h'}^{\setminus}})_{h' \in \mathcal{H}\setminus\{h\}}) \in \Psi_h(\widetilde{p}, \widetilde{t}_{\setminus h})$ is a solution to maximization problem (A3).

2. Suppose our claim is false, i.e., there exists $(x_h^*, \tau^*_{\mathcal{B}_h}) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ such that $(x_h^*, \tau^*_{\mathcal{B}_h}) \in \Gamma_h(\widetilde{p}, \widetilde{t}_{\setminus h}(\mathcal{B}))$ and

$$u_h^{\mathcal{B}}\left(x_h^*, \tau^*_{\mathcal{B}_h}, (\widetilde{t}_{\mathcal{B}_{h'}})_{h' \neq h}\right) > u_h^{\mathcal{B}}(\widetilde{x}_h, \widetilde{\tau}_{\mathcal{B}_h}, (\widetilde{t}_{\mathcal{B}_{h'}})_{h' \neq h}) \tag{A8}$$

Choose $\widehat{x}_h = x_h^*$ and $\widehat{t}_h = (\tau^*_{\mathcal{B}_h}, (\widehat{t}_{\mathcal{B}_h^{\setminus}} = 0))$. Observe that we can rewrite (A8) as follows,

$$u_h^{\mathcal{B}}(\widehat{x}_h, \widehat{\tau}_{\mathcal{B}_h}, (\widetilde{t}_{\mathcal{B}_{h'}})_{h' \in \mathcal{H}\setminus\{h\}}) > u_h^{\mathcal{B}}(\widetilde{x}_h, \widetilde{t}_{\mathcal{B}_h}, (\widetilde{\tau}_{\mathcal{B}_{h'}})_{h' \in \mathcal{H}\setminus\{h\}}). \tag{A9}$$

Then, it is easy to contradict the fact that $(\widetilde{x}_h, \widetilde{t}_h)$ solves problem (A3) at $\widetilde{t}_{\backslash h} \in T_{\backslash h}$ using

$$(\widehat{x}_h, (\widehat{t}_{\mathcal{B}_h}, \widehat{t}_{\mathcal{B}^{\backslash}_h}), (\widetilde{t}_{\mathcal{B}_{h'}}, \widetilde{t}_{\mathcal{B}^{\backslash}_{h'}})_{h' \in \mathcal{H}\backslash\{h\}}) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)}.$$

3.

Suppose otherwise, i.e., there exists $(\widehat{x}_h, \widehat{t}_h) \in \Psi_h(\widetilde{p}, \widetilde{t}_{\backslash h} := ((\widetilde{\tau}_{\mathcal{B}_{h'}})_{h' \neq h}, (\widetilde{t}_{\mathcal{B}^{\backslash}_{h'}} = 0)_{h' \neq h}))$ such that $u^{\mathcal{H}}_h(\widehat{x}_h, (\widehat{t}_{\mathcal{B}_h}, \widehat{t}_{\mathcal{B}^{\backslash}_h}), ((\widetilde{\tau}_{\mathcal{B}_{h'}})_{h' \neq h}, (\widetilde{t}_{\mathcal{B}^{\backslash}_{h'}} = 0)_{h' \neq h})) > u^{\mathcal{H}}_h(\widetilde{x}_h, (\widetilde{t}_{\mathcal{B}_h}, \widetilde{t}_{\mathcal{B}^{\backslash}_h}), ((\widetilde{\tau}_{\mathcal{B}_{h'}})_{h' \neq h}, (\widetilde{t}_{\mathcal{B}^{\backslash}_{h'}} = 0)_{h' \neq h}))$.

Then, from the proof in point 1 in the Statement of the present Proposition, we have $(\widehat{x}_h, \widehat{t}_h) := (\widehat{x}_h, (\widehat{t}_{hh'})_{h' \in \mathcal{B}_h}, (\widehat{t}_{hh'} = 0)_{h' \in \mathcal{B}^{\backslash}_h})$ and then $\widehat{t}_{\mathcal{B}^{\backslash}_h} = 0$. From (A6) and (A7), for any $h \in \mathcal{H}$, $\widetilde{t}_{\mathcal{B}^{\backslash}_h} = 0$. Defined $\widehat{\tau}_{\mathcal{B}_h} = \widehat{t}_{\mathcal{B}_h}$, we can show $(\widehat{x}_h, \widehat{\tau}_{\mathcal{B}_h})$ contradicts the assumption that $(\widetilde{x}_h, \widetilde{\tau}_{\mathcal{B}_h})$ is a solution to Problem (A4) at $(\widetilde{\tau}_{\mathcal{B}_{h'}})_{h' \neq h}$. □

Proposition 15 *1. If $(\widetilde{x}, \widetilde{t}, \widetilde{p})$ is an equilibrium, then for any $h \in \mathcal{H}$ and any $h' \in \mathcal{B}^{\backslash}_h$, we have $\widetilde{t}_{hh'} = 0$.*

2. $(\widetilde{x}, \widetilde{t}, \widetilde{p}) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{H(H-1)C}_+ \times S$ is an equilibrium $\Leftrightarrow$ $(\widetilde{x}, (\widetilde{t}_{hh'})_{h' \in \mathcal{B}_h}, \widetilde{p}) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{C \sum_h B_h}_+ \times S$ is a $\mathcal{B}$ -equilibrium.

Proof It easily follows from Proposition 14. □

Appendix B: Set-valued functions defined using function inequalities

Definition 20 Let a nonempty, convex subset X of $\mathbb{R}^n$ and a function $f : X \to \mathbb{R}$ be given. We say that f is

- Locally NonSatiated, or LNS, if $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x' \in B(x, \varepsilon) \cap X$ such that $u(x') > u(x)$;
- NonSatiated, or NS, if $\forall x \in X \exists x' \in X$ such that $u(x') > u(x)$;
- semistrictly quasi-concave if for any $x, y \in X$ and any $\lambda \in (0, 1)$, $f(x) > f(y) \Rightarrow f((1-\lambda)x + \lambda y) > f(y)$.

Proposition 16 (Villanacci 2022, Corollary 42, page 15) *If X is a convex metric space and $u : X \to \mathbb{R}$ is continuous, then u is semistrictly quasi-concave and NonSatiated $\Leftrightarrow$ u is quasi-concave and Locally NonSatiated.*

Proposition 17 *Let the following functions and sets be given: for any $j \in \{1, \dots, m\}$, $f_j : \mathbb{R}^C \longrightarrow \mathbb{R}$, $x \mapsto f_j(x)$; $f : \mathbb{R}^C \to \mathbb{R}^m$, $x \longmapsto (f_j(x))^m_{j=1}$; $B = \{x \in \mathbb{R}^C : f(x) \geq 0\}$ and $\widetilde{B} = \{x \in \mathbb{R}^C : f(x) >> 0\}$. If $\widetilde{B} \neq \varnothing$ and for any $j \in \{1, \dots m\}$, f_j is continuous, Locally NonSatiated and quasi-concave, then 1. $\widetilde{B} = Int(B)$, and 2. $B = Cl(\widetilde{B})$.*

Proof For any $j \in \{1, \dots, m\}$, define $B_j := \{x \in \mathbb{R}^C : f_j(x) \geq 0\}$ and $\widetilde{B}_j := \{x \in \mathbb{R}^C : f_j(x) > 0\}$; observe that $B = \cap^m_{j=1} B_j$ and $\widetilde{B} = \cap^m_{j=1} \widetilde{B}_j$.

1. First of all observe that if f_j is is semistrictly quasi-concave, continuous and nonsatiated, then

$$\varnothing \neq \widetilde{B}_j = Int(B_j). \tag{B10}$$

The above result is proven in Lemma 11 page 7 in Border (2017) or in Proposition 38 in Villanacci (2022). Therefore,

$$\widetilde{B} = \cap_{j=1}^{m} \widetilde{B}_j = \cap_{j=1}^{m} Int(B_j) = Int(\cap_{j=1}^{m}(B_j)) = Int(B).$$

2. Since it is false that $\cap_{j=1}^{m} Cl(B_j) = Cl(\cap_{j=1}^{m}(B_j))$, we cannot use an approach similar to the above one.

By assumption, B is closed and $\widetilde{B} \subseteq B$; then $Cl(\widetilde{B}) \subseteq Cl(B) = B$. We are left with showing that $B \subseteq Cl(\widetilde{B})$. We want to show that: $x \in B \Rightarrow \forall \varepsilon > 0$, $\mathcal{B}(x,\varepsilon) \cap B \neq \varnothing$. Suppose otherwise, i.e., $x \in B$, or $f(x) \geq 0$ and $\exists \varepsilon > 0$ such that $\mathcal{B}(x,\varepsilon) \cap B = \varnothing$, or, for any $j = 1, \ldots, m$, $f_j(x) \geq 0$ and $\exists \varepsilon > 0$ such that $\forall y \in \mathcal{B}(x,\varepsilon)$, $\exists j^* \in \{1, \ldots, m\}$ such that $f_{j^*}(y) < 0$.

Then, $\exists x \in B \subseteq \mathbb{R}^C$, $\exists j^* \in \{1, \ldots, m\}$, $\exists \varepsilon \in \mathbb{R}_{++}$ such that $\forall y \in \mathcal{B}(x,\varepsilon)$, we have $f_{j^*}(x) \geq 0 > f_{j^*}(y)$.

Then x is a local maximum point for f_{j^*} which contradicts the fact that f_{j^*} is Locally Nonsatiated. □

Proposition 18 *Let a subset* Π *of* $\mathbb{R}^p$ *and a function* $f : \Pi \times \mathbb{R}^C \longrightarrow \mathbb{R}^m$, $x \mapsto \left(f_j(\pi, x)\right)_{j=1}^{m}$ *be given (with* $f_j : \Pi \times \mathbb{R}^C \longrightarrow \mathbb{R}$*). Let also the following set valued function be given.*

$$\begin{array}{l} B : \Pi \longrightarrow\longrightarrow \mathbb{R}^C, \;\; \pi \mapsto\mapsto \left\{x \in \mathbb{R}^C : f(\pi, x) \geq 0\right\}; \\ \widetilde{B} : \Pi \longrightarrow\longrightarrow \mathbb{R}^C, \;\; \pi \mapsto\mapsto \left\{x \in \mathbb{R}^C : f(\pi, x) >> 0\right\}. \end{array}$$

I. If $\widetilde{B}$ *is non-empty valued,* f *is continuous and for any* $j = 1, \ldots m$ *and for any* $\pi \in \Pi$, *either* $f_{j|\{\pi\}}$ *is NonSatiated and semistrictly quasi-concave, or* $f_{j|\{\pi\}}$ *is Locally NonSatiated and quasi-concave, then* B *is non-empty valued convex valued, closed graph and lower hemicontinuous.*

II. If in addition either a. B *is compact valued or b.* $Im(B)$ *is contained in a compact set, then* B *is upper hemicontinuous.*

Proof I.

1. B is non-empty valued.

Since $\widetilde{B} \subseteq B$ by definition, and $\widetilde{B} \neq \varnothing$, by assumption, then the desired result follows.

2. B is convex valued.

Since for any $j = 1, \ldots m$ and for any $\pi \in \Pi$, $f_{j|\{\pi\}}$is quasi-concave, then $\left\{x \in \mathbb{R}^n : f_{j|\{\pi\}}(x) \geq 0\right\}$ is convex and then $B(\pi) = \cap_{j=1}^{m} \left\{x \in \mathbb{R}^n : f_{j|\{\pi\}}(x) \geq 0\right\}$ is convex as well.

3. B is closed graph.

We want to show that for and $(\pi^n, x^n)_{n\in\mathbb{N}} \in \left(\Pi \times \mathbb{R}^C\right)^\infty$ such that for any $n \in \mathbb{N}$, $x_n \in B(\pi^n)$ and such that $(\pi^n, x^n) \longrightarrow (\pi, x)$, then $x \in B(\pi)$.

Since for any $n \in \mathbb{N}$, $x_n \in B(\pi^n)$, we do have that for any $n \in \mathbb{N}$, $f(\pi^n, x^n) \geq 0$. Taking limits of both sides of that inequality and using the continuity of f, we do have $f(x, \pi) \geq 0$, i.e., $x \in B(\pi)$, as desired.

4. B is lower hemicontinuous.

First of all observe that from Proposition 2.38, page 50 in Hu and Papageorgiou (1997), it is enough to show that a. $\widetilde{B}$ is lower hemicontinuous, and b. $B = Cl(\widetilde{B})$.

a. We now want to use a well known characterization of lower hemicontinuity.[10] Indeed, we want to show that

for any sequence $(\pi^n)_{n\in\mathbb{N}} \in \Pi^\infty$ such that $\pi^n \to \pi$ and any $x \in \widetilde{B}(\pi)$, there exists a sequence $(x^n)_{n\in\mathbb{N}} \in \left(\mathbb{R}^C\right)^\infty$ such that $\forall n \in \mathbb{N}$, $(x^n) \in \widetilde{B}(\pi^n)$ and $x^n \to x$.

Observe that $f(\pi^n, x) \longrightarrow f(\pi, x) >> 0$ where strict inequalities follows from the fact that $x \in \widetilde{B}(\pi)$. Then, there exists $N \in \mathbb{N}$ such that for any $n > N$, $f(\pi^n, x) >> 0$. Then, for any $n > N$, $x \in B(\pi^n)$ and taking $x^n = x$ for any $n > N$ (and arbitrary values of x^n for $n \leq N$) completes the proof.

b. It is the content of Proposition 17, identifying f with $f_{|\{\pi\}}$.

II. Conclusion a. and b. follow from the four results contained in I. above and Lemma 1, page 33 and result at bottom page 23 in Hildebrand (1974). □

Appendix C: Extension of continuous quasi-concave and concave functions

C.0.1 The case of Lipschitz and concave functions

Definition 21 A function $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is said to be L-Lipschitz (continuous) on S or to satisfy a Lipschitz condition on S if

$$\exists L \in \mathbb{R}_{++} \text{ such that } \forall x^1, x^2 \in S, \quad \left\| f\left(x^1\right) - f\left(x^2\right) \right\| \leq L \cdot \left\| x^1 - x^2 \right\|. \tag{C12}$$

Remark 17 A continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ is not necessarily Lipschitz, the standard counter-example being $f(x) = \sqrt{x}$. Below we present some sufficient conditions for a function to be Lipschitz.

Proposition 19 *Let an open, convex set S in $\mathbb{R}^m$ and a function $f : S \to \mathbb{R}^n$ be given. If f is differentiable and $\exists \gamma > 0$ such that $\forall x \in A \subseteq S$, $\|Df(x)\| < \gamma$, then f is Lipschitz on A.*

Proposition 20 *If A is an open subset of $\mathbb{R}^n$ and $f : A \longrightarrow \mathbb{R}$ is a C^1 function, then f is Lipschitz on any non-empty compact subset C of A.*

Example 1 $v : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$, $v(x_1, x_2) = \log(x_1 + 1) + \log(x_2 + 1)$ is Lipschitz.

[10] See Proposition 4, p. 229 in Ok (2007) and Theorem AIII.2, p. 197 in Hildebrand and Kirman (1976).

5.1 C.1 A sufficient condition for Lipschitz continuity

Proposition 21 *Let a function* $f : \mathbb{R}_+^n \longrightarrow \mathbb{R}$, $x = (x_i)_{i=1}^n \mapsto f(x)$ *be given. Then,* f *is Lipschitz continuous if and only if* $\exists L \in \mathbb{R}_{++}$ *such that for any* $i \in \{1, \dots n\}$, $x_{\setminus i} := (x_j)_{j \in \{1,\dots n\} \setminus \{i\}} \in \mathbb{R}_+^{n-1}$, $f_{\{x_{\setminus i}\}} : \mathbb{R} \longrightarrow \mathbb{R}$, $x_i \mapsto f(x_i, x_{\setminus i})$ *is L-Lipschitz continuous.*

Proof Well known. □

Proposition 22 *Let a concave, continuous and increasing function* $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $x \mapsto f(x)$ *be given. If* $f'(0^+)$ *is finite, then* f *is* $f'(0^+)$ *-Lipschitz continuous.*

Proof Obvious. □

Corollary 1 *Let a function* $f : \mathbb{R}_+^n \longrightarrow \mathbb{R}$, $x = (x_i)_{i=1}^n \mapsto f(x)$ *be given. If* f *is continuous, concave, increasing and* $\exists L \in \mathbb{R}_{++}$ *such that for any* $i \in \{1, \dots n\}$, $x_{\setminus i} \in \mathbb{R}_+^{n-1}$, $f'_{\{x_{\setminus i}\}}(0^+) \leq L$, *then* f *is Lipschitz continuous.*

C.1.1 Extending the real function of real variable *v*

Proposition 23 *If* $v : (0,1) \to \mathbb{R}$, $t \longmapsto v(t)$ *is (continuous,) increasing, concave and satisfies the condition*

$$\exists \varepsilon > 0 \text{ and } k > 0 \text{ such that } \forall t \in (0, \varepsilon), v'(t^-) < k, \tag{C13}$$

then there exists a continuous, concave, increasing function $V : \mathbb{R} \to \mathbb{R}$, $t \longmapsto V(t)$ *which is an extension of* v.

The proof is outlined below and goes through several steps.

Proof v is continuous. For any $t \in (a,b)$, $v'(t^+), v'(t^-) \in [0,k]$; v is bounded; there exists $v_0 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 0^+} v(t) = v_0$; there exists $v_1 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 1^-} v(t) = v_1$.

The function $\widehat{v} : [0,1] \longrightarrow \mathbb{R}$,

$$\widehat{v}(t) = \begin{cases} v_0 & \text{if } t = 0 \\ v(t) & \text{if } t \in (0,1) \\ v_1 & \text{if } t = 1 \end{cases}$$

is continuous. Defined $v'_- : (0,1) \longrightarrow \mathbb{R} t \mapsto v'(t^-) := \lim_{h \longrightarrow 0^-} \frac{v(t+h) - v(t)}{h}$, i.e., the left derivative in t, then there exists $v'_0 \in \mathbb{R}_+$ such that $\lim_{t \longrightarrow 0^+} v'_-(t) = v'_0$.

Defined $v'_+ : (0,1) \longrightarrow \mathbb{R} t \mapsto v'(t^+)$, there exists $v'_1 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 1^-} v'(t^+) = v'_1$.

The function V defined below is the desired extension.

$$V : \mathbb{R} \longrightarrow \mathbb{R}, \qquad V(t) = \begin{cases} v_0 + v'_0 \cdot t & \text{if } t \leq 0 \\ \widehat{v}(t) & \text{if } t \in [0,1] \\ v_1 + v'_1 \cdot t & \text{if } t \geq 1. \end{cases}$$

To show that V is continuous and increasing is straightforward. To show that V is concave uses the following facts. The derivative set-valued function associated with V is denoted and defined as follows:

$$p : \mathbb{R} \longrightarrow\!\!\longrightarrow \mathbb{R},$$
$$t \mapsto\!\!\mapsto \begin{cases} \{v_0'\} & \text{if } t \leq 0 \\ \left[v_-'\left(t^+\right), v\left(t^-\right)\right] & \text{if } t \in (0, 1) \\ \{v_1'\} & \text{if } t \geq 1. \end{cases}$$

It follows that a. for any $t_0, t_1 \in \mathbb{R}$, if $t_0 < t_1$, then for any $p_0 \in p(t_0)$ and any $p_1 \in p(t_1)$, we have $p_0 \geq p_1$, and b. for any $t_0, t_1 \in \mathbb{R}$, for any $p_{t_0} \in p(t_0)$, $V(t_1) \leq V(t_0) + p_{t_0}(t_1 - t_0)$. □

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