Topology and its Applications 389 (2026) 109872

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

Totally paracompact spaces and the Menger covering property ☆

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ARTICLE INFO

Article history:
Received 19 December 2025
Received in revised form 27 May 2026
Accepted 28 May 2026
Available online 1 June 2026

MSC:
primary 54D20
secondary 03E17

Keywords:
Total paracompactness
Menger property
GO-space
Topological games

ABSTRACT

A topological space is totally paracompact if any base of this space contains a locally finite subcover. We focus on a problem of Curtis whether in the class of regular Lindelöf spaces total paracompactness is equivalent to the Menger covering property. To this end we consider topological spaces with certain dense subsets. It follows from our results that the above equivalence holds in the class of Lindelöf GO-spaces defined on subsets of reals. We also provide a game-theoretical proof that any regular Menger space is totally paracompact and show that in the class of first-countable spaces the Menger game and a partial open neighborhood assignment game of Aurichi are equivalent. We also show that if $\mathfrak{b} = \omega_1$, then there is an uncountable subspace of the Sorgenfrey line whose all finite powers are Lindelöf, which is a strengthening of a famous result due to Michael.

1. Introduction

By *space* we mean a Hausdorff infinite topological space. A space X is *totally paracompact* [12], if every open base of X contains a locally finite cover of X. In 1924, Menger introduced the following covering property as an attempt to characterize σ-compactness in the class of metric spaces. A metric space X is *Menger-basis* if for any basis $\mathcal{B}$ of X, there are sets $B_0, B_1, \ldots \in \mathcal{B}$ whose diameters tend to zero and the family $\{ B_n : n \in \omega \}$ covers X. After that Hurewicz reformulated the above definition, extending it to all topological spaces: a space X is *Menger*, if for every sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of X there are finite sets $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots$ such that the family $\bigcup\{ \mathcal{F}_n : n \in \omega \}$ is a cover of X [13]. The Hurewicz definition

☆ The research of the third author was funded by the National Science Center, Poland Weave-UNISONO call in the Weave programme Project: Set-theoretic aspects of topological selections 2021/03/Y/ST1/00122 and also by the University of Messina.
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https://doi.org/10.1016/j.topol.2026.109872

of the Menger property is equivalent to the Menger-basis property in the class of metric spaces. The Menger property is one of the central properties considered in the topological selections theory.

Our investigation is motivated by the following result of Curtis. Notice that in Curtis's paper, as in many other old works [1], the Menger property is referred to as the Hurewicz property.

Theorem 1.1 *(Curtis [8, Theorem 3.1]). Every regular Menger space is totally paracompact.*

Notice that there are non-regular countable spaces that may not be paracompact, e.g., Bing's space [18].

A space is *Lindelöf* if any open cover of this space contains a countable subcover (notice that in this approach, we do not request that the space is regular). Since any Menger space is Lindelöf, in the light of the above result, Curtis formulated the following problem.

Problem 1.2 *(Curtis [8]). Is it true that any Lindelöf totally paracompact space is Menger?*

Thus far it is known that there is a positive solution to Problem 1.2 in the class of metrizable spaces [15] and in the class of generalized ordered spaces defined on subsets of the real line [17]. In Section 2, we show that total paracompactness is equivalent to the Menger property in the class of Lindelöf spaces having a σ-compact dense subset. We also consider another classes of spaces, where it holds. Our results capture all known classes of spaces, where the Curtis conjecture is true. Since the Menger property can be characterized using a topological game [13,19], in Section 4, we provide a game-theoretical proof of Theorem 1.1 which seems to be much simpler than the original one (modulo game-characterization of the Menger property). Section 5 is devoted to relations between a partial open neighborhood assignment game of Aurichi [2] and the Menger game (see definitions below). In particular, we show that in the class of first-countable spaces these two games are equivalent. The Aurichi game comes on stage naturally when considering Menger spaces and totally paracompact spaces in the context of the so-called *D-spaces* and an old problem whether every regular Lindelöf space is a D-space [10] (see Section 5 for definitions and more details). We finish our paper with comments and open problems.

2. Menger vs. total paracompactness

We establish the following partial answer to Problem 1.2. Recall that a paracompact space with a dense σ-compact subspace is Lindelöf [11, Theorem 5.1.25].

Theorem 2.1. *Every totally paracompact space with a σ-compact dense subset is Menger.*

Proof. Assume that X is a totally paracompact space with a σ-compact dense subset $\bigcup_{n\in\omega} C_n$, where each set C_n is compact. Fix a sequence $\mathcal{U}_0, \mathcal{U}_1, \dots$ of open covers of X. We may assume that the families $\mathcal{U}_n$ are closed under finite unions. Fix $x \in X$. For each $n \in \omega$, let $\mathcal{B}_n(x)$ be the family of all open neighborhoods V of x such that V is contained in some set from $\mathcal{U}_n$ and $V \cap C_n \neq \emptyset$. Then the collection $\mathcal{B}(x) := \bigcup_{n\in\omega} \mathcal{B}_n(x)$ is a local base at x. It follows that the family $\mathcal{B} := \bigcup_{x\in X} \mathcal{B}(x)$ is a base for X. By the assumption, there exists a locally finite cover $\mathcal{B}' \subseteq \mathcal{B}$ of X. Fix $n \in \omega$. The family $\mathcal{B}'$ is locally finite and it covers the compact set C_n. Thus, the set

$$\{\, B \in \mathcal{B}' : B \cap C_n \neq \emptyset \,\}$$

is finite and the set

$$\mathcal{B}'_n := \mathcal{B}' \cap \bigcup_{x\in X} \mathcal{B}_n(x)$$

is finite, too. By the definition, each element from the union $\bigcup_{x\in X}\mathcal{B}_n(x)$ is contained in a set from $\mathcal{U}_n$. Since the family $\mathcal{U}_n$ is closed under finite unions, there is a set $U_n \in \mathcal{U}_n$ such that $\bigcup \mathcal{B}'_n \subseteq U_n$. Since the family $\mathcal{B}' = \bigcup_{n\in\omega}\mathcal{B}'_n$ covers X, the family $\{U_n : n \in \omega\}$ covers X, too. □

Corollary 2.2. *[Lelek [15]] Every separable totally paracompact space is Menger.*

Corollary 2.3. *Every metrizable totally paracompact Lindelöf space is Menger.*

A space X is *totally metacompact* [9], if every open base of X contains a point-finite cover of X. Clearly, every totally paracompact space is totally metacompact. We asked if we could replace total paracompactness in the statement of Theorem 2.1 with total metacompactness. The following example gives a negative answer to this question.

Example 2.4. There is a totally metacompact, not paracompact, not Lindelöf space having a σ-compact dense subset: Consider $X := (\omega + 1) \times (\omega_1 + 1) \setminus \{(\omega, \omega_1)\}$ with a topology inherited from the Tychonoff product of the one-point compactification of ω and the one-point compactification of the discrete space ω_1. Such a space is not normal, and thus it is neither paracompact nor Lindelöf. The set $\omega \times (\omega_1 + 1)$ is a σ-compact dense subset of X. The space X is totally metacompact [7].

Recall that every metacompact separable space is Lindelöf. Then we found another partial answer to the Curtis question with the following result. The proof of it is a modification of the proof of Theorem 2.1.

Theorem 2.5. *Every separable totally metacompact space is Menger.*

Corollary 2.6. *For a separable space X, the following assertions are equivalent.*

(1) *The space X is regular and Menger.*
(2) *The space X is totally paracompact.*
(3) *The space X is regular and totally metacompact.*

A space X is *scattered* if for every nonempty closed set $A \subseteq X$ there is a point $a \in A$ which is isolated in the relative topology of A. Let X be a space. Define $X^{(0)} := X$. Fix an ordinal number α and assume that the sets $X^{(\beta)}$ have been defined for all ordinal numbers $\beta < \alpha$. If $\alpha = \beta + 1$, for some β, then

$$X^{(\alpha)} := X^{(\beta)} \setminus \{\, x \in X^{(\beta)} : x \text{ is isolated in } X^{(\beta)} \,\}.$$

If α is a limit ordinal, then

$$X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}.$$

A space X is scattered if and only if, there is an ordinal number α such that $X^{(\alpha)} = \emptyset$.

Proposition 2.7. *Let X be a Lindelöf space such that the space $X^{(\alpha)}$ is Menger for some ordinal number α. Then the space X is Menger.*

Proof. Let X be a Lindelöf space and α be the minimal ordinal number such that $X^{(\alpha)}$ is Menger. The proof is by induction on α. If $\alpha = 0$, then there is nothing to prove. Fix an ordinal number α and assume that for each Lindelöf space Y, if $Y^{(\beta)}$ is Menger, for some $\beta < \alpha$, then the space Y is Menger. Fix a sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of X. Let

$$X' := X \setminus X^{(\alpha)}.$$

There are finite sets $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots$ such that the family $\bigcup_{n\in\omega} \mathcal{F}_n$ covers $X^{(\alpha)}$. Let $U := \bigcup\bigcup_{n\in\omega} \mathcal{F}_n$. Fix $x \in X \setminus U$. Then there is an open neighborhood V_x of x in X such that $\overline{V_x}^{(\beta)} = \emptyset$. Thus, the set $\overline{V_x}$ is Menger, by the assumption. Since the space $X \setminus U$ is Lindelöf as a closed subspace of X, there is a countable set $C \subseteq X \setminus U$ such that

$$X \setminus U \subseteq \bigcup_{x\in C} \overline{V_x}.$$

Then the space $X \setminus U$ is Menger as a countable union of Menger spaces $\overline{V_x} \setminus U$ for $x \in C$. There are finite sets $\mathcal{F}'_0 \subseteq \mathcal{U}_0, \mathcal{F}'_1 \subseteq \mathcal{U}_1, \ldots$ such that the family $\bigcup_{n\in\omega} \mathcal{F}'_n$ covers $X \setminus U$. Consequently the family $\bigcup_{n\in\omega} \mathcal{F}_n \cup \mathcal{F}'_n$ covers X and the sets $\mathcal{F}_n \cup \mathcal{F}'_n$ are finite subsets of $\mathcal{U}_n$. □

Corollary 2.8. *Every totally metacompact regular Lindelöf space X such that $X^{(\alpha)}$ is separable, for some ordinal number α, is Menger.*

Proof. If X is totally metacompact, then for every ordinal α, the set $X^{(\alpha)}$ is also totally metacompact. By Theorem 2.5, the space $X^{(\alpha)}$ is Menger. Apply Proposition 2.7. □

3. GO-spaces

Sakai [17] considered Problem 1.2 for subspaces of the Sorgenfrey line and more generally in the class of generalized ordered spaces which are defined on subsets of the real line (see definition below). This class of spaces has a long history in general topology and it has been well studied. A *real generalized-ordered space* (or abbreviated *real GO-space*) is a triple $(X, <, \tau)$, where $X \subseteq \mathbb{R}$, the relation $<$ is the usual order on $\mathbb{R}$ restricted to X and τ is a Hausdorff topology on X generated by some intervals (possibly degenerated). For simplicity we write a real GO-space $(X, <, \tau)$ as X. Every real GO-space X can be split as

$$X = R_X \cup E_X \cup L_X \cup I_X,$$

where:

$$\begin{aligned}
I_X &:= \{\, x \in X : \{x\} \text{ is open in } X \,\},\\
R_X &:= \{\, x \in X \setminus I_X : [x, \infty) \text{ is open in } X \,\},\\
L_X &:= \{\, x \in X \setminus I_X : (-\infty, x] \text{ is open in } X \,\},\\
E_X &:= \{\, x \in X \setminus I_X : \{\, (a, b) : a, b \in X, x \in (a, b) \,\} \text{ is a base at } x \,\}.
\end{aligned}$$

We get the following corollary from the previous section.

Corollary 3.1. *Every totally metacompact Lindelöf real GO-space is Menger.*

Proof. There is an ordinal number α such that $X^{(\alpha)} = X^{(\alpha+1)}$. It follows that $X^{(\alpha)}$ is a closed subspace of X with no isolated points. Any real GO-space with no isolated points is separable. Apply Theorem 2.8. □

Corollary 3.2. *For a Lindelöf real GO-space X, the following assertions are equivalent.*

(1) *The space X is Menger.*

(2) *The space X is totally paracompact.*
(3) *The space X is totally metacompact.*

Remark 3.3. The above corollary was obtained also by Sakai [17, Theorem 5.2], but his proof is much more technical than our.

Now we consider real GO-spaces defined on the entire real line. So, let $X = \mathbb{R}$, $X = R_X \cup E_X \cup L_X \cup I_X$, and let τ be some topology τ that comes from the partition. For a subset $C \subseteq X$, define

$$\begin{aligned} i(C) &:= \{\, x \in X : \{x\} \text{ is isolated in } \tau | C \,\}, \\ r(C) &:= (R_X \cap C) \setminus i(C), \\ l(C) &:= (L_X \cap C) \setminus i(C), \\ e(C) &:= E_X \cap C. \end{aligned}$$

Recall the following result of Benett–Balogh.

Theorem 3.4 *(Benett–Balogh [4]). Let X be a real GO-space defined on the entire real line. The space X is totally paracompact if and only if for every perfect set C in the standard topology on the real line, the set $i(C)$ is not dense and co-dense in C in the standard topology.*

Theorem 3.5. *A real GO-space defined on the entire real line is Menger if and only if it is σ-compact.*

Proof. Every σ-compact space is Menger. So, assume that X is a Menger real GO-space defined on the entire real line. Let $\mathcal{U}$ be the collection of all intervals (a, b), where $a, b \in \mathbb{R}$ and $a < b$, such that $(a, b) \cap X$ is σ-compact in X. Then the set $U := \bigcup \mathcal{U}$ is σ-compact in X. Towards a contradiction, assume that the set $C := X \setminus U$ is nonempty. Since U is open in the standard topology, we have that the set C is closed in the standard topology. Now we prove that C is perfect. Assume that there is an interval (a, b) such that $(a, b) \cap C = \{x\}$ for some $x \in \mathbb{R}$. Since $(a, x) \subseteq U$, the set U is σ-compact in X and (a, x) is an F_σ-set, the set (a, x) is σ-compact in X. Similarly, the set (x, b) is σ-compact in X. Thus, the set $(a, b) \cap X$ is σ-compact in X, too. It follows that $(a, b) \subseteq U$, a contradiction.

The set $C \cap R_X$ is not dense in C with the standard topology. On the contrary, since C is perfect, there is a copy C' of the Cantor set inside C such that the sets $i(C') \subseteq R_X \cap C$ and $r(C') \subseteq R_X \cap C$ are dense in C' with respect to the standard topology on C'. Then the set $i(C')$ of isolated points of C' is dense and co-dense in C'. By Theorem 3.4 and Corollary 3.2 we get a contradiction. Similarly, the set $C \cap L_X$ is not dense in C with the standard topology.

By the above observations, there is an interval $[a, b]$ in $\mathbb{R}$ such that the set $C' := [a, b] \cap C$ is perfect in the standard topology and it is disjoint with $R_X \cup L_X$. Thus, we may assume that $C = C'$. Now we prove that the set $i(C)$ is dense in C with respect to the standard topology. On the contrary, let $[a, b]$ be an interval in $\mathbb{R}$ such that the set $[a, b] \cap C$ is perfect in the standard topology and it is disjoint with $i(C)$. Then the topologies on $[a, b] \cap C$ inherited from $\mathbb{R}$ and X are the same. It follows that $(a, b) \cap X$ is σ-compact, and thus $(a, b) \in \mathcal{U}$, a contradiction.

The set $i(C)$ is also co-dense in C with respect to the standard topology. On the contrary, there is an interval $[a, b]$ in $\mathbb{R}$ such that the set $C' := [a, b] \cap C$ is perfect in the standard topology and it is contained in $i(C)$. It follows that C' is a closed and discrete subspace of X. Since X is Lindelöf, the set C' is countable, a contradiction.

Finally, we have that the set $i(C)$ is dense and co-dense in C in the standard topology. By Theorem 3.4 and Corollary 3.2 we get a contradiction.

We conclude that $C = \emptyset$, and thus the space X is σ-compact. □

4. Curtis's result via topological games

Many classical covering properties, starting with the Menger property, admit natural game-theoretic counterpart [21,19,3]. Recall the following definition.

Definition 4.1. A *Menger game* played on a space X, denoted by $\mathrm{M}(X)$, is a game with two players, ALICE and BOB, with an inning for each natural number n. In round n ALICE plays an open cover $\mathcal{U}_n$ of X and BOB replies with a finite set $\mathcal{F}_n \subseteq \mathcal{U}_n$. BOB wins the game, if the family $\bigcup_{n\in\omega} \mathcal{F}_n$ covers X and ALICE wins otherwise.

Theorem 4.2 *([13,19]). A space X is Menger if and only if ALICE has no winning strategy in the game* $\mathrm{M}(X)$.

We provide a game-theoretical proof of Theorem 1.1 which seems to be much simpler than the original one (modulo the above game-characterization of the Menger property). We need the following straightforward observation.

Observation 4.3. Let $\mathcal{B}'$ be an open cover of a space X and A' be a closed subset of X. If

- $A' \subseteq B'$ for some $B' \in \mathcal{B}'$,
- $B \cap A' = \emptyset$ for every $B \in \mathcal{B}' \setminus \{B'\}$, and
- $\{\, V_B : B \in \mathcal{B}' \,\}$ is an open cover of X,

then $A' \subseteq V_{B'}$.

Theorem 4.4 *([8, Theorem 3.1.]). Every regular Menger space is totally paracompact.*

Proof. Let X be a regular Menger space and $\mathcal{B}$ be a fixed base for X.

Round 0: By Lindelöfness, there exists a countable subcover, say $\mathcal{B}_0$, of $\mathcal{B}$. Again, by Lindelöfness, there is an open cover $\{\, V_B^0 : B \in \mathcal{B}_0 \,\}$ such that $\overline{V_B^0} \subseteq B$ for all $B \in \mathcal{B}_0$. ALICE plays $\mathcal{V}_0 := \{\, V_B^0 : B \in \mathcal{B}_0 \,\}$. BOB replies with $\{V_B^0 : B \in \mathcal{F}_0\}$, where $\mathcal{F}_0$ is a finite subset of $\mathcal{B}_0$. Put $A_0 := \bigcup_{B\in\mathcal{F}_0} \overline{V_B^0}$.

Round 1: There is a countable subcover $\mathcal{B}_1$ of $\mathcal{B}$ such that $\mathcal{F}_0 \subseteq \mathcal{B}_1$ and for every set $B \in \mathcal{B}_1 \setminus \mathcal{F}_0$, we have $B \cap A_0 = \emptyset$. There is an open cover $\{V_B^1 : B \in \mathcal{B}_1\}$ such that $\overline{V_B^1} \subseteq B$ for all $B \in \mathcal{B}_1$. Put $V_1 := \bigcup_{B\in\mathcal{F}_0} V_B^1$. By Observation 4.3, we have that $A_0 \subseteq V_1 \subseteq \overline{V_1} \subseteq \bigcup \mathcal{F}_0$. ALICE plays $\mathcal{V}_1 := \{V_1 \cup V_B^1 : B \in \mathcal{B}_1 \setminus \mathcal{F}_0\}$. BOB replies with $\{\, V_1 \cup V_B^1 : B \in \mathcal{F}_1 \setminus \mathcal{F}_0 \,\} \subseteq \mathcal{V}_1$, where $\mathcal{F}_1$ is a finite subset of $\mathcal{B}_1$. We may assume that $\mathcal{F}_0 \subseteq \mathcal{F}_1$. Put

$$A_1 := \bigcup_{B\in\mathcal{F}_1\setminus\mathcal{F}_0} \overline{V_1 \cup V_B^1} = \bigcup_{B\in\mathcal{F}_1} \overline{V_B^1}.$$

Fix a natural number $n > 1$.

Round n: There is a countable subcover $\mathcal{B}_n$ of $\mathcal{B}$ such that $\mathcal{F}_{n-1} \subseteq \mathcal{B}_n$ and for every set $B \in \mathcal{B}_n \setminus \mathcal{F}_{n-1}$, we have $B \cap A_{n-1} = \emptyset$. There is an open cover $\{V_B^n : B \in \mathcal{B}_n\}$ such that $\overline{V_B^n} \subseteq B$ for all $B \in \mathcal{B}_n$. Put $V_n := \bigcup_{B\in\mathcal{F}_{n-1}} V_B^n$. By Observation 4.3, we have that $A_{n-1} \subseteq V_n \subseteq \overline{V_n} \subseteq \bigcup \mathcal{F}_{n-1}$. ALICE plays $\mathcal{V}_n := \{V_n \cup V_B^n : B \in \mathcal{B}_n \setminus \mathcal{F}_{n-1}\}$. BOB replies with a family $\{\, V_n \cup V_B^n : B \in \mathcal{F}_n \setminus \mathcal{F}_{n-1} \,\} \subseteq \mathcal{V}_n$, where $\mathcal{F}_n$ is a finite subset of $\mathcal{B}_n$. We may assume that $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$. Put

$$A_n := \bigcup_{B\in\mathcal{F}_n\setminus\mathcal{F}_{n-1}} \overline{V_n \cup V_B^n} = \bigcup_{B\in\mathcal{F}_n} \overline{V_B^n}.$$

Since Alice has no winning strategy in $\mathrm{M}(X)$, there is a play, where Alice uses the above defined strategy and the game is won by Bob, i.e.,

$$\bigcup\{V_B^n : B \in \mathcal{F}_n, n \in \omega\}$$

is an open cover of X. For every $n \in \omega$ and $B \in \mathcal{F}_n$ we have $V_B^n \subseteq B$. It follows that $\bigcup_{n\in\omega} \mathcal{F}_n \subseteq \mathcal{B}$ is an open cover of X. Fix $x \in X$. There is a natural number n such that $x \in V_n$. For $k > n$ and $B \in \mathcal{F}_k \setminus \mathcal{F}_n$ we have $B \cap V_n = \emptyset$. It follows that the family $\bigcup_{n\in\omega} \mathcal{F}_n$ is locally finite. Thus, the space X is totally paracompact. □

5. Partial open neighborhood assignment game of Aurichi

Let X be a space. An *open neighborhood assignment of* X is a family $\{ U_x : x \in X \}$ of open sets in X such that $x \in U_x$ for all $x \in X$. The space X is a *D-space* [10] if for any open neighborhood assignment $\{ U_x : x \in X \}$ of X, there is a closed and discrete in X space $D \subseteq X$ such that the family $\{ U_x : x \in D \}$ covers X. In the literature, many references to a result that any Menger space is a D-space lead to a paper of Aurichi [2]. It turns out that this result, in the class of regular spaces, follows directly from the much older Theorem 1.1 (Aurichi does not assume that a space is regular). We provide a proof for the sake of completeness.

Proposition 5.1. *Every totally paracompact space is a D-space.*

Proof. Let X be a space and $\{U_x : x \in X\}$ be an open neighborhood assignment of X. For $x \in X$, let $\mathcal{B}_x$ be a local base at x such that $\bigcup \mathcal{B}_x \subseteq U_x$. Then $\mathcal{B} = \bigcup_{x\in X} \mathcal{B}_x$ is a base for X. By the assumption, there is a family $\mathcal{B}' \subset \mathcal{B}$ which is a locally finite cover of X. Clearly, the sets $\mathcal{B}' \cap \mathcal{B}_x$ are finite for each $x \in X$. Then the set $D = \{x \in X : \mathcal{B}' \cap \mathcal{B}_x \neq \emptyset\}$ is closed and discrete in X and the family $\{U_x : x \in D\}$ covers X. □

In order to prove that any Menger space is a D-space, Aurichi used the following topological game. Let X be a space. A *partial open neighborhood assignment in* X is a family $\{ V_x : x \in Y \}$, where $Y \subseteq X$ and each set V_x is an open neighborhood of x. For a partial open neighborhood assignment $\{ V_x : x \in Y \}$ in X and a set $D \subseteq Y$ define $V_D := \bigcup_{x\in D} V_x$. Partial open neighborhood assignments $\{ V_x : x \in Y \}$ and $\{ V'_x : x \in Y' \}$ in X are *compatible* if $V_x = V'_x$ for each $x \in Y \cap Y'$. Since the ambient space X will be clear from the context, a partial open neighborhood assignment in X will be just called a *partial open neighborhood assignment*.

Definition 5.2 *(Aurichi [2, Definition 2.1]).* Let X be a space. A *partial open neighborhood assignment game* played on X, denoted by $\mathrm{PONAG}(X)$, is a game with two players, Alice and Bob, with an inning per each natural number n. In round 0, Alice plays a partial open neighborhood assignment $\mathcal{U}_0 = \{ V_x : x \in Y_0 \}$ for some $Y_0 \subseteq X$ such that $\mathcal{U}_0$ covers X. Then Bob replies with a closed and discrete in X space $D_0 \subseteq Y_0$. In round $n > 0$, Alice plays a partial open neighborhood assignment $\mathcal{U}_n = \{ V_x : x \in Y_n \}$ in X such that

- $Y_n \subseteq X \setminus \bigcup_{k<n} V_{D_k}$,
- $\mathcal{U}_n$ covers $X \setminus \bigcup_{k<n} V_{D_k}$.

Bob replies with a closed and discrete in X space $D_n \subseteq Y_n$.

Bob wins the game, if the family $\bigcup_{n\in\omega}\{ V_x : x \in D_n \}$ covers X, otherwise Alice wins.

Recall some important results from the Aurichi paper from our point of view.

Theorem 5.3. *[2, Proposition 2.6] If X is a Menger space, then Alice has no winning strategy in the game* $\mathrm{PONAG}(X)$.

Theorem 5.4. *[2, Proposition 2.3] Let X be a space. If* Alice *has no winning strategy in the game* PONAG(X)*, then X is a D-space.*

It turns out that in a wide class of topological spaces, the implication from Theorem 5.3 is reversible.

Theorem 5.5. *Let X be a Lindelöf space such that each closed non-Menger subspace of X contains a convergent sequence.* Alice *has a winning strategy in the game* PONAG(X) *if and only if* Alice *has a winning strategy in the game* M(X).

Proof. ($\Rightarrow$): By Theorem 5.3, the space X is not Menger. Apply Theorem 4.2.

($\Leftarrow$): Fix a winning strategy σ for Alice in the game M(X). Play the game M(X) according to σ in order to show that we may assume some additional properties of moves for both players.

Round 0: Let $\mathcal{U}_0$ be a move of Alice played according to the strategy σ. Since the space X is Lindelöf, we may assume that $\mathcal{U}_0 = \{ U^0_k : k \in \omega \}$ is an increasing cover of X. By the assumption, there is a convergent sequence $Y_0 = \{ y^0_k : k \in \omega \} \subseteq X$. Since the cover $\mathcal{U}_0$ is increasing, a cofinal subset of Y_0 is contained in a single set from $\mathcal{U}_0$. Thus, we may assume that $Y_0 \subseteq U^0_0$. If not, then it is enough to replace Y_0 by its cofinal subset and $\mathcal{U}_0$ by its cofinal subfamily. Then, we may assume that Bob replies with a single set $U_0 \in \mathcal{U}_0$.

Round 1: Let $\mathcal{U}_1$ be the next move of Alice according to σ. We may assume that $\mathcal{U}_1 = \{ U^1_k : k \in \omega \}$ is an increasing open cover of X such that $U_0 \subseteq U^1_0$. Since Alice has a winning strategy in M(X), the space $X \setminus U_0$ cannot be Menger. Indeed, the strategy σ indicates a strategy for Alice in M$(X \setminus U_0)$. If $X \setminus U_0$ is Menger, then by Theorem 4.2 this indicated strategy cannot be a winning one which causes that σ is not a winning strategy, a contradiction. By the assumption, the space $X \setminus U_0$ contains a convergent sequence $Y_1 = \{ y^1_k : k \in \omega \}$. Since the cover $\mathcal{U}_1$ is increasing, a cofinal subset of Y_1 is contained in a single set from $\mathcal{U}_1$. Thus, we may assume that $Y_1 \subseteq U^1_0 \setminus U_0$. If not, then it is enough to replace Y_1 by its cofinal subset and $\mathcal{U}_1$ by its cofinal subfamily.

Fix $n > 0$ and let U_{n-1} be the last set picked by Bob in round $n-1$ in M(X).

Round n: Let $\mathcal{U}_n$ be a family played by Alice, according to σ. In a similar manner as in round 1, we may assume that $\mathcal{U}_n = \{ U^n_k : k \in \omega \}$ is an increasing open cover of X such that $U_{n-1} \subseteq U^n_0$ and there is a convergent sequence $Y_n \subseteq U^n_0 \setminus U_{n-1}$. Then Bob replies with a set $U_n \in \mathcal{U}_n$.

Formally, if Alice has a winning strategy in the game M(X), then Alice has a winning strategy in M(X) satisfying all the above additional conditions. Thus, we assume that the strategy σ already has all these properties. If

$$(\mathcal{U}_0, U_0, \mathcal{U}_1, U_1, \dots)$$

is a play in the game M(X), where Alice uses the strategy σ and in each inning Bob picks a single set, as above, then the family $\{ U_n : n \in \omega \}$ is increasing and it does not cover X.

Now we translate the above strategy σ to a strategy for Alice in PONAG(X) as follows.

Round 0: Let $\mathcal{U}_0 = \{ U^0_k : k \in \omega \}$ be the family played by Alice according to σ in M(X) and the set $Y_0 = \{ y^0_k : k \in \omega \} \subseteq U^0_0$ be a convergent sequence as above. Let $V_{y^0_k} := U^0_k$ for $k \in \omega$. In PONAG(X), Alice plays a partial open neighborhood assignment $\{ V_{y^0_k} : y^0_k \in Y_0 \}$ and Bob replies with a closed and discrete subspace $D_0 \subseteq Y_0$ in X. Since Y_0 is a convergent sequence, the set D_0 is finite. Since the family $\mathcal{U}_0$ is increasing, there is a set $U_0 \in \mathcal{U}_0$ such that $V_{D_0} = U_0$. In M(X), Bob replies with U_0.

Round 1: Let $\mathcal{U}_1$ be the next move of Alice according to σ in M(X) and $Y_1 \subseteq U^1_0 \setminus U_0$ be convergent sequence as above. Let $V_{y^1_k} := U^1_k$ for $k \in \omega$. In PONAG(X), Alice plays a partial open neighborhood assignment $\{ V_{y^1_k} : y^1_k \in Y_1 \}$ and Bob replies with a closed and discrete subspace $D_1 \subseteq Y_1$ in X. Since Y_1 is a convergent sequence, the set D_1 is finite. Since the family $\mathcal{U}_1$ is increasing, there is a set $U_1 \in \mathcal{U}_1$ such that $V_{D_1} = U_1$. In M(X), Bob replies with U_1.

Fix $n > 0$ and suppose that open sets $U_0 \subseteq \cdots \subseteq U_{n-1}$, partial open neighborhood assignments $\{ V_x : x \in Y_k \}$ and finite sets $D_k \subseteq Y_k \subseteq U_k \setminus U_{k-1}$ have already been defined for $0 \le k \le n$.

Round n: Let $\mathcal{U}_n$ be a family played by Alice, according to σ in $\mathrm{M}(X)$ and $Y_n = \{ y_k^n : k \in \omega \} \subseteq U_0^n \setminus U_{n-1}$ be a convergent sequence as above. Let $V_{y_k^n} := U_k^n$ for $k \in \omega$. In $\mathrm{PONAG}(X)$, Alice plays a partial open neighborhood assignment $\{ V_{y_k^n} : y_k^n \in Y_n \}$ and Bob replies with a closed and discrete subspace $D_n \subseteq Y_n$ in X. Since Y_n is a convergent sequence, the set D_n is finite. Since the family $\mathcal{U}_n$ is increasing, there is a set $U_n \in \mathcal{U}_n$ such that $V_{D_n} = U_n$. In $\mathrm{M}(X)$, Bob replies with U_n.

Since σ is a winning strategy for Alice in $\mathrm{M}(X)$, the family $\{ U_n : n \in \omega \}$ does not cover X. We have

$$\bigcup_{n\in\omega} U_n = \bigcup_{n\in\omega} V_{D_n},$$

and thus the above defined strategy for Alice in $\mathrm{PONAG}(X)$ is a winning strategy. □

Proposition 5.6. *Let X be a space. If Bob has a winning strategy in $\mathrm{M}(X)$, then Bob has a winning strategy in $\mathrm{PONAG}(X)$.*

Proof. Let σ be a winning strategy for Bob in $\mathrm{M}(X)$. Define a strategy for Bob in $\mathrm{PONAG}(X)$ as follows.

Round 0: Assume that in $\mathrm{PONAG}(X)$, Alice plays a partial open neighborhood assignment $\mathcal{U}_0 = \{ V_x^0 : x \in Y_0 \}$ for some set $Y_0 \subseteq X$. A family $\mathcal{U}_0' := \mathcal{U}_0$ is an open cover of X. Assume that $\mathcal{U}_0'$ is a family played by Alice in $\mathrm{M}(X)$. Let $\mathcal{F}_0' \subseteq \mathcal{U}_0'$ be a finite set played by Bob in $\mathrm{M}(X)$, according to σ, as a response to $\mathcal{U}_0'$. There is a finite set $D_0 \subseteq Y_0$ such that $\bigcup \mathcal{F}_0' = V_{D_0}$. Then in $\mathrm{PONAG}(X)$, Bob plays the set D_0.

Round 1: Assume that, in $\mathrm{PONAG}(X)$, Alice plays an open neighborhood assignment $\mathcal{U}_1 = \{ V_x^1 : x \in Y_1 \}$ for some set $Y_1 \subseteq X \setminus V_{D_0}$. Since the family $\mathcal{U}_1$ covers $X \setminus V_{D_0}$, a family $\mathcal{U}_1' := \{ V_x^1 \cup V_{D_0} : x \in Y_1 \}$ is an open cover of X. Assume that $\mathcal{U}_1'$ is the next move of Alice in $\mathrm{M}(X)$. Let $\mathcal{F}_1' \subseteq \mathcal{U}_1'$ be a finite set played by Bob in $\mathrm{M}(X)$, according to σ, as a response to $\mathcal{U}_1'$. There is a finite set $D_1 \subseteq Y_1$ such that $\mathcal{F}_1' = \{ V_x^1 \cup V_{D_0} : x \in D_1 \}$. We have $\bigcup \mathcal{F}_1' = V_{D_0} \cup V_{D_1}$. Then in $\mathrm{PONAG}(X)$, Bob plays the set D_1.

Fix $n > 1$ and assume that open neighborhoods assignments $\{ V_x : x \in Y_k \}$ played by Alice and sets $D_k \subseteq Y_k$ played by Bob have been already defined for all $0 \le k \le n$.

Round n: Assume that Alice plays an open neighborhood assignment $\mathcal{U}_n = \{ V_x^n : x \in Y_n \}$ in $\mathrm{PONAG}(X)$, for some set $Y_n \subseteq X \setminus \bigcup_{k<n} V_{D_k}$. Since the family $\mathcal{U}_n$ covers $X \setminus \bigcup_{k<n} V_{D_k}$, a family

$$\mathcal{U}_n' := \{ V_x^n \cup \bigcup_{k<n} V_{D_k} : x \in Y_n \}$$

is an open cover of X. Assume that $\mathcal{U}_n'$ is the next move of Alice in $\mathrm{M}(X)$. Let $\mathcal{F}_n' \subseteq \mathcal{U}_n'$ be a finite set played by Bob in $\mathrm{M}(X)$, according to σ, as a response to $\mathcal{U}_n'$. There is a finite set $D_n \subseteq Y_n$ such that

$$\mathcal{F}_n' = \{ V_x^n \cup \bigcup_{k<n} V_{D_k} : x \in D_n \}.$$

We have $\bigcup \mathcal{F}_n' = \bigcup_{k\le n} V_{D_k}$. Then in $\mathrm{PONAG}(X)$, Bob plays the set D_n.

Since σ is a winning strategy for Bob in $\mathrm{M}(X)$, the family $\bigcup_{n\in\omega} \mathcal{F}_n'$ covers X. Since $\bigcup \mathcal{F}_n' = \bigcup_{k\le n} V_{D_k}$ for each n, the family $\{ V_{D_n} : n \in \omega \}$ covers X. It follows that the strategy defined above for Bob in $\mathrm{PONAG}(X)$ is a winning strategy. □

Recall that a space X is Fréchet-Urysohn if for every $A \subseteq X$ and $p \in \overline{A}$ there exists a sequence in A converging to p.

Proposition 5.7. *Let X be a Lindelöf Fréchet-Urysohn space. If Bob has a winning strategy in $\mathrm{PONAG}(X)$, then Bob has a winning strategy in $\mathrm{M}(X)$.*

Proof. Let σ be a winning strategy for Bob in PONAG(X). We construct a winning strategy for Bob in M(X) as follows.

Round 0: Let $\mathcal{U}_0$ be a family played by Alice in M(X). Since the space X is Lindelöf, we may assume that $\mathcal{U}_0 = \{ U_k^0 : k \in \omega \}$ is an increasing cover of X. We may also assume that there is a non-isolated point y_0 in X. Indeed, because otherwise, the space is countable and we are done. By the Fréchet property, there is a sequence $Y_0 = \{ y_k^0 : k \in \omega \} \subseteq X$ converging to y_0. For simplicity we may assume that $y_0 \in U_0^0$. Let $V_{y_k^0} := U_k^0$ for all k. Assume that in PONAG(X) Alice plays the family $\mathcal{U}_0' := \{ V_{y_k^0} : y_k^0 \in Y_0 \}$. Let $D_0 \subseteq Y_0$ be a set played by Bob according to σ, in response to $\mathcal{U}_0'$. Since the set D_0 is a subsequence of a convergent sequence and it is closed in X, the set D_0 is finite. There is a set $U_0 \in \mathcal{U}_0$ such that $U_0 = V_{D_0}$ and Bob plays the set U_0 in M(X) as a response to $\mathcal{U}_0$.

Round 1: Assume that $\mathcal{U}_1$ is a family played by Alice in M(X). Since the space X is Lindelöf, we may assume that $\mathcal{U}_1 = \{ U_k^1 : k \in \omega \}$ is an increasing cover of X and also that $U_0 \subseteq U_0^1$. As before we may assume that there is a non-isolated point y_1 in $X \setminus V_{D_0}$. By the Fréchet property, there is a sequence $Y_1 = \{ y_k^1 : k \in \omega \} \subseteq X \setminus V_{D_0}$ converging to y_1. Let $V_{y_k^1} := U_k^1$ for all k. Assume that in PONAG(X) Alice plays the family $\mathcal{U}_1' := \{ V_{y_k^1} : y_k^1 \in Y_1 \}$. Let $D_1 \subseteq Y_1$ be a set picked by Bob according to σ, in response to $\mathcal{U}_1'$. Since the set D_1 is a subsequence of a convergent sequence and it is closed in X, the set D_1 is finite. There is a set $U_1 \in \mathcal{U}_1$ such that $U_1 = V_{D_1}$ and Bob plays the set U_1 in M(X) as a response to $\mathcal{U}_1$.

Fix $n > 1$ and assume that

- families $\mathcal{U}_k$ played by Alice in M(X),
- sets U_k played by Bob in M(X),
- convergent sequences $Y_k := \{ y_i^k : i \in \omega \}$
- partial open neighborhood assignments $\{ V_{y_i^k} : y_i^k \} \in Y_k$ played by Alice in PONAG(X)
- sets $D_k \subseteq Y_k$ played by Bob in PONAG(X) according to the strategy σ in response to $\mathcal{U}_k'$ such that $U_k = V_{D_k}$

have already been defined for all $0 \leq k < n$ and $U_0 \subseteq \cdots \subseteq U_{n-1}$.

Round n: Assume that $\mathcal{U}_n$ is a family played by Alice in M(X). Since the space X is Lindelöf, we may assume that $\mathcal{U}_n = \{ U_k^n : k \in \omega \}$ is an increasing cover of X and also that $U_{n-1} \subseteq U_0^n$. We may assume that there is a non-isolated point y_n in $X \setminus V_{D_n}$. By the Fréchet property, there is a sequence $Y_n = \{ y_k^n : k \in \omega \} \subseteq X \setminus V_{D_{n-1}}$ converging to y_n. Let $V_{y_k^n} := U_k^n$ for all y_k^n. Assume that in PONAG(X), Alice plays the family $\mathcal{U}_n' := \{ V_{y_k^n} : y_n^k \in Y_n \}$. Let $D_n \subseteq Y_n$ be a set picked by Bob according to σ, in response to $\mathcal{U}_n'$. Since the set D_n is a subsequence of a convergent sequence and it is closed in X, the set D_n is finite. There is a set $U_n \in \mathcal{U}_n$ such that $U_n = V_{D_n}$ and Bob plays the set U_n in M(X) as a response to $\mathcal{U}_n$.

Since σ is a winning strategy for Bob in PONAG(X), the family $\{ V_{D_n} : n \in \omega \}$ covers X. Since $U_n = V_{D_n}$ for all n, the family $\{ U_n : n \in \omega \}$ also covers X, and thus the defined above strategy for Bob in M(X) is a winning strategy. □

Remark 5.8. The proof of Theorem 5.5 and Proposition 5.7 remain true if Bob has a winning strategy in a modified version of PONAG, where in each round n, Bob in response to an Alice's partial open neighborhood assignment $\{ V_x : x \in Y_n \}$, plays just a closed subset $D_n \subseteq Y_n$ in X (not necessarily closed and discrete in the ambient space X).

Corollary 5.9. *Let X be a Lindelöf Fréchet space. Then* PONAG(X) *and* M(X) *are equivalent games.*

By a result of Telgarsky [21], if a space X is perfectly normal and Bob has a winning strategy in $\mathrm{M}(X)$, then the space X is σ-compact. Combination of this fact together with our results gives the following corollary.

Corollary 5.10. *Let X be a perfectly normal space. Then the following assertions are equivalent.*

(1) *Bob has a winning strategy in* $\mathrm{M}(X)$.
(2) *Bob has a winning strategy in* $\mathrm{PONAG}(X)$.
(3) *The space X is σ-compact.*

6. Comments and open problems

Subspaces of the Sorgenfrey line and Lindelöfness in all finite powers. By the result of Michael [16], assuming that the Continuum Hypothesis holds, there is an uncountable subspace of the Sorgenfrey line, whose all finite powers are Lindelöf. On the other hand, by the result of Todorcevic [22], if the open colorings axiom holds (a statement independent from ZFC), each uncountable subspace of the Sorgenfrey line has a non-Lindelöf square.

A space X is *Hurewicz* [13], if for each sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of X, there are finite sets $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots$ such that the family $\{\bigcup \mathcal{F}_n : n \in \omega\}$ is a γ-*cover* of X, i.e., the sets $\{n : x \in \bigcup \mathcal{F}_n\}$ are cofinite for all $x \in X$. We have the following implications

$$\sigma\text{-compactness} \longrightarrow \text{Hurewicz} \longrightarrow \text{Menger}$$

and none of these implications is reversible, even in the class of subspaces of the real line [14,5,6,23].

Let $\mathrm{P}(\omega)$ be the powerset of ω. We identify each set from $\mathrm{P}(\omega)$ with its characteristic function, an element of the Cantor cube 2^ω. Let $[\omega]^\omega$ be the family of all infinite subsets of ω and Fin be the family of all finite subsets of ω. Identifying each set $x \in [\omega]^\omega$ with the increasing enumeration of its elements, we have that $x \in \omega^\omega$. For $x, y \in [\omega]^\omega$, we write $x \leq^* y$ if the set $\{n : x(n) \leq y(n)\}$ is cofinite. A set $A \subseteq [\omega]^\omega$ is *unbounded*, if for each $y \in [\omega]^\omega$, there is $x \in A$ such that $x \not\leq^* y$. Let $\mathfrak{b}$ be the minimal cardinality of an unbounded set in $[\omega]^\omega$. We have $\omega_1 \leq \mathfrak{b} \leq \mathfrak{c}$ and it is independent from ZFC that $\mathfrak{b} = \omega_1 < \mathfrak{c}$. It turns out that the above mentioned Michael result can be obtained using much milder hypothesis than the Continuum Hypothesis.

Example 6.1. Assume that $\mathfrak{b} = \omega_1$. There is an uncountable subspace of the Sorgenfrey line whose all finite powers are Hurewicz (in particular, they are Lindelöf).

The set $\mathrm{P}(\omega)$ can be treated as a subset of the real line using the standard homeomorphism between 2^ω and the standard middle-third Cantor set C in the real line. Using this standard homeomorphism and the above identification of elements in $\mathrm{P}(\omega)$ and 2^ω, we see that a counterpart in $\mathrm{P}(\omega)$ of the set of all left-endpoints of intervals which are engaged in the construction of this standard middle-third Cantor set C is the set Fin. Thus, we may assume that $\mathrm{P}(\omega) = C$. Let $\mathrm{P}(\omega)_{\mathrm{M}}$ be the set $\mathrm{P}(\omega)$ with a topology, where open neighborhoods of points from Fin are inherited from the standard topology on $\mathbb{R}$ and the points from $[\omega]^\omega$ are isolated.

Let $X = \{x_\alpha : \alpha < \mathfrak{b}\} \subseteq [\omega]^\omega$ be an unbounded set in $[\omega]^\omega$ such that $x_\alpha \leq^* x_\beta$ for all $\alpha < \beta < \mathfrak{b}$ (such a set exists in ZFC). Let $(X \cup \mathrm{Fin})_{\mathrm{M}}$ be the set $X \cup \mathrm{Fin}$ with a topology inherited from $\mathrm{P}(\omega)_{\mathrm{M}}$. Now let $(X \cup \mathrm{Fin})_{\mathrm{S}}$ be the set $X \cup \mathrm{Fin}$ with a topology inherited from the Sorgenfrey line. Observe that in such a case the basic open sets for the points of Fin are exactly the same, when consider Fin as a subspace of $(X \cup \mathrm{Fin})_{\mathrm{M}}$ and $(X \cup \mathrm{Fin})_{\mathrm{S}}$. Thus, the identity map from $(X \cup \mathrm{Fin})_{\mathrm{M}}$ onto $(X \cup \mathrm{Fin})_{\mathrm{S}}$ is continuous.

By the result of Tsaban and the third named author [20], since $\mathfrak{b} = \omega_1$, all finite powers of $(X \cup \mathrm{Fin})_{\mathrm{M}}$ are Hurewicz. Since $(X \cup \mathrm{Fin})_{\mathrm{S}}$ is a continuous image of $(X \cup \mathrm{Fin})_{\mathrm{M}}$ and the Hurewicz property is preserved by continuous mappings, all finite powers of $(X \cup \mathrm{Fin})_{\mathrm{S}}$ are Hurewicz (and thus Lindelöf).

6.1. Open problems

We finish the paper with the following problems emerging from our investigations. A wider class than σ-compact spaces is the class of Hurewicz spaces. In the light of Theorem 2.1 the following problem arises.

Problem 6.2. *Let X be a totally paracompact space with a dense Hurewicz subspace. Is the space X Menger? What if this dense subset is a Lindelöf space of size smaller than $\mathfrak{b}$? What if the Martin Axiom holds plus the negation of the Continuum Hypothesis?*

Problem 6.3. *Are the Menger game and the partial open neighborhood assignment game equivalent in the class of Lindelöf spaces?*

Problem 6.4. *Let X be a Menger space and assume that Bob has a winning strategy in* $\mathrm{PONAG}(X)$*. Does it follow that Bob has a winning strategy in* $\mathrm{M}(X)$*?*

We would like also to know whether there is some relation between the partial open neighborhood assignment game and the total paracompactness.

Problem 6.5. *Let X be a regular space.*

(1) *Does Alice have no winning strategy in* $\mathrm{PONAG}(X)$ *imply that X is totally paracompact?*
(2) *Does Bob have a winning strategy in* $\mathrm{PONAG}(X)$ *imply that X is totally paracompact?*

Acknowledgements

The authors thank the National Group for Algebric and Geometric Structures, and their Applications (GNSAGA-INdAM) for its invaluable support. Parts of the work reported here were carried out during two visits of the third named author at the University of Messina. These visits were partially supported by the University of Messina. The third named author thanks the other authors for their kind hospitality. All the authors would like to thank the referee for the careful reading of the manuscript and for the valuable suggestions provided.

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