UNIVERSITY OF MESSINA
PHD JOINT PROGRAM:
UNIVERSITY OF CATANIA - UNIVERSITY OF PALERMO
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES,
PHYSICAL SCIENCES AND EARTH SCIENCES
XXXVIII CYCLE

DOCTORAL THESIS

A probabilistic analysis of compound and iterated conditionals under coherence: theoretical aspects, applications, and related topics

SSD: MATH-03/B

Author:
Lydia CASTRONOVO

Supervisor:
Prof. Giuseppe SANFILIPPO

PhD Coordinator:
PROF. FRANCESCO OLIVERI

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics and Computational Sciences

A.A. 2024/2025

Declaration of Authorship

I, Lydia Castronovo, declare that this thesis titled "A probabilistic analysis of compound and iterated conditionals under coherence: theoretical aspects, applications, and related topics" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: Lydia Castronovo

Date: 21 / 12 / 2025

“*Questo non è il Vietnam, è ~~il bowling~~ la Matematica, ci sono delle regole. [...] Amico mio, stai per entrare in una valle di lacrime.*”

The Big Lebowski (semicit.)

UNIVERSITY OF MESSINA

Abstract

Department of Mathematical and Computer Sciences,
Physical Sciences and Earth Sciences

Doctor of Philosophy

A probabilistic analysis of compound and iterated conditionals under coherence: theoretical aspects, applications, and related topics

by LYDIA CASTRONOVO

This dissertation analyses various aspects on conditional events, compound conditionals, and coherence in the framework of subjective probability. We investigate logical and probabilistic properties of iterated conditionals defined as trivalent objects and as suitable conditional random quantities. We check the validity of two generalised versions of Bayes' Rule for iterated conditionals and we study the p-validity of generalised versions of *Modus Ponens* and *two-premise centering* for iterated conditionals. Then, we study a generalisation of the operations of conjunction and disjunction of conditional events in the context of conditional random quantities, where these new objects, instead of depending on the probabilities of the two conditional events, depend on two arbitrary values a, b in the unit interval. We show that they are connected by a generalised version of the De Morgan's law and, by means of a geometrical approach, we compute the lower and upper bounds on these new objects both in the precise and the imprecise case. Moreover, some particular cases, obtained for specific values of a and b or in case of some logical relations, are analysed. We also show how the subjective approach to conditional probability and compound conditionals can be applied to the theory of fuzzy sets. Then, we continue the analysis of the algebraic approach to conditionals by defining the Boolean algebra of iterated conditionals, and with the study of some (probabilistic) properties related to the concept of "canonical extension". Finally we show how, thanks to coherence and proper scoring rules, the concepts of entropy and extropy can be extended from partitions to arbitrary family of events, and we find the associated Bregman divergences.

Acknowledgements

I would like to start by thanking my supervisor Professor Giuseppe Sanfilippo, in first place for proposing me to apply for this program, and second for the attention and the hard work in guiding me. Among all the bad things this Ph.D. gave me there is a really good one that I need to mention. It gave me the possibility to spend the most beautiful four months in Barcelona, where not only I enjoyed doing research since a long time, but I also met brilliant, passionate, and inspiring mathematicians. A special thanks to Lluis Godo and Tommaso Flaminio for giving me the possibility of studing at IIIA-CSIC and for being so patient with me during meeting, and to Valeria and Francesco for welcoming me immediately and making me feel at home. But most of my journey has been in Palermo, where I also have the most of the people I really have to spend a word for. Thank you Bruno, Alessandro and Manuel for sharing your experience and being always supportive and positive, and Federica for also being the most incredible guide in Barcelona. Special thanks to Professor Marco Tabacchi and the "tabacchini" Giuseppe Filippone, and Mario for giving me the possibility to work in a research team and on topics far from my research comfort zone. I can't go any further without mentioning the two people who have made these two years truly special for me. Giuseppe Failla, I probably couldn't survive these three years and I can not even imagine this Ph.D. without you. You are the greatest friend and collegue I could have ever asked for, and a passionate and exceptional mathematician, never let anyone make you think otherwise (XXXVIII cycle for ever!). Gianmarco, your presence has turned my life upside down and made it colorful again. Thank you for always being by my side, for making me brave and for inspiring me in every aspects of my life.

I thank the universe for also having wonderful and supportive people outside the academia. Francesco, Ivan, and Gabriella, you have been my anchor to reality outside the university world. Thank you for being such a strong and essential presence in my life. Thanks to Teresa for showing me that distance in friendship means just longer phone call to catch up. I know that our university paths divided many years ago, but your friendship make me always feel understood. Angela, thank you for your friendship and for sharing my love for spicy food and my new obsession for Korea, 사랑해.

Thanks also to Alessandra, Simona, Vittoria, and Elisa for always being the dearest friends I could wish for.

Last but definitely not least, I want to thank my lovely and irreplaceable family. Mum, Dad, you are the most amazing parents a girl can desire, your passion for your work, your love for culture and your curiosity for life made me want to always dream big. Also thank you for the weirdest Christmas gifts ever as a child, I cannot exclude that part of the responsability of this thesis is due to the book "Pensare i numeri". I also have as a brother the most brilliant and wonderful human being that ever touched this planet Earth. Claudio, my admiration and love for you are everlasting.

P.S. I also want to thanks my (not so-) little turtle Punk and my bimbo-cane Rohan for teaching me the importance and effectiveness of pet therapy.

Contents

List of Figures

To my grandparents Antonina, Rosario, Cristina, Onofrio, and Rosa
To my uncle Mario, to whom I owe my name

Introduction

This thesis is the result of research carried out over the three years of PhD in Mathematics and Computational Sciences. The aim is to bring together research ([17, 20]) and conference papers ([18, 19, 22]) on the main line of research carried out during this PhD, that is on the study of conditional events, compound and iterated conditionals, and coherence in the framework of subjective probability.

Conditionals are typically expressed by "if-then" sentences and they are useful and important tools in many fields. The analysis of conditionals and how to combine them is a relevant topic from the more theoretical areas, such as mathematics and philosophy ([1, 32, 71]), to the applicative ones, like psychology and artificial intelligence ([15, 34, 81]). The study of conditional events is not limited to how to combine them through conjunction and disjunction. Indeed, objects of interest are certainly iterated conditionals, structures where both the antecedent and the consequent are conditionals. Compound and iterated conditionals are widely used in natural language to describe decisions and inferences based on incomplete or uncertain information. Among the various tools for managing conditional uncertainty, subjective probability theory allows to evaluate uncertainty on a conditional by means of a conditional probability assessment, interpreted as a degree of belief. In this theory, given an individual in a certain state of knowledge, the probability $P(E)$ attributed to an event E is a measure of the individual's degree of belief in the occurrence of E. A key point in the theory of subjective probability is how to consistently assess probabilities operationally. To do so, de Finetti proposed a coherence principle based on a suitable betting scheme ([33]). The subjective theory has been extended to conditionals by means of conditional probability assessments. In this framework, given two events A and B, the conditional *if A then B* is represented by a conditional event designated as $B|A$. This is a three-valued object which can be assessed as *true*, *false*, or *void* ([32]). In this formulation, the probability of the conditional *if A then B* is interpreted by the conditional probability $P(B|A)$. In the literature, not only conditional events but also logical operations among them have been usually defined in three-valued logic frameworks (see, e.g., [9, 24, 36, 66]). For what concern two-valued conditionals, an approach is given by the Boolean algebra of conditionals studied in [40], a super-structure that contains the three-valued algebra as a substructure. A different approach to compound and iterated conditionals has been developed in the setting of coherence in the recent papers [53, 56, 58]. In this approach, conjoined, disjoined, and iterated conditionals are defined not as three-valued objects, but rather as suitable conditional random quantities having possibly more than three values in the unit interval $[0, 1]$.

This dissertation is structured as follows. In Chapter 1, some introductory topics will be covered. In particular we will recall some basics on events, conditional events, coherence and its geometrical characterization, conjunction and disjunction of conditional events both in trivalent logics and in a conditional random quantities framework, and we briefly remind some notions on coherence for imprecise assessments. Then, in Chapter 2 we present the research paper [17], which is the extended version of the conference paper [21] presented at the "15th international conference on Scalable Uncertainty Management (SUM 2022)". The work is an analysis of different notion of iterated conditionals (structures where both the antecedent and the consequent are conditional events), some of them already known in the literature (Cooper-Calabrese in Section 2.2, de Finetti in Section 2.3, Farrell in Section 2.4) while others defined as random quantities with a particular structure (Section 2.5) depending on different definitions of conjunction. We check the validity of some basic properties, the Import-Export principle and two generalised versions of the Bayes' Rule for iterated conditioning. Then, we study the p-validity of generalised versions of Modus Ponens and two-premise centering for iterated conditionals. Subsequently, [20] is exposed in Chapter 3. This work is a reorganized and extended version of the conference paper [16] presented at the "13th International Symposium on Imprecise Probabilities: Theories and Applications (ISIPTA 2023)". In this paper, we propose a generalisation of the definitions of conjunction and disjunction of two conditional events as conditional random quantities. More precisely, these new defined structures, instead of depending on the probabilities of the two conditional events, are built using two values a and b in $[0,1]$. Then, we consider some logical and probabilistic relations between them and we study the coherent assessments both in the precise and the imprecise case. In Chapter 4, we present [18], where we analyse a definition of fuzzy set build through conditional events where the membership function is defined as a conditional probability. Then, the definitions of complement and intersection of fuzzy sets are given based on the operations of conditional events as random quantities. In Chapter 5, we continue the study on the Boolean algebra of conditional started in [40, 42] by iterating the construction to build the Boolean algebra of iterated conditionals. In particular we study how some properties of a probability on these Boolean algebras of conditionals can be characterized in terms of satisfiability of known principles of its "canonical extensions" to the algebra of iterated conditionals. Finally in Chapter 6, we consider the theory of the proper scoring rules and its relation with the study of coherence in term of penalty criterion. More precisely, we consider the asymmetric logarithmic scoring rules for the probability of an event and thanks to it we suitably extend the definitions of entropy and extropy, usually given for a partition of events, to the case of arbitrary families of events, and we find the associated Bregman divergences.

Chapter 1

Events, Conditionals and Coherence

We start this first chapter with some basic notions needed throughout this thesis, that are fundamental in particular for Chapter 2, Chapter 3, and Chapter 4. In the first part, we recall some preliminaries on events, conditional events, the notion of indicator of a conditional event, and the geometrical characterization of coherence. Then, we recall the definitions of conjunction and disjunction of conditionals both in the context of trivalent logics and in the framework of conditional random quantities and we remind some well-known properties satisfied by events, conditionals and iterated conditionals (structures where both antecedent and consequent are conditionals). We conclude the chapter briefly mentioning the notion of coherence in the case of imprecise assessments.
To help readability, preliminaries on Boolean algebra of conditionals and scoring rules are recalled in the chapter where they are needed (Chapter 5 and Chapter 6, respectively).

In what follows we use the same symbol to refer to an event A, a two-valued logical entity which can be *true*, or *false*, and its indicator, which takes value 1 in the first case and 0 in the second. We use the symbols Ω and $\varnothing$ to refer to the sure event and the impossible event, respectively. The negation of an event A is denoted by $\overline{A}$. Given two events A and B, we denote by $A \wedge B$ (resp., $A \vee B$), or simply by AB, their conjunction (resp., disjunction). When, an event A logically implies an event B, i.e., $A\overline{B} = \varnothing$, we write $A \subseteq B$. We say that n events $E_1, \dots, E_n$ are logically independent when there are no logical relations among them. Given two events A and H, with $H \neq \varnothing$, the conditional event $A|H$ is a three-valued logical entity which is *true*, or *false*, or *void*, according to whether AH is true, or $\overline{A}H$ is true, or $\overline{H}$ is true, respectively. The negation of a conditional event $A|H$ is defined as $\overline{A|H} = \overline{A}|H$. Moreover, we say that $A|H$ logically implies $B|K$, denoted by $A|H \subseteq B|K$, if and only if AH logically implies BK and $\bar{B}K$ logically implies $\bar{A}H$ (Goodman-Nguyen inclusion relation, [65], see also [82]) that is ([65]),

$$A|H \subseteq B|K \iff AH \subseteq BK \text{ and } \bar{B}K \subseteq \bar{A}H. \tag{1.1}$$

1.1 Conditional prevision and coherence

We denote by X a random quantity and by $\mathbb{P}(X)$ its prevision. In the betting scheme, given any event $H \neq \varnothing$, agreeing to the betting metaphor, if you assess that $\mu = \mathbb{P}(X|H)$ then for any given $s \in \mathbb{R}$ you are willing to pay an amount $s\mu$ and to receive sX, or $s\mu$, according to whether H is true, or false (bet called off), respectively. As we will see the conjunction of two conditional events can be seen as a conditional random quantity with a finite number of possible (numerical) values. Then, in what follows, for any given conditional random quantity $X|H$, we assume that, when H is true, the set of possible values of X is a finite set of real numbers and we say that $X|H$ is a finite conditional random quantity.

Given a prevision function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite conditional random quantities, consider a finite subfamily $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{P} = (\mu_1, \dots, \mu_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the conditional random quantity $X_i|H_i$, $i \in \{1, \dots, n\}$. With the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i (X_i - \mu_i)$. We denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$.

Definition 1. The function $\mathbb{P}$ defined on $\mathcal{K}$ is *coherent* if and only if, $\forall n \geqslant 1$, $\forall\, s_1, \dots, s_n$, $\forall\, \mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$, it holds that: $\min\, \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant \max\, \mathcal{G}_{\mathcal{H}_n}$.

In other words, $\mathbb{P}$ on $\mathcal{K}$ is incoherent if and only if there exists a finite combination of n bets such that, after discarding the case where all the bets are called off, the values of the random gain are all positive or all negative. In the particular case where $\mathcal{K}$ is a family of conditional events, then Definition 1 becomes the well known definition of coherence for a conditional probability function, denoted as P. Notice that, in the general case where the conditional random quantities in $\mathcal{K}$ are bounded but possibly infinitely valued, the condition $\min \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant \max \mathcal{G}_{\mathcal{H}_n}$ in Definition 1 becomes $\inf \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant \sup \mathcal{G}_{\mathcal{H}_n}$.

Following the approach given in [27, 53, 75], once a coherent assessment $\mu = \mathbb{P}(X|H)$ is specified, the conditional random quantity $X|H$ is not looked at as the restriction of X to H, but is defined as X, or μ, according to whether H is true, or $\overline{H}$ is true; that is,

$$X|H = XH + \mu\overline{H}. \tag{1.2}$$

Notice that the representation (1.2) is not circular. Once the value $\mu = \mathbb{P}(X|H)$ is (coherently) specified by the betting scheme, the object $X|H$ in (1.2) is (subjectively) determined. We observe that $\mathbb{P}(XH + \mu\overline{H}) = \mathbb{P}(X|H)$. Indeed,

$$\mathbb{P}(XH + \mu\overline{H}) = \mathbb{P}(X|H)P(H) + \mu P(\overline{H}) = \mu. \tag{1.3}$$

Remark 1. *Given any (finite) conditional random quantity $X|H$, with $\{x_1, \dots, x_r\}$ set of possible values of X when H is true, and a prevision assessment μ on $X|H$, by Definition 1 coherence requires that $\mu \in [\min\{x_1, \dots, x_r\}, \max\{x_1, \dots, x_r\}]$. Indeed, if you pay $\mu < \min\{x_1, \dots, x_r\}$ in a bet on $X|H$, when H is true it holds that $X - \mu > 0$ and hence $\min \mathcal{G}_H > 0$. Likewise, if you pay $\mu > \max\{x_1, \dots, x_r\}$ in a bet on $X|H$, when H is true it holds that $X - \mu < 0$ and hence $\max \mathcal{G}_H < 0$. Thus, $X|H = XH + \mu\overline{H} \in [\min\{x_1, \dots, x_r\}, \max\{x_1, \dots, x_r\}]$.*

In the particular case when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = P(A|H)$, where $P(A|H)$ is the conditional probability on $A|H$. Given a conditional event $A|H$ and a probability assessment $P(A|H) = x$, the indicator of $A|H$, denoted by the same symbol, is

$$A|H = AH + x\overline{H} = AH + x(1 - H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ x, & \text{if } \overline{H} \text{ is true.} \end{cases} \tag{1.4}$$

Notice that $\mathbb{P}(AH + x\overline{H}) = xP(H) + xP(\overline{H}) = x$. For the indicator of the negation of $A|H$ it holds that $\overline{A}|H = 1 - A|H$. Given two conditional events $A|H$ and $B|K$, for every coherent assessment (x, y) on $\{A|H, B|K\}$, it holds that ([58, formula (15)])

$$AH + x\overline{H} \leqslant BK + y\overline{K} \Longleftrightarrow A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B,$$

that is, between the numerical values of $A|H$ and $B|K$, under coherence it holds that

$$A|H \leqslant B|K \iff A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B. \tag{1.5}$$

We conclude this subsection by recalling a result ([53, Theorem 4]), needed later in Section 3.1, showing that if two conditional random quantities $X|H$, $Y|K$ coincide when $H \vee K$ is true, then $X|H$ and $Y|K$ also coincide when $H \vee K$ is false, and hence $X|H$ coincides with $Y|K$ in all cases.

Theorem 1. *Given any events $H \neq \emptyset$ and $K \neq \emptyset$, and any random quantities X and Y, let Π be the set of the coherent prevision assessments $\mathbb{P}[X|H] = \mu$ and $\mathbb{P}[Y|K] = \nu$.*
(i) Assume that, for every $(\mu, \nu) \in \Pi$, $X|H = Y|K$ when $H \vee K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.
(ii) For every $(\mu, \nu) \in \Pi$, $X|H = Y|K$ when $H \vee K$ is true if and only if $X|H = Y|K$.

Remark 2. *Theorem 1 has been generalised in [56, Theorem 6] by replacing the symbol "$=$" by "$\leqslant$" in statements (i) and (ii). In other words, if $X|H \leqslant Y|K$ when $H \vee K$ is true, then $\mathbb{P}[X|H] \leqslant \mathbb{P}[Y|K]$ and hence $X|H \leqslant Y|K$ in all cases.*

1.2 Axiomatic conditional probability

In this section, we briefly recall the definition of a conditional probability from an axiomatic point of view.
We observe that, given a function P defined on $\mathcal{B} \times \mathcal{H}$, where $\mathcal{B}$ and $\mathcal{H}$ are arbitrary families of events, with $H \neq \emptyset$ for every $H \in \mathcal{H}$, P satisfies the properties of a conditional probability if the following conditions hold.

(i) $P(E|H) \geqslant 0$ and $P(H|H) = 1$ for every $E \in \mathcal{B}$ and $H \in \mathcal{H}$;

(ii) $P[(E_1 \vee E_2)|H] = P(E_1|H) + P(E_2|H)$, if $E_1E_2H = \emptyset$, $E_1, E_2 \in \mathcal{B}$, $H \in \mathcal{H}$;

(iii) $P(EH|K) = P(E|HK)P(H|K)$, for every $E, EH, H \in \mathcal{B}$ and $K, HK \in \mathcal{H}$.

We recall that, if P is coherent, then the properties $(i), (ii)$, and (iii) are satisfied; however, these properties are in general not sufficient for coherence of P. We list below some conditions which, together with $(i), (ii)$, and (iii), imply coherence of P (see, e.g.,[28, 35, 70, 87]):
(a) $\mathcal{B}$ is a boolean algebra and $\mathcal{H} = \mathcal{B}^0 = \mathcal{B}\backslash\{\emptyset\}$;
(b) $\mathcal{B}$ is a boolean algebra and $\mathcal{H} \cup \{\emptyset\}$ is a subalgebra of $\mathcal{B}$;

(c) $\mathcal{B}$ is a boolean algebra and $\mathcal{H}$ is an additive (or P-quasi additive) subset of $\mathcal{B}\backslash\{\emptyset\}$;
(d) $\mathcal{B}$ is a boolean algebra, $\mathcal{H}$ is a nonempty subset of $\mathcal{B}\backslash\{\emptyset\}$, and the following condition is satisfied ([31])

$$\prod_{i=1}^{n} P(E_i|H_i) = \prod_{i=1}^{n} P(E_i|H_{i+1}),$$

where $E_i \in \mathcal{B}$, $H_i \in \mathcal{H}$, $E_i \subseteq H_iH_{i+1}$, $i = 1, \dots, n$, and $H_{n+1} = H_1$.
In particular, in [28] the following (equivalent) definition of coherence is given

Definition 2. *The assessment $P(\cdot|\cdot)$ on an arbitrary family of conditional events $\mathcal{K}$ is coherent if P can be extended as a conditional probability (i.e., which satisfies $(i), (ii)$, and (iii)) on $\mathcal{B} \times \mathcal{B}^0 \supseteq \mathcal{K}$, with $\mathcal{B}$ algebra and $\mathcal{H} \cup \{\emptyset\}$ an additive subset of $\mathcal{B}$.*

In our approach, when P is coherent, it is called a *conditional probability*. In particular, when $\mathcal{B}$ is an algebra and $\mathcal{H} = \mathcal{B}^0$, P is said a *full conditional probability* ([35]).

Remark 3. *We recall that if a function P defined on an arbitrary family of conditional events $\mathcal{K}$ is coherent, then it can always be extended to a full conditional probability P' defined on $\mathcal{B} \times \mathcal{B}^0 \supseteq \mathcal{K}$.*

Notice that, when $K = \Omega$, from (iii) it follows that $P(EH) = P(E|H)P(H)$, where $P(\cdot) = P(\cdot|\Omega)$. In particular, $P(E|H) = \frac{P(EH)}{P(H)}$, when $P(H) > 0$.

1.3 Geometrical characterization of coherence

Given a family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, for each $i \in \{1, \ldots, n\}$ we denote by $\{x_{i1}, \ldots, x_{ir_i}\}$ the set of possible values of X_i when H_i is true; then, we set $A_{ij} = (X_i = x_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, r_i$. We set $C_0 = \overline{H}_1 \cdots \overline{H}_n$ (it may be $C_0 = \emptyset$) and we denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Hence $\bigwedge_{i=1}^n (A_{i1} \vee \cdots \vee A_{ir_i} \vee \overline{H}_i) = \bigvee_{h=0}^m C_h$. With each C_h, $h \in \{1, \ldots, m\}$, we associate a vector $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}$, $j = 1, \ldots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq \overline{H}_i$; with C_0 we associate $Q_0 = \mathcal{P} = (\mu_1, \ldots, \mu_n)$. Denoting by $\mathcal{I}$ the convex hull of $Q_1, \ldots, Q_m$, the condition $\mathcal{P} \in \mathcal{I}$ amounts to the existence of a vector $(\lambda_1, \ldots, \lambda_m)$ such that: $\sum_{h=1}^m \lambda_h Q_h = \mathcal{P}$, $\sum_{h=1}^m \lambda_h = 1$, $\lambda_h \geqslant 0$, $\forall h$; in other words, $\mathcal{P} \in \mathcal{I}$ is equivalent to the solvability of the system (Σ), associated with $(\mathcal{F}, \mathcal{P})$,

$$(\Sigma) \quad \begin{cases} \sum_{h=1}^m \lambda_h q_{hi} = \mu_i \,, \; i \in \{1, \ldots, n\} \,, \\ \sum_{h=1}^m \lambda_h = 1, \;\; \lambda_h \geqslant 0 \,, \;\; h \in \{1, \ldots, m\} \,. \end{cases} \tag{1.6}$$

Given the assessment $\mathcal{P} = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, let S be the set of solutions $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of system (Σ). We point out that the solvability of system (Σ) is a necessary (but not sufficient) condition for coherence of $\mathcal{P}$ on $\mathcal{F}$. When (Σ) is solvable, and hence $S \neq \emptyset$, we define:

$$\begin{array}{l} \Phi_i(\Lambda) = \Phi_i(\lambda_1, \ldots, \lambda_m) = \sum_{r: C_r \subseteq H_i} \lambda_r, \; ; \; \Lambda \in S \,, \\ M_i = \max_{\Lambda \in S}, \Phi_i(\Lambda), \; i \in \{1, \ldots, n\} \,; \\ I_0 = \{i \,:\, M_i = 0\} \,, \mathcal{F}_0 = \{X_i|H_i \,, i \in I_0\}, \\ \mathcal{P}_0 = (\mu_i, \, i \in I_0). \end{array} \tag{1.7}$$

For what concerns the probabilistic meaning of I_0, it holds that $i \in I_0$ if and only if the (unique) coherent extension of $\mathcal{P}$ to $H_i|\mathcal{H}_n$ is zero. Then, the following theorem can be proved ([4, Thm 3]):

Theorem 2. A conditional prevision assessment $\mathcal{P} = (\mu_1, \ldots, \mu_n)$ on the family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) defined in (1.6) is solvable; (ii) if $I_0 \neq \emptyset$, then $\mathcal{P}_0$ is coherent.

Remark 4. *We observe that if Λ is a solution of System (Σ), associated with the pair $(\mathcal{F}, \mathcal{P})$, such that $\Phi_j(\Lambda) > 0$, $j = 1, \ldots, n$, then it holds that $I_0 = \emptyset$ and hence by Theorem 2 the assessment $\mathcal{P}$ on $\mathcal{F}$ is coherent.*

Let $\mathcal{S}'$ be a nonempty subset of the set of solutions $\mathcal{S}$ of system (Σ). We denote by I_0' the set I_0 defined as in (1.7), where $\mathcal{S}$ is replaced by $\mathcal{S}'$, that is

$$I_0' = \{i : M_i' = 0\}, \text{ where } M_i' = \max_{\Lambda \in \mathcal{S}'} \Phi_i(\Lambda) \,, \quad i \in \{1, \dots, n\}. \tag{1.8}$$

Moreover, we denote by $(\mathcal{F}_0', \mathcal{P}_0')$ the pair associated with I_0'. Then, we obtain

Theorem 3. *The assessment $\mathcal{P}$ on $\mathcal{F}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) associated with the pair $(\mathcal{F}, \mathcal{P})$ is solvable; (ii) if $I_0' \neq \emptyset$, then $\mathcal{P}_0'$ is coherent.*

Of course, the previous results can be used in the case of a probability assessment, which will be denoted by $\mathcal{P}$, on a family of conditional events $\mathcal{F}$. More precisely, given a family $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ of n conditional events, we observe that, for each i, the family $\{E_iH_i, \bar{E}_iH_i, \bar{H}_i\}$ is a partition of Ω. Then, $\Omega = \bigwedge_{i=1}^{n}(E_iH_i \vee \bar{E}_iH_i \vee \bar{H}_i)$. By applying the distributive property it follows that Ω can also be written as the disjunction of 3^n logical conjunctions, some of which may be impossible. The remaining ones, denoted by $C_0, C_1, \dots, C_m$, where $C_0 = \bar{H}_1 \cdots \bar{H}_n$ (if non-impossible), are the constituents generated by $\mathcal{F}$. Of course, $m + 1 \leqslant 3^n$, with $m + 1 = 3^n$ when, for example, the events in $\mathcal{F}$ are logically independent. Let $\mathcal{P} = (p_1, \dots, p_n)$ be a probability assessment on $\mathcal{F}$. In this case, for each constituent C_h, $h = 1, \dots, m$, it holds that $Q_h = (q_{h1}, \dots, q_{hn})$, where $q_{hi} = 1$, or 0, or p_i, according to whether $C_h \subseteq E_iH_i$, or $C_h \subseteq \bar{E}_iH_i$, or $C_h \subseteq \bar{H}_i$. The point $Q_0 = \mathcal{P}$ is associated with C_0. In the following example, which will be useful in Section 2.2 to prove Theorem 8, we illustrate how the constituents and the associated points are generated in order to check the coherence of a probability assessment.

Example 1. *Let $\mathcal{F} = \{E_1|H_1, E_2|H_2\} = \{B|K, B|(K \wedge (\bar{H} \vee A))\}$, where A, B, H, K are four logically independent events, and let $\mathcal{P} = (p_1, p_2)$ be a probability assessment on $\mathcal{F}$. We verify that $\mathcal{P}$ is coherent for every $(p_1, p_2) \in [0,1]^2$. It holds that*

$$\begin{aligned}
&\Omega = (E_1H_1 \vee \bar{E}_1H_1 \vee \bar{H}_1) \wedge (E_2H_2 \vee \bar{E}_2H_2 \vee \bar{H}_2) = \\
&= [BK \vee \bar{B}K \vee \bar{K}] \wedge [(BK \wedge (\bar{H} \vee A)) \vee (\bar{B}K \wedge (\bar{H} \vee A)) \vee (\bar{K} \vee \bar{A}H)] = \\
&= (BK \wedge (\bar{H} \vee A)) \vee \emptyset \vee \bar{A}HBK \vee \emptyset \vee (\bar{B}K \wedge (\bar{H} \vee A)) \vee \bar{A}H\bar{B}K \vee \emptyset \vee \emptyset \vee (\bar{K} \vee \bar{A}H\bar{K}) = \\
&= C_1 \vee C_2 \vee C_3 \vee C_4 \vee C_0,
\end{aligned}$$

where the constituents are

$$\begin{aligned}
&C_1 = BK \wedge (\bar{H} \vee A) = AHBK \vee \bar{H}BK,\ C_2 = \bar{A}HBK, C_3 = \bar{B}K \wedge (\bar{H} \vee A) = AH\bar{B}K \vee \bar{H}\,\bar{B}K, \\
&C_4 = \bar{A}H\bar{B}K,\ C_0 = \bar{K} \vee \bar{A}H\bar{K} = \bar{K}.
\end{aligned}$$

The points Q_h's associated with the pair $(\mathcal{F}, \mathcal{P})$ are

$$Q_1 = (1,1),\ Q_2 = (1, p_2),\ Q_3 = (0,0),\ Q_4 = (0, p_2),\ Q_0 = \mathcal{P} = (p_1, p_2).$$

We denote by $\mathcal{I}$ the convex hull of points Q_1, Q_2, Q_3, Q_4 (see Figure 1.1).) The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ is

$$\begin{cases}
\lambda_1 + \lambda_2 = p_1, \\
\lambda_1 + p_2\lambda_2 + p_2\lambda_4 = p_2, \\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\
\lambda_i \geqslant 0,\ \forall i = 1, \dots, 4.
\end{cases} \tag{1.9}$$

We observe that $\mathcal{P} = (p_1, p_2) = p_1 Q_2 + (1 - p_1) Q_4$ and hence the system (1.9) is solvable and a solution is $\Lambda = (0, p_1, 0, 1 - p_1)$, with $p_1 \in [0,1]$.

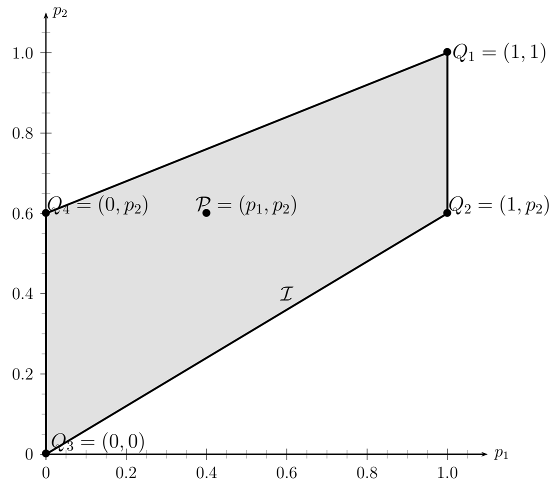

FIGURE 1.1: Convex hull of the points Q_1, Q_2, Q_3, Q_4 associated with the pair $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \{B|K, B|(K \wedge (\bar{H} \vee A))\}$ and $\mathcal{P} = (p_1, p_2)$. In the figure the numerical values are: $p_1 = 0.4$ and $p_2 = 0.6$.

By considering the function ϕ as defined in (1.7), it holds that

$$\phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,$$

$$\phi_2(\Lambda) = \sum_{h:C_h \subseteq (K \wedge (A \vee \bar{H}))} \lambda_h = \lambda_1 + \lambda_3 = 0.$$

We set $\mathcal{S}' = \{\Lambda\}$ and we get $\mathcal{I}'_0 = \{2\}$. We observe that the sub-assessment p_2 on $B|(K \wedge (\bar{H} \vee A))$ is coherent for every $p_2 \in [0, 1]$. Thus, by Theorem 3, the assessment $\mathcal{P} = (p_1, p_2)$ on $\mathcal{F}$ is coherent $\forall (p_1, p_2) \in [0, 1]^2$.

The next example, which is related to the compound prevision theorem listed in Section 4.1, illustrates how to check coherence for (conditional) prevision assessments.

Example 2. [1]*[Compound prevision theorem] Let $\mathcal{F} = \{XH, X|H, H\}$, where H is an event, with $H \neq \emptyset$, $H \neq \Omega$, and X is a finite random quantity. We denote by $\{x_1, \ldots, x_n\}$ the possible values of X when H is true. We also set $x' = \min\{x_1, \ldots, x_n\}$ and $x'' = \max\{x_1, \ldots, x_n\}$. Let $\mathcal{P} = (z, \mu, p)$ be a prevision assessment on $\mathcal{F}$. We verify that $\mathcal{P}$ is coherent if and only if $\mu \in [x', x'']$, $p \in [0, 1]$, and $z = \mu p$. Of course, coherence requires that $p \in [0, 1]$. It holds that*

$$\Omega = C_1 \vee \cdots \vee C_n \vee C_{n+1},$$

where the constituents are

$$C_1 = (X = x_1), \cdots, C_n = (X = x_n),\ C_{n+1} = \bar{H}.$$

The points Q_h's associated with the pair $(\mathcal{F}, \mathcal{P})$ are

$$Q_1 = (q_{11}, q_{12}, q_{13}) = (x_1, x_1, 1), \ldots,\ Q_n = (q_{n1}, q_{n2}, q_{n3}) = (x_n, x_n, 1),$$
$$Q_{n+1} = (q_{(n+1)1}, q_{(n+1)2}, q_{(n+1)3}) = (0, \mu, 0).$$

[1]This example was inspired by Angelo Gilio's talk "On Coherence and Conditionals" presented at the workshop "Reasoning and uncertainty: probabilistic, logical, and psychological perspectives", Regensburg, August 9-10, 2022.

We denote by $\mathcal{I}$ the convex hull of points $Q_1, \dots, Q_{n+1}$. The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ is

$$\begin{cases} \lambda_1 x_1 + \cdots + \lambda_n x_n = z, \\ \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1}\mu = \mu, \\ \lambda_1 + \cdots + \lambda_n = p, \\ \lambda_1 + \cdots + \lambda_{n+1} = 1, \\ \lambda_i \geqslant 0,\ i = 1, \dots, n+1. \end{cases} \Longleftrightarrow \begin{cases} \lambda_1 x_1 + \cdots + \lambda_n x_n = z, \\ z = \mu p, \\ \lambda_1 + \cdots + \lambda_n = p, \\ \lambda_{n+1} = 1 - p, \\ \lambda_i \geqslant 0,\ i = 1, \dots, n+1. \end{cases} \tag{1.10}$$

We distinguish two cases: (i) $p = 0$; and $p \in (0,1]$.
Case (i). System (1.10) is solvable only if $z = 0$, with the unique solution given by $\Lambda = (\lambda_1, \dots, \lambda_n, \lambda_{n+1}) = (0, \dots, 0, 1)$. By considering the function ϕ as defined in (1.7), it holds that

$$\phi_1(\Lambda) = \phi_3(\Lambda) = \sum_{h: C_h \subseteq \Omega} \lambda_h = \lambda_1 + \cdots + \lambda_{n+1} = 1 > 0$$

and

$$\phi_2(\Lambda) = \sum_{h: C_h \subseteq H} \lambda_h = \lambda_1 + \cdots + \lambda_n = p = 0.$$

Then $I_0 = \{2\}$. We consider the sub-assessment $\mathcal{P}_0 = (\mu)$ on $\mathcal{F}_0 = \{X|H\}$. It can be easily proved that $\mu = \mathbb{P}(X|H)$ is coherent if and only if $\mu \in [x', x'']$. Then, by Theorem 2, the assessment $\mathcal{P} = (0, \mu, 0)$ on $\mathcal{F}$ is coherent if and only if $\mu \in [x', x'']$.
Case (ii). In this case System (1.10) becomes

$$\begin{cases} z = \mu p, \\ \mu = \frac{z}{p} = \frac{\lambda_1 x_1 + \cdots + \lambda_n x_n}{\lambda_1 + \cdots + \lambda_n}, \\ \lambda_1 + \cdots + \lambda_n = p, \\ \lambda_{n+1} = 1 - p, \\ \lambda_i \geqslant 0,\ i = 1, \dots, n+1, \end{cases} \tag{1.11}$$

which is solvable when $z = \mu p$ and $\mu \in [x', x'']$, because μ is a convex linear combination of $\{x_1, \dots, x_n\}$ with weights $\frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n}, \cdots, \frac{\lambda_n}{\lambda_1 + \cdots + \lambda_n}$. For each solution Λ of System (1.11) it holds that $\phi_1(\Lambda) = \phi_3(\Lambda) = 1 > 0$ and $\phi_2(\Lambda) = p > 0$. Then, as $I_0 = \varnothing$, by Theorem 2 the assessment (z, μ, p), with $p \in (0,1]$ is coherent if and only $\mu \in [x', x'']$ and $z = \mu p$. Therefore $\mathcal{P}$ is coherent if and only if $\mu \in [x', x'']$, $p \in [0,1]$, and $z = \mu p$. We now show that, in agreement to Definition 1, an incoherent assessment leads to the existence of a combination of bets where the values of the random gain are all positive or all negative (Dutch Book). We set $\mathcal{P} = (z, \mu, p)$, with $p \in [0,1]$, $\mu \in [x', x'']$ and $z \neq \mu p$. We observe that the points $Q_1, \dots, Q_{n+1}$ belong to the plane $\pi : -\mathcal{X} + \mathcal{Y} + \mu\mathcal{Z} = \mu$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are the axes coordinates. Moreover, $\mathcal{P}$ does not belong to the convex hull $\mathcal{I}$ because, as $z \neq \mu p$, System (1.10) is not solvable. Now, by considering the function $f(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = -\mathcal{X} + \mathcal{Y} + \mu\mathcal{Z} - \mu$, we observe that for each constant k the equation $f(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = k$ represents a plane which is parallel to π and coincides with π when $k = 0$. Then, $f(Q_1) = \cdots = f(Q_{n+1}) = 0$. Moreover, $f(Q_h) - f(\mathcal{P}) = z - \mu p \neq 0, h = 1, \dots, n+1$. We recall that for $h = 1, \dots, n+1$, the value g_h, of the random gain

$$G = s_1(XH - z) + s_2 H(X - \mu) + s_3(H - p)$$

associated to the constituent C_h is

$$g_h = s_1(q_{h1} - z) + s_2 H(q_{h2} - \mu) + s_3(q_{h3} - p).$$

We observe that, by setting the stakes $s_1 = -1$, $s_2 = 1$, $s_3 = -\mu$, it holds that $g_h =$

$f(Q_h) - f(\mathcal{P}) = z - \mu p \neq 0$, $h = 1, \dots, n+1$. Therefore, $\min g_h \cdot \max g_h > 0$, when $s_1 = -1$, $s_2 = 1$, $s_3 = -\mu$, that is a combination of bets where the value of the random gain are all positive or all negative.

Remark 5. *By Example 2 coherence requires that (*compound prevision theorem*)*

$$\mathbb{P}[XH] = \mathbb{P}[X|H]P(H). \tag{1.12}$$

*In particular, when X is (the indicator of) an event E, Equation (1.12) becomes (*compound probability theorem*)*

$$P(E \wedge H) = P(E|H)P(H). \tag{1.13}$$

Moreover, under logical independence of E and H, Example 2 shows that $P(E|H)$ coincides with the ratio $\frac{P(E \wedge H)}{P(H)}$ when $P(H) > 0$, and $P(E|H)$ can be any value in $[0,1]$ when $P(H) = 0$. Then, differently from the "standard" approach where $P(E|H)$ is defined only when when $P(H) > 0$ by the ratio $\frac{P(E \wedge H)}{P(H)}$, in the coherence-based approach the conditional probability $P(E|H)$ is a primitive notion which is properly defined even if $P(H) = 0$.

We recall the following extension theorem for conditional previsions, which is a generalisation of de Finetti's fundamental theorem of probability to conditional random quantities (see, e.g.,[70, 87, 96])

Theorem 4. *Let $\mathcal{P} = (\mu_1, \dots, \mu_n)$ be a coherent prevision assessment on a family of bounded conditional random quantities $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$. Moreover, let $X|H$ be a further bounded conditional random quantity. Then, there exists a suitable closed interval $[\mu', \mu'']$ such that the extension $\mu = \mathbb{P}(X|H)$ is coherent if and only if $\mu \in [\mu', \mu'']$.*

1.4 Conjunction and disjunction of two conditional events

When we deal with the conjunction and disjunction of conditional events various approaches can be found in the literature. Traditionally these operations are looked in the framework of trivalent logics, in which different definitions are possible. We recall some notions of conjunction among conditional events in some trivalent logics: Kleene-Lukasiewicz-Heyting conjunction ($\wedge_K$), or de Finetti conjunction ([32]); Lukasiewicz conjunction ($\wedge_L$); Bochvar internal conjunction, or Kleene weak conjunction ($\wedge_B$); Sobocinski conjunction, or quasi conjunction ($\wedge_S$). In all these definitions the result of the conjunction is still a conditional event with set of truth values $\{\text{true}, \text{false}, \text{void}\}$ (see, e.g., [24, 25]). We list below in an explicit way the five conjunctions and the associated disjunctions obtained by De Morgan's law ([60]):

1. $(A|H) \wedge_K (B|K) = AHBK|(AHBK \vee \overline{A}H \vee \overline{B}K)$,
   $(A|H) \vee_K (B|K) = (AH \vee BK)|(\bar{A}H\bar{B}K \vee AH \vee BK)$;

2. $(A|H) \wedge_L (B|K) = AHBK|(AHBK \vee \overline{A}\,H \vee \overline{B}K \vee \overline{H}\,\overline{K})$,
   $(A|H) \vee_L (B|K) = (AH \vee BK)|(\bar{A}H\bar{B}K \vee AH \vee BK \vee \overline{H}\,\overline{K})$;

3. $(A|H) \wedge_B (B|K) = AHBK|HK$,
   $(A|H) \vee_B (B|K) = (A \vee B)|HK$;

4. $(A|H) \wedge_S (B|K) = ((AH \vee \overline{H}) \wedge (BK \vee \overline{K}))|(H \vee K)$,
   $(A|H) \vee_S (B|K) = (AH \vee BK)|(H \vee K)$.

Another possibility is to study conjunction and disjunction of two conditional events as a suitable conditional random quantity in a betting-scheme context ([53, 56], see also [71, 80]).

Definition 3. *Given two conditional events $A|H$, $B|K$ and a (coherent) probability assessment $P(A|H) = x$, $P(B|K) = y$, the conjunction $(A|H) \wedge_{gs} (B|K)$, where the subindex gs stays for Gilio-Sanfilippo, is defined as the following conditional random quantity*

$$(A|H) \wedge_{gs} (B|K) = (AHBK + x\overline{H}BK + yAH\overline{K})|(H \vee K). \tag{1.14}$$

Then, the prevision of the conjunction is the following

$$\begin{aligned}\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = \quad & P(AHBK|(H \vee K)) + P(A|H)P(\bar{H}BK|(H \vee K)) \\ & + P(B|K)P(AH\bar{K}|(H \vee K)).\end{aligned}$$

In particular, when $P(H \vee K) > 0$, formula (1.4) becomes the well known formula of McGee ([80]) and Kaufmann ([71])

$$\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = \frac{P(AHBK) + P(A|H)P(\bar{H}BK) + P(B|K)P(AH\bar{K})}{P(H \vee K)}.$$

Remark 6. *Notice that the conjunction in (1.14) can be represented as $X|H$ in (1.2) and, once the (coherent) assessment (x, y, z), where $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$, is given, the conjunction is (subjectively) determined by*

$$(A|H) \wedge_{gs} (B|K) = AHBK + x\overline{H}BK + yAH\overline{K} + z_{gs}\overline{H}\,\overline{K}.$$

Then, the set of possible values of $(A|H) \wedge_{gs} (B|K)$, i.e. $\{1, 0, x, y, z_{gs}\}$, is associated to a given (coherent) subjective assessment (x, y, z_{gs}).

Within the betting scheme, by starting with a coherent assessment (x, y) on $\{A|H, B|K\}$, if you extend (x, y) (in a coherent way) by adding the assessment $\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = z_{gs}$, then you agree to pay z_{gs}, by receiving the random amount

$$(A|H) \wedge_{gs} (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}BK \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ z_{gs}, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

In other words, you receive:

- 1, if both conditional events $A|H$ and $B|K$ are true;
- 0, if $A|H$ or $B|K$ is false;
- $x = P(A|H)$, if $A|H$ is void and $B|K$ is true;
- $y = P(B|K)$, if $A|H$ is true and $B|K$ is void;
- z_{gs}, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void.

Notice that, in some particular case, the conjunction $(A|H) \wedge_{gs} (B|K)$, which is a five-valued object, reduces to a conditional event, that is a three-valued object.

Theorem 5. *[53, Theorem 7] Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset$, $K \neq \emptyset$, the extension $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ is coherent if and only if the Fréchet-Hoeffding bounds are satisfied, that is $z_{gs} \in [z'_{gs}, z''_{gs}]$, where*

$$z'_{gs} = \max\{x + y - 1, 0\}, \quad z''_{gs} = \min\{x, y\}. \tag{1.15}$$

Remark 7. *In case of some logical dependencies, for the interval $[z'_{gs}, z''_{gs}]$ of coherent extensions z_{gs} it holds that $[z'_{gs}, z''_{gs}] \subseteq [\max\{x+y-1,0\}, \min\{x,y\}]$. In particular, if $HK = \emptyset$, it holds that $(A|H) \wedge_{gs} (B|K) = (A|H) \cdot (B|K)$ and hence $z_{gs} = z'_{gs} = z''_{gs} = xy$. See [52] for other cases of logical dependencies.*

Definition 3 can be extended to the case of n conditional events as follows ([56])

Definition 4. *Let n conditional events $E_1|H_1, \dots, E_n|H_n$ be given. For each nonempty strict subset S of $\{1,\dots,n\}$, let x_S be a prevision assessment on $\bigwedge_{i\in S}(E_i|H_i)$. Then, the conjunction $(E_1|H_1) \wedge_{gs} \cdots \wedge_{gs} (E_n|H_n)$ is the conditional random quantity $\mathcal{C}_{1\cdots n}$ defined as*

$$\mathcal{C}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigwedge_{i=1}^n E_iH_i \text{ is true,} \\ 0, & \text{if } \bigvee_{i=1}^n \bar{E}_iH_i \text{ is true,} \\ x_S, & \text{if } (\bigwedge_{i\in S} \bar{H}_i) \wedge (\bigwedge_{i\notin S} E_iH_i) \text{ is true, } \emptyset \neq S \subset \{1,2,\dots,n\}, \\ x_{1\cdots n}, & \text{if } \bigwedge_{i=1}^n \bar{H}_i \text{ is true,} \end{cases} \tag{1.16}$$

where $x_{1\cdots n} = x_{\{1,\dots,n\}} = \mathbb{P}(\mathcal{C}_{1\cdots n})$.

For $n = 1$ we obtain $\mathcal{C}_1 = E_1|H_1$. In Definition 4 each possible value x_S of $\mathcal{C}_{1\cdots n}$, $\emptyset \neq S \subset \{1,\dots,n\}$, is evaluated when defining (in a previous step) the conjunction $\mathcal{C}_S = \bigwedge_{i\in S}(E_i|H_i)$. Then, after the conditional prevision $x_{1\cdots n}$ is evaluated, $\mathcal{C}_{1\cdots n}$ is completely specified. Of course, we require coherence for the prevision assessment $(x_S, \emptyset \neq S \subseteq \{1,\dots,n\})$, so that $\mathcal{C}_{1\cdots n} \in [0,1]$. In the framework of the betting scheme, $x_{1\cdots n}$ is the amount that you agree to pay with the proviso that you will receive:

- the amount 1, if all conditional events are true;
- the amount 0, if at least one of the conditional events is false;
- the amount x_S equal to the prevision of the conjunction of that conditional events which are void, otherwise. In particular you receive back $x_{1\cdots n}$ when all conditional events are void.

As we can see from (1.16), the conjunction $\mathcal{C}_{1\cdots n}$ assumes values in the interval $[0,1]$ and is (in general) a (2^n+1)-valued object because the number of nonempty subsets S, and hence the number of possible values x_S, is $2^n - 1$.

Remark 8. *It can be verified that*

$$x_{1\cdots n} = \mathbb{P}[(\textstyle\bigwedge_{i=1}^n E_iH_i + \sum_{\emptyset\neq S\subset\{1,2\dots,n\}} x_S(\bigwedge_{i\in S}\bar{H}_i) \wedge (\bigwedge_{i\notin S} E_iH_i))|(\bigvee_{i=1}^n H_i)] \tag{1.17}$$

and

$$\mathcal{C}_{1\cdots n} = [\bigwedge_{i=1}^n E_iH_i + \sum_{\emptyset\neq S\subset\{1,2\dots,n\}} x_S(\bigwedge_{i\in S}\bar{H}_i) \wedge (\bigwedge_{i\notin S} E_iH_i)]|(\bigvee_{i=1}^n H_i).$$

A similar comment can be done for x_S and $\mathcal{C}_S$, for each nonempty subset $S \subset \{1,2\dots,n\}$.

We recall a result which shows that the prevision of the conjunction on n conditional events satisfies the Fréchet-Hoeffding bounds ([56, Theorem13]).

Theorem 6. *Let n conditional events $E_1|H_1, \dots, E_n|H_n$ be given, with $x_i = P(E_i|H_i)$, $i = 1,\dots,n$ and $x_{1\cdots n} = \mathbb{P}(\mathcal{C}_{1\cdots n})$. Then*

$$\max\{x_1 + \cdots + x_n - n + 1, 0\} \leqslant x_{1\cdots n} \leqslant \min\{x_1,\dots,x_n\}.$$

In [52, Theorem 10] we have shown, under logical independence, the sharpness of the Fréchet-Hoeffding bounds.

Remark 9. *Given a finite family $\mathcal{F}$ of conditional events, their conjunction is also denoted by $\mathcal{C}(\mathcal{F})$. We recall that in [56], given two finite families of conditional events $\mathcal{F}_1$ and $\mathcal{F}_2$, the object $\mathcal{C}(\mathcal{F}_1) \wedge \mathcal{C}(\mathcal{F}_2)$ is defined as $\mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2)$. Then, conjunction satisfies the commutativity and associativity properties ([56, Propositions 1 and 2]). Moreover, the operation of conjunction satisfies the monotonicity property ([56, Theorem7]), that is $\mathcal{C}_{1\cdots n+1} \leqslant \mathcal{C}_{1\cdots n}$. Then,*

$$\mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) \leqslant \mathcal{C}(\mathcal{F}_1), \;\; \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) \leqslant \mathcal{C}(\mathcal{F}_2). \tag{1.18}$$

Following the same idea it is possible to define the disjunction of two conditional events.

Definition 5. *Given two conditional events $A|H$, $B|K$ and a (coherent) probability assessment $P(A|H) = x$, $P(B|K) = y$, the disjunction $(A|H) \vee_{gs} (B|K)$ is defined as the following conditional random quantity*

$$(A|H) \vee_{gs} (B|K) = ((AH \vee BK) + x\bar{H}\,\bar{B}K + y\bar{A}H\bar{K})|(H \vee K). \tag{1.19}$$

Within the betting scheme, by starting with a coherent assessment (x, y) on $\{A|H, B|K\}$, if you extend (x, y) (in a coherent way) by adding the assessment $\mathbb{P}[(A|H) \vee_{gs} (B|K)] = w_{gs}$, then you agree to pay w_{gs}, by receiving the random amount

$$(A|H) \vee_{gs} (B|K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \overline{A}H \wedge \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}\,\overline{B}K \text{ is true,} \\ y, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\ w_{gs}, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

We recall that De Morgan's Laws are satisfied and hence it holds that

$$\overline{(A|H) \vee_{gs} (B|K)} = (\bar{A}|H) \wedge_{gs} (\bar{B}|K) \tag{1.20}$$

and

$$\overline{(A|H) \wedge_{gs} (B|K)} = (\bar{A}|H) \vee_{gs} (\bar{B}|K), \tag{1.21}$$

where the negations $\overline{(A|H) \vee_{gs} (B|K)}$ and $\overline{(A|H) \wedge_{gs} (B|K)}$ are defined as $\overline{(A|H) \vee_{gs} (B|K)} = 1 - (A|H) \vee_{gs} (B|K)$ and $\overline{(A|H) \wedge_{gs} (B|K)} = 1 - (A|H) \wedge_{gs} (B|K)$, respectively. We also observe that the prevision sum rule is satisfied, that is

$$A|H + B|K = (A|H) \wedge_{gs} (B|K) + (A|H) \vee_{gs} (B|K)$$

and hence

$$P(A|H) + P(B|K) = \mathbb{P}[(A|H) \wedge_{gs} (B|K)] + \mathbb{P}[(A|H) \vee_{gs} (B|K)]. \tag{1.22}$$

From (1.22), by exploiting the Fréchet-Hoeffding bounds for the conjunction (1.15), we obtain the Fréchet-Hoeffding bounds for the disjunction :

$$w'_{gs} = \max\{x, y\}, \; w''_{gs} = \min\{x + y, 1\}. \tag{1.23}$$

The operations above are all commutative and associative. By setting $P(A|H) = x$, $P(B|K) = y$, $P((A|H) \wedge_i (B|K)) = z_i$, $i \in \{K, L, B, S\}$, and

	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$	$\wedge_{gs}$
$AHBK$	1	1	1	1	1	1	1
$AH\bar{B}K$	1	0	0	0	0	0	0
$AH\bar{K}$	1	y	z_K	z_L	z_B	1	y
$\bar{A}HBK$	0	1	0	0	0	0	0
$\bar{A}H\bar{B}K$	0	0	0	0	0	0	0
$\bar{A}H\bar{K}$	0	y	0	0	z_B	0	0
$\bar{H}BK$	x	1	z_K	z_L	z_B	1	x
$\bar{H}\bar{B}K$	x	0	0	0	z_B	0	0
$\bar{H}\bar{K}$	x	y	z_K	0	z_B	z_S	z_{gs}

TABLE 1.1: Numerical values (of the indicator) of the conjunctions $(A|H) \wedge_i (B|K)$, $i \in \{K, L, B, S, gs\}$. The triplet (x, y, z_i) denotes a coherent assessment on $\{A|H, B|K, (A|H) \wedge_i (B|K)\}$.

$\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = z_{gs}$, based on (1.4) and on (1.2) the conjunctions $(A|H) \wedge_i (B|K)$, $i \in \{K, L, B, S, gs\}$ can be also looked at as random quantities with set of possible value illustrated in Table 1.1. A similar interpretation can also be given for the associated disjunctions. In Table 1.2 we list the lower and upper bounds for the coherent extension $z_i = P((A|H) \wedge_i (B|K))$, $i \in \{K, L, B, S\}$ ([88]) and $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ ([53]) of the given assessment $P(A|H) = x$ and $P(B|K) = y$. Notice that, differently from conditional events which are three-valued objects, the conjunction $(A|H) \wedge_{gs} (B|K)$ (and the associated disjunction) is no longer a three-valued object, but a five-valued object with values in $[0, 1]$. In betting terms, the prevision $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ represents the amount you agree to pay, with the proviso that you will receive the random quantity $AHBK + x\bar{H}BK + yAH\bar{K}$, if $H \vee K$ is true, z_{gs} if $\bar{H}\bar{K}$ is true. In other words by paying z_{gs} you receive: 1, if both conditional events are true; 0, if at least one of the conditional events is false; the probability of the conditional event that is void if one conditional event is void and the other one is true; the amount z_{gs} you paid if both conditional events are void. The notion of conjunction $\wedge_{gs}$ (and disjunction $\vee_{gs}$) among conditional events has been generalised to the case of n conditional events in [56]. For some applications see, .e.g., [90, 91]. Developments of this approach to general compound conditionals has been given in [41]. Differently from the other notions of conjunctions and disjunction, $\wedge_{gs}$ and $\vee_{gs}$ preserves the classical logical and probabilistic properties valid for unconditional events (see, e.g., [58, 61]). In particular, the Fréchet-Hoeffding bounds, i.e., the lower and upper bounds for the conjunction $z' = \max\{x + y - 1, 0\}, z'' = \min\{x, y\}$, and for the disjunction $w' = \max\{x, y\}, z'' = \min\{x + y, 1\}$, obtained under logical independence in the unconditional case for the coherent extensions of $P(A) = x$ and $P(B) = y$ to $z = P(AB)$ and $w = P(A \vee B)$, respectively, when A and B are replaced by $A|H$ and $B|K$, are only satisfied by z_{gs} (see Table 1.2) and w_{gs} (see Table 1.3).

1.5 Imprecise assessments

We close this chapter by recalling the notion of coherence, for imprecise, or set-valued, prevision assessments. As in this paper we limit our analysis to conditional random quantities with possible values in the unit interval, in our case the imprecise assessments are subsets of the unitary hypercube.

Conjunction	Lower bound	Upper bound
$(A\|H) \wedge_K (B\|K)$	$z'_K = 0$	$z''_K = \min\{x, y\}$
$(A\|H) \wedge_L (B\|K)$	$z'_L = 0$	$z''_L = \min\{x, y\}$
$(A\|H) \wedge_B (B\|K)$	$z'_B = 0$	$z''_B = 1$
$(A\|H) \wedge_S (B\|K)$	$z'_S = \max\{x + y - 1, 0\}$	$z''_S = \begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x, y) \neq (1, 1) \\ 1, & \text{if } (x, y) = (1, 1) \end{cases}$
$(A\|H) \wedge_{gs} (B\|K)$	$z'_{gs} = \max\{x + y - 1, 0\}$	$z''_{gs} = \min\{x, y\}$

TABLE 1.2: Lower and upper bounds for the coherent extensions $z_i = P((A|H) \wedge_i (B|K))$, $i \in \{K, L, B, S\}$ and $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ of the given assessment $P(A|H) = x$ and $P(B|K) = y$.

Disjunction	Lower bound	Upper bound
$(A\|H) \vee_K (B\|K)$	$w'_K = \max\{x, y\}$	$w''_K = 1$
$(A\|H) \vee_L (B\|K)$	$w'_L = \max\{x, y\}$	$w''_L = 1$
$(A\|H) \vee_B (B\|K)$	$w'_B = 0$	$w''_B = 1$
$(A\|H) \vee_S (B\|K)$	$w'_S = \begin{cases} \frac{xy}{x+y-xy}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$	$w''_S = \min\{x + y, 1\}$
$(A\|H) \vee_{gs} (B\|K)$	$w'_{gs} = \max\{x, y\}$	$w''_{gs} = \min\{x + y, 1\}$

TABLE 1.3: Lower and upper bounds for the coherent extensions $w_i = P((A|H) \vee_i (B|K))$, $i \in \{K, L, B, S\}$ and $w_{gs} = \mathbb{P}[(A|H) \vee_{gs} (B|K)]$ of the given assessment $P(A|H) = x$ and $P(B|K) = y$.

Definition 6. *Let $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$ be a family of n conditional random quantities. An* imprecise, or set-valued, assessment $\mathcal{A} \subseteq [0,1]^n$ *on $\mathcal{F}$ is a set of precise assessments $\mathcal{P}$ on $\mathcal{F}$.*

Let $\mathcal{A}$ be an imprecise assessment on $\mathcal{F}$. For each $j \in \{1, 2, \dots, n\}$, the projection $\rho_j(\mathcal{A})$ of $\mathcal{A}$ onto the j-th coordinate, is an imprecise assessment on $X_j|H_j$ and is defined as

$$\rho_j(\mathcal{A}) = \{x_j : \mu_j = x_j, \text{ for some } (\mu_1, \dots, \mu_n) \in \mathcal{A}\}.$$

Definition 7. *Let $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$ be a family of n conditional random quantities. An imprecise assessment $\mathcal{A}$ on $\mathcal{F}$ is* coherent *if and only if, for every $j \in \{1, \dots, n\}$ and for every $x_j \in \rho_j(\mathcal{A})$, there exists a coherent precise assessment $\mathcal{P} = (\mu_1, \dots, \mu_n)$ on $\mathcal{F}$, such that $\mathcal{P} \in \mathcal{A}$ and $\mu_j = x_j$.*

In the context of imprecise assessments the notions of g-coherence and total coherence have been also introduced ([47, 49]).

Remark 10. *We observe that an interval-valued prevision assessment $\mathcal{A} = [l_1, u_1] \times \cdots \times [l_n, u_n]$ is an imprecise prevision assessment. In this case it holds that $\rho_j(\mathcal{A}) = [l_j, u_j]$. Then, based on Definition 7, the imprecise prevision assessment $\mathcal{A}$ on $\mathcal{F}$ is coherent if and only if, given any $j \in \{1, \dots, n\}$ and any $x_j \in [l_j, u_j]$, there exists a coherent precise prevision assessment $\mathcal{P} = (\mu_1, \dots, \mu_n)$ on $\mathcal{F}$, with $l_i \leqslant \mu_i \leqslant u_i$, $i = 1, \dots, n$, and such that $\mu_j = x_j$.*

Chapter 2

A probabilistic analysis of selected notions of iterated conditioning under coherence

The results of this chapter are taken from [17].

The idea behind this second chapter is to deepen the analyses of the iterated conditionals both in trivalent logics and in the framework of conditional random quantities. To carry out this study, we begin by considering some classical (logical and probabilistic) properties that are true for events and (basic) conditional and investigate them in the case of compound and iterated conditional (Section 2.1). For what concern the trivalued logics, we analyse the iterated conditionals in the logic of Cooper-Calabrese (Section 2.2), de Finetti (Section 2.3), and Farrell (Section 2.4). Then, in Section 2.5 we study some iterated conditioning in the framework of conditional random quantities under coherence. We also consider the validity of two generalized versions of Bayes' Rule (Section 2.5.6), and we check the p-validity of two generalized inference rules: Modus Ponens and two-premise centering (Section 2.6).

2.1 Some basic properties and Import-Export principle

In this section, after recalling some basic logical and probabilistic properties satisfied by events and conditional events, we rewrite them for compound and iterated conditionals, by replacing events with conditional events. We also recall the Import-Export principle and its connection with the Lewis' triviality results.

Given two events A and B, with $A \neq \emptyset$, it is well-known the validity of following properties

1. $B|A = AB|A$;
2. $AB \leqslant B|A$ and hence $P(AB) \leqslant P(B|A)$;
3. $P(AB) = P(B|A)P(A)$ (*compound probability theorem*, see Remark 5);
4. if A and B are logically independent, by setting $P(A) = x$ and $P(B) = y$, the extension $\mu = P(B|A)$ is coherent if and only if $\mu \in [\mu', \mu'']$, where (see, e.g. [90, Theorem 6])

$$\mu' = \begin{cases} \frac{\max\{x+y-1,0\}}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \quad , \quad \mu'' = \begin{cases} \frac{\min\{x,y\}}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \tag{2.1}$$

Another classical property that can be checked for the iterated conditional is the Import-Export principle. Given three events B, K, A, with $AK \neq \emptyset$, we say that the Import-Export principle ([80]) is satisfied for the iterated conditional $(B|K)|A$, if

$$(B|K)|A = B|AK. \tag{2.2}$$

We also recall that the validity of the Import-Export principle (together with the validity of the total probability theorem) could lead to the counter-intuitive consequences related to Lewis' triviality results ([79], see also [91]). Indeed, assuming that the total probability theorem holds for iterated conditionals, that is

$$P(C|A) = P((C|A) \wedge C) + P((C|A) \wedge \bar{C}) = P((C|A)|C)P(C) + P((C|A)|\bar{C})P(\bar{C});$$

if the Import-Export principle is valid, by applying (2.2) and by observing that $P(C|AC) = 1$ and $P(C|A\bar{C}) = 0$, it follows that

$$P(C|A) = P(C|AC)P(C) + P(C|A\bar{C})P(\bar{C}) = P(C), \tag{2.3}$$

which of course is not valid in general for conditional events. Then, the non validity of the Import-Export principle may avoid Lewis' triviality results.

In Chapter 1 we recalled some basic properties satisfied by events and conditional events. The main idea in this second chapter is to understand if, when we go from events to conditional events and from conditionals to iterated conditionals, these properties are still valid (in their generalised versions if necessary) for these structures.
To do so, in Properties 1-4 (Section 2.1) we replace events A, B by conditional events $A|H, B|K$, and for the compound conditionals the symbol of probability P by the symbol of prevision $\mathbb{P}$, the properties 1, 2, 3, and 4 become:

P1. $(B|K)|(A|H) = [(A|H) \wedge (B|K)]|(A|H)$;

P2. $(A|H) \wedge (B|K) \leqslant (B|K)|(A|H)$ and hence $\mathbb{P}[(A|H) \wedge (B|K)] \leqslant \mathbb{P}[(B|K)|(A|H)]$;

P3. $\mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(B|K)|(A|H)]P(A|H)$ (*compound formula for iterated conditional*);

P4. if A, B, H, K are logically independent events, denoting $P(A|H) = x$ and $P(B|H) = y$, the extension μ on $(B|K)|(A|H)$ is coherent if and only if $\mu \in [\mu', \mu'']$, where μ' and μ'' are given in formula (2.1).

We recall that when a compound or iterated conditional reduces to a conditional event, its prevision coincides with its probability.
We will show in Example 3, Example 4, and Section 2.6 that when some of the properties P1-P4 are not satisfied, some counterintuitive probabilistic results are obtained. These are avoided by a suitable notion of iterated conditioning which satisfies properties P1-P4.

In section 2.2, section 2.3, and section 2.4, we will check the validity of the previous properties for notions of compound and iterated conditionals introduced in different trivalent logics as suitable conditional events (in these cases, in properties P2 and P4, the symbol of prevision $\mathbb{P}$ is replaced by the symbol of probability P, because the involved objects are conditional events). Then, in Section 2.5 we will check the basic properties for the iterated conditionals, defined as conditional random quantities, built using the structure $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\overline{\bigcirc}$ and the different notions of conjunction in trivalent logic recalled in Section 1.4.

2.2 The iterated conditional in the trivalent logic of Cooper-Calabrese

In this section, in the framework of a trivalent logic, we study the validity of the Import-Export principle and of the properties P1-P4 for the notion of iterated conditional, here denoted by $(B|K)|_C(A|H)$, studied by Cooper ([30]) and by Calabrese ([12], see also [11, 10]). We also give two results on the set of all coherent assessments for suitable families of conditional events which include the iterated conditional $(B|K)|_C(A|H)$.

We recall that the notions of conjunction and disjunction of conditionals used by Cooper and Calabrese coincide with $\wedge_S$ and $\vee_S$, respectively [1].

Definition 8. *Given any pair of conditional events $A|H$ and $B|K$, the iterated conditional $(B|K)|_C(A|H)$ is defined as the following conditional event*

$$(B|K)|_C(A|H) = B|(K \wedge (\bar{H} \vee A)). \tag{2.4}$$

We observe that in (2.4) the conditioning event is the conjunction of the conditioning event K of the consequent $B|K$ and the material conditional $\bar{H} \vee A$ associated with the antecedent $A|H$.

Remark 11 (Import-Export principle for $|_C$). *By applying Definition 8 with $H = \Omega$, it holds that*

$$(B|K)|_C A = ABK|AK = B|AK, \tag{2.5}$$

which shows that the Import-Export principle is satisfied by $|_C$.

2.2.1 Property P1

We observe that

$$\begin{aligned}((A|H) \wedge_S (B|K))|_C(A|H) &= ((AHBK \vee AH\bar{K} \vee \bar{H}BK)|(H \vee K))|_C(A|H) = \\ &= (AHBK \vee AH\bar{K} \vee \bar{H}BK)|(AK \vee \bar{H}K \vee AH\bar{K}).\end{aligned} \tag{2.6}$$

From (2.4) and (2.6) it follows that $(B|K)|_C(A|H) \neq ((A|H) \wedge_S (B|K))|_C(A|H)$. Indeed, as illustrated by Table 2.1, when the constituent $AH\bar{K}$ is true, it holds that $(B|K)|_C(A|H)$ is void, while $((A|H) \wedge_S (B|K))|_C(A|H)$ is true. Then, property P1 is not satisfied by the pair $(\wedge_S, |_C)$.

C_h	$(A\|H) \wedge_S (B\|K)$	$(B\|K)\|_C(A\|H)$	$((A\|H) \wedge_S (B\|K))\|_C(A\|H)$
$AHBK \vee \bar{H}BK$	True	True	True
$AH\bar{B}K \vee \bar{H}\bar{B}K$	False	False	False
$AH\bar{K}$	True	Void	True
$\bar{A}H$	False	Void	Void
$\bar{H}\bar{K}$	Void	Void	Void

TABLE 2.1: Truth values of $(A|H) \wedge_S (B|K)$, $(B|K)|_C(A|H)$, and $((A|H) \wedge_S (B|K))|_C(A|H)$.

2.2.2 Property P2

From Table 2.1 we also obtain that property P2 is not satisfied by $(\wedge_S, |_C)$. Indeed, under logical independence, when $AH\bar{K}$ is true, it holds that $(A|H) \wedge_S (B|K)$ is true,

[1] Note that also Cantwell ([14]) defines the iterated conditional as Cooper and Calabrese, but, unlike them, in his trivalent logic conjunction and disjunction are defined by $\wedge_K$ and $\vee_K$, respectively.

while $(B|K)|_C(A|H)$ is void. Then, since $AH \neq \emptyset$, $K \nsubseteq B$, and $((A|H) \wedge_S (B|K)) \nsubseteq (B|K)|_C(A|H)$, from (1.5) it follows that $((A|H) \wedge_S (B|K)) \nleq (B|K)|_C(A|H)$.

2.2.3 Property P3

Now we focus our attention on the following result regarding the coherence of a probability assessment on $\{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$ (Theorem 7), then we use this result in order to check the validity of property P3 for the pair $(\wedge_S, |_C)$.

Theorem 7. *Let A, B, H, K be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = xy$ and $z'' = max(x, y)$.*

Proof. The constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are (see also Table 2.2)

$$C_1 = AHBK, C_2 = \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} = \bar{A}H, C_3 = \bar{H}BK, C_4 = AH\bar{B}K,$$
$$C_5 = \bar{H}\bar{B}K, C_6 = AH\bar{K}, C_0 = \bar{H}\bar{K},$$

and

$$Q_1 = (1, 1, 1),\ Q_2 = (0, y, 0),\ Q_3 = (x, 1, 1), Q_4 = (1, 0, 0),$$
$$Q_5 = (x, 0, 0),\ Q_6 = (1, y, 1), \mathcal{P} = Q_0 = (x, y, z).$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

	C_h	$A\|H$	$(B\|K)\|_C(A\|H)$	$(A\|H) \wedge_S (B\|K)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$\bar{A}H$	0	y	0	Q_2
C_3	$\bar{H}BK$	x	1	1	Q_3
C_4	$AH\bar{B}K$	1	0	0	Q_4
C_5	$\bar{H}\bar{B}K$	x	0	0	Q_5
C_6	$AH\bar{K}$	1	y	1	Q_6
C_0	$\bar{H}\bar{K}$	x	y	z	Q_0

TABLE 2.2: Constituents C_h and points Q_h associated with $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$, the probability assessment $\mathcal{P} = (x, y, z)$.

$$\begin{cases} \lambda_1 + x\lambda_3 + \lambda_4 + x\lambda_5 + \lambda_6 = x, \\ \lambda_1 + y\lambda_2 + \lambda_3 + y\lambda_6 = y, \\ \lambda_1 + \lambda_3 + \lambda_6 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \\ \lambda_i \geqslant 0,\ i = 1, \dots, 6. \end{cases} \tag{2.7}$$

Lower bound We first prove that the assessment (x, y, xy) is coherent for every $(x, y) \in [0, 1]^2$. Then, in order to prove that $z' = xy$ is the lower bound for $z = P((A|H) \wedge_S (B|K))$, we verify that the assessment (x, y, z) is not coherent when $z < z' = xy$.
We observe that $\mathcal{P} = (x, y, xy) = xyQ_1 + (1 - x)Q_2 + x(1 - y)Q_4$, so a solution of (2.7) is given by $\Lambda = (xy, 1 - x, 0, x(1 - y), 0, 0)$.

Then, by considering the function ϕ as defined in (1.7), it holds that

$$\begin{aligned}\phi_1(\Lambda) &= \textstyle\sum_{h:C_h\subseteq H}\lambda_h = \lambda_1+\lambda_2+\lambda_4+\lambda_6 = xy+(1-x)+x(1-y)=1>0,\\ \phi_2(\Lambda) &= \textstyle\sum_{h:C_h\subseteq (A\vee \bar{H})\vee K}\lambda_h = \lambda_1+\lambda_3+\lambda_4+\lambda_5 = xy+x(1-y)=x,\\ \phi_3(\Lambda) &= \textstyle\sum_{h:C_h\subseteq H\vee K}\lambda_h = \lambda_1+\cdots+\lambda_6 = 1>0.\end{aligned}$$

Let $\mathcal{S}'=\{(xy,1-x,0,x(1-y),0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.7). We have that $M_1'=1$, $M_2'=x$, $M_3'=1$ (as defined in (1.8)). We distinguish two cases: (i) $x>0$, (ii) $x=0$. In the case (i) we get $M_1'>0$, $M_2'>0$, $M_3'>0$ and then $I_0'=\emptyset$. By Theorem 2, the assessment (x,y,xy) is coherent $\forall (x,y)\in[0,1]^2$. In the case (ii) we have that $M_1'>0$, $M_2'=0$, $M_3'>0$, hence $I_0'=2$. We observe that the sub-assessment $\mathcal{P}_0'=y$ on $\mathcal{F}_0'=\{(B|K)|_C(A|H)\}$ is coherent for every $y\in[0,1]$. Then, by Theorem 2, the assessment (x,y,xy) on $\mathcal{F}$ is coherent $\forall (x,y)\in[0,1]^2$.

In order to verify that $z'=xy$ is the lower bound for z, we observe that the points Q_1,Q_2,Q_4 belong to the plane $\pi: yX+Y-Z=y$, where X,Y,Z are the axes coordinates.
Now, by considering the function $f(X,Y,Z)=yX+Y-Z$, we observe that for each constant k the equation $f(X,Y,Z)=k$ represents a plane which is parallel to π and coincides with π when $k=y$. We also observe that $f(Q_1)=f(Q_2)=f(Q_4)=y$, $f(Q_3)=f(Q_5)=xy\leqslant y$ and $f(Q_6)=f(1,y,1)=y+y-1=2y-1\leqslant y$.
Then, for every $\mathcal{P}=\sum_{h=1}^6\lambda_hQ_h$, with $\sum_{h=1}^6\lambda_h=1$ and $\lambda_h\geqslant 0$, that is $\mathcal{P}\in\mathcal{I}$, it holds that $f(\mathcal{P})=f(\sum_{h=1}^6\lambda_hQ_h)=\sum_{h=1}^6\lambda_hf(Q_h)\leqslant y$. On the other side, given $a>0$, by considering $\mathcal{P}=(x,y,xy-a)$ it holds that $f(\mathcal{P})=y+a>y$ and hence $\mathcal{P}=(x,y,xy-a)\notin\mathcal{I}$. Therefore, for any given $a>0$ the assessment $(x,y,xy-a)$ is not coherent because $(x,y,xy-a)\notin\mathcal{I}$. Then the lower bound of $z=P((A|H)\wedge_S(B|K))$ is $z'=xy$.

Upper bound We verify that the assessment $(x,y,\max(x,y))$ on $\mathcal{F}$ is coherent for every $(x,y)\in[0,1]^2$. Moreover, we show that $z''=\max(x,y)$ is the upper bound for $z=P((A|H)\wedge_S(B|K))$ by showing that any assessment (x,y,z) on $\mathcal{F}$ with $(x,y)\in[0,1]^2$ and $z>\max\{x,y\}$ is not coherent. We distinguish two cases: (i) $x\geqslant y$, (ii) $x<y$.
(i) We have that $\max(x,y)=x$ and hence

$$\mathcal{P}=(x,y,x)=(1-x)Q_2+xQ_6.$$

Then, the vector $\Lambda=(0,1-x,0,0,0,x)$ is a solution of (2.7). Moreover, it holds that

$$\begin{aligned}\phi_1(\Lambda) &= \sum_{h:C_h\subseteq H}\lambda_h = \lambda_1+\lambda_2+\lambda_4+\lambda_6 = 1-x+x=1>0,\\ \phi_2(\Lambda) &= \sum_{h:C_h\subseteq (K\wedge(A\vee\bar{H}))}\lambda_h = \lambda_1+\lambda_3+\lambda_4+\lambda_5=0,\\ \phi_3(\Lambda) &= \sum_{h:C_h\subseteq(H\vee K)}\lambda_h = \lambda_1+\cdots+\lambda_6=1>0.\end{aligned}$$

Let $\mathcal{S}'=\{(0,1-x,0,0,0,x)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.7). We have that $M_1'=1$, $M_2'=0$, $M_3'=1$. It follows that $\mathcal{I}_0'=2$. As the sub-assessment $\mathcal{P}_0'=y$ on $\mathcal{F}_0'=\{(B|K)|_C(A|H)\}$ is coherent $\forall y\in[0,1]$, by Theorem 2, it follows that the assessment $(x,y,\max(x,y))$ is coherent.

In order to verify that $z'' = \max(x,y) = x$ is the upper bound for z, we observe that if $\max\{x,y\} = 1$, then $(1,y,z)$ with $z > 1$ is incoherent. Let us assume that $y \leqslant x < 1$. We observe that the points Q_2, Q_3, Q_6 belong to the plane $\pi : X + \frac{1-x}{1-y}Y - Z = \frac{y(1-x)}{1-y}$, where X, Y, Z are the axes coordinates.
Now, by considering the function $f(X,Y,Z) = X + \frac{1-x}{1-y}Y - Z - \frac{y(1-x)}{1-y}$, we observe that $f(Q_1) = 1-x > 0$, $f(Q_2) = f(Q_3) = f(Q_6) = 0$, $f(Q_4) = 1 - y\frac{1-x}{1-y} \geqslant 0$, $f(Q_5) = \frac{x-y}{1-y} \geqslant 0$. Then, for every $\mathcal{P} = \sum_{h=1}^{6} \lambda_h Q_h$, with $\sum_{h=1}^{6} \lambda_h = 1$ and $\lambda_h \geqslant 0$, that is $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geqslant 0$. On the other side, given $z > x$, by considering $\mathcal{P} = (x,y,z)$ it holds that $f(\mathcal{P}) = x - z < 0$ and hence $\mathcal{P} = (x,y,z) \notin \mathcal{I}$. Therefore, for any given $z > x$ the assessment (x,y,z) is not coherent because $(x,y,z) \notin \mathcal{I}$. Then the upper bound on z is $z'' = z = \max\{x,y\}$.
(ii) In this case $\max(x,y) = y$. We prove that the assessment $(x,y,\max(x,y))$ is coherent. We observe that

$$(x,y,y) = yQ_3 + (1-y)Q_5.$$

Then, the vector $\Lambda = (0,0,y,0,1-y,0)$ is a solution of (2.7). We have

$$\phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = 0; \quad \phi_2(\Lambda) = \sum_{h:C_h \subseteq (K \wedge (A \vee \bar{H}))} \lambda_h = 1 > 0;$$
$$\phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq (H \vee K)} \lambda_h = 1 > 0.$$

Let $\mathcal{S}' = \{(0,0,y,0,1-y,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.7). We have that $M_1' = 0$, $M_2' = 1$, $M_3' = 1$. It follows that $\mathcal{I}_0' = 1$. The sub-assessment x on $\{A|H\}$ is coherent $\forall x \in [0,1]$. Then, by Theorem 2, the assessment $(x,y,\max(x,y))$ on $\mathcal{F}$ is coherent $\forall (x,y) \in [0,1]^2$.

In order to verify that $z'' = \max(x,y) = y$ is the upper bound for z, we observe that if $\max\{x,y\} = 1$, then $(x,1,z)$ with $z > 1$ is incoherent. Let us assume that $x \leqslant y < 1$. We observe that the points Q_3, Q_5, Q_6 belong to the plane $\pi : \frac{1-y}{1-x}X - \frac{x(1-y)}{1-x} + Y - Z = 0$, where X, Y, Z are the axes coordinates.
Now, by considering the function $f(X,Y,Z) = \frac{1-y}{1-x}X - \frac{x(1-y)}{1-x} + Y - Z$, we observe that $f(Q_1) = f(Q_4) = 1-y > 0$, $f(Q_2) = \frac{y-x}{1-x} > 0$ $f(Q_3) = f(Q_5) = f(Q_6) = 0$. Then, for every $\mathcal{P} = \sum_{h=1}^{6} \lambda_h Q_h$, with $\sum_{h=1}^{6} \lambda_h = 1$ and $\lambda_h \geqslant 0$, that is $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geqslant 0$. On the other side, given $z > y$, by considering $\mathcal{P} = (x,y,z)$ it holds that $f(\mathcal{P}) = y - z < 0$ and hence $\mathcal{P} = (x,y,z) \notin \mathcal{I}$. Therefore, for any given $z > y$ the assessment (x,y,z) is not coherent because $(x,y,z) \notin \mathcal{I}$. Then the upper bound on z is $z'' = y = \max\{x,y\}$.
Finally, for each given $(x,y) \in [0,1]$, as (x,y,z') and (x,y,z'') are coherent, by the Fundamental theorem of probability, any assessment (x,y,z) with $z \in [z',z'']$ is coherent too. Then, the assessment (x,y,z) on $\mathcal{F}$ is coherent for every $(x,y) \in [0,1]^2$ and $z \in [z',z'']$. □

Remark 12 (Property P3). *From Theorem 7 it follows that any probability assessment $(x,y,z) \in [0,1]^3$ on $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$, for which $z = xy$, is coherent. In other words, any assessment which satisfies property P3 is coherent. For instance, the assessment $(1,0,0)$ on $\mathcal{F}$ is coherent. However, when $xy < \max\{x,y\}$, there are coherent probability assessments which do not satisfy property P3. Indeed, any assessment (x,y,z), with $xy < z \leqslant \max\{x,y\}$, is coherent. Then, property P3 is not satisfied in general by the pair $(\wedge_S, |_C)$. For instance the assessment $(1,0,1)$ on $\mathcal{F}$ is coherent. However, the* *assessment* $(1, 0, 1)$, *on* $\{A, B|A, AB\}$ *is not coherent because does not satisfy the compound probability theorem.*

We illustrate in the following example some counterintuitive (probabilistic) aspects of the invalidity of properties P1, P2, and P3.

Example 3. *Let two independent random quantities* X *and* Y *be given. We assume that* $X \in \{1, \dots, 6\}$, *with* $P(X = h) = \frac{1}{6}$, $h = 1, \dots, 6$, *and* $Y \in \{0, 1\}$, *with* $P(Y = 0) = P(Y = 1) = \frac{1}{2}$. *We set* $A = (X \geqslant 3)$, $H = (X \geqslant 2)$, $B = (Y = 1)$, *and* $K = (Y = 0)$. *From (2.4), as* $B = \bar{K}$, *it follows that*

$$(B|K)|_C(A|H) = \bar{K}|(K \wedge (\bar{H} \vee A)) = \varnothing|(K \wedge (\bar{H} \vee A)). \tag{2.8}$$

Moreover, since

$$(A|H) \wedge_S (B|K) = (\bar{H} \vee AH) \wedge (\bar{K} \vee BK)|(H \vee K) = (AH\bar{K})|(H \vee K), \tag{2.9}$$

and $(H \vee K) \wedge (\bar{H} \vee A) = AH \vee \bar{H}K$, *it holds that*

$$\begin{aligned}((A|H) \wedge_S (B|K))|_C(A|H) = (AH\bar{K}|(H \vee K))|_C(A|H) = AH\bar{K}|((H \vee K) \wedge (\bar{H} \vee A)) = \\ = AH\bar{K}|(AH \vee \bar{H}K).\end{aligned} \tag{2.10}$$

From (2.8) and (2.10) it follows that $((A|H) \wedge_S (B|K))|_C(A|H) \neq (B|K)|_C(A|H)$, *and hence property P1 is not satisfied. Concerning the probabilistic aspects we observe that* $P((B|K)|_C(A|H)) = 0$, *while*

$$\begin{aligned} P(((A|H) \wedge_S (B|K))|_C(A|H)) &= P(AH\bar{K}|(AH \vee \bar{H}K)) = \tfrac{P(A\bar{K})}{P(AH)+P(\bar{H}K)} \\ &= \frac{\frac{4}{6}\frac{1}{2}}{\frac{4}{6}+\frac{1}{6}\frac{1}{2}} = \tfrac{4}{9} \neq 0. \end{aligned}$$

From (2.8) and (2.9) it holds that $(A|H) \wedge_S (B|K) \nleqslant (B|K)|_C(A|H)$ *and hence property P2 is not satisfied. Moreover,*

$$\begin{aligned} P((A|H) \wedge_S (B|K)) &= P(AH\bar{K}|(H \vee K)) = \tfrac{P(A\bar{K})}{P(H \vee K)} \\ &= \frac{\frac{4}{6}\cdot\frac{1}{2}}{\frac{5}{6}+\frac{1}{2}-\frac{5}{6}\cdot\frac{1}{2}} = \tfrac{4}{11} > 0 = P((B|K)|_C(A|H)). \end{aligned}$$

We also obtain (the counterintuitive result) that the probability of the conjunction is greater than the probability of one of the two conjuncts, indeed it holds that $P((A|H) \wedge_S (B|K)) = \frac{4}{11} > 0 = P(B|K)$. *Therefore,*

$$0 = P(A|H)P((B|K)|_C(A|H)) \neq P((A|H) \wedge_S (B|K)) = \frac{4}{11}$$

and hence property P3 is not satisfied.

2.2.4 Property P4

We check the validity of property P4 for the iterated conditioning $|_C$ by studying the set of all coherent probability assessments on the family $\{A|H, B|K, (B|K)|_C(A|H)\}$ (Theorem 8).

Theorem 8. *Let* A, B, H, K *be any logically independent events. The probability assessments* $\mathcal{P} = (x, y, z)$ *on the family of conditional events* $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$ *is coherent for every* $(x, y, z) \in [0, 1]^3$.

Proof. We recall that $(B|K)|_C(A|H) = B|(K \wedge (\bar{H} \vee A))$. Then, the constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, B|(K \wedge (\bar{H} \vee A))\}$ are (see also Table 2.3)

$$C_1 = AHBK,\ C_2 = AH\bar{B}K,\ C_3 = AH\bar{K},\ C_4 = \bar{A}HBK,\ C_5 = \bar{A}H\bar{B}K,$$
$$C_6 = \bar{A}H\bar{K},\ C_7 = \bar{H}BK,\ C_8 = \bar{H}\,\bar{B}K,\ C_0 = \bar{H}\,\bar{K},$$

and

$$Q_1 = (1,1,1), Q_2 = (1,0,0), Q_3 = (1,y,z), Q_4 = (0,1,z), Q_5 = (0,0,z),$$
$$Q_6 = (0,y,z), Q_7 = (x,1,1), Q_8 = (x,0,0), \mathcal{P} = Q_0 = (x,y,z).$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

	C_h	$A\|H$	$B\|K$	$(B\|K)\|_C(A\|H)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$H\bar{B}K$	1	0	0	Q_2
C_3	$AH\bar{K}$	1	y	z	Q_3
C_4	$\bar{A}HBK$	0	1	z	Q_4
C_5	$\bar{A}H\bar{B}K$	0	0	z	Q_5
C_6	$\bar{A}H\bar{K}$	0	y	z	Q_6
C_7	$\bar{H}BK$	x	1	1	Q_7
C_8	$\bar{H}\,\bar{B}K$	x	0	0	Q_8
C_0	$\bar{H}\,\bar{K}$	x	y	z	Q_0

TABLE 2.3: Constituents and points Q_h' associated with $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$, the probability assessment $\mathcal{P} = (x, y, z)$.

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + x\lambda_7 + x\lambda_8 = x, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y, \\ \lambda_1 + z\lambda_3 + z\lambda_4 + z\lambda_5 + z\lambda_6 + \lambda_7 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1, \\ \lambda_i \geqslant 0,\ i = 1, \dots, 8. \end{cases} \tag{2.11}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_3, Q_6; indeed $(x, y, z) = xQ_3 + (1-x)Q_6 = x(1, y, z) + (1-x)(0, y, z)$. The vector $\Lambda = (0, 0, x, 0, 0, 1-x, 0, 0)$ is a solution of (2.11), with

$$\begin{aligned} \phi_1(\Lambda) &= \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 > 0, \\ \phi_2(\Lambda) &= \sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 0, \\ \phi_3(\Lambda) &= \sum_{h:C_h \subseteq ((A \vee \bar{H}) \wedge K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_7 + \lambda_8 = 0. \end{aligned}$$

We set $\mathcal{S}' = \{\Lambda\}$ and we get $\mathcal{I}'_0 = \{2, 3\}$. So we obtain $\mathcal{F}'_0 = \{B|K, B|(K \wedge (\bar{H} \vee A))\}$ and $\mathcal{P}'_0 = (y, z)$. By recalling Example 1 the sub-assessment (y, z) on $\{B|K, B|(K \wedge (\bar{H} \vee A))\}$ is coherent for every $(y, z) \in [0,1]^2$. Thus, by Theorem 3, the assessment (x, y, z) on $\mathcal{F}$ is coherent $\forall (x, y, z) \in [0,1]^3$. $\square$

Remark 13 (Property P4). *We observe that the probability propagation rule valid for unconditional events (Property P4) is no longer valid for Cooper-Calabrese's iterated conditional. Indeed, from Theorem 8, any probability assessment* (x, y, z) *on* $\mathcal{F} =$

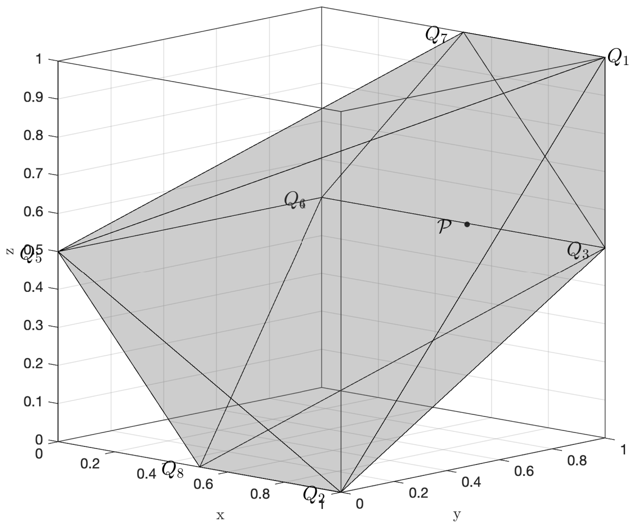

FIGURE 2.1: Convex hull of the points Q_i for $i = 1,\ldots,8$ associated with the pair $(\mathcal{F},\mathcal{P})$, where $\mathcal{P} = (x,y,z)$ and $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$. In the figure the numerical values are: $x = 0.5$, $y = 1$, $z = 0.5$.

$\{A|H, B|K, (B|K)|_C(A|H)\}$, with $(x,y,z) \in [0,1]^3$ is coherent. For instance, the assessment $(1,1,0)$ on $\mathcal{F}$ is coherent, while it is not coherent on $\{A, B, B|A\}$.

We briefly sum up here the results obtained in this section (see also Table 6.1) for the iterated conditioning $|_C$. We verified that $|_C$ does not satisfy any of the desirable properties P1-P4 we considered in Section 4.1. More precisely, we verified that $(B|K)|_C(A|H) \neq ((A|H) \wedge_S (B|K))|_C(A|H)$. Thus, P1 does not hold. Furthermore, $((A|H) \wedge_S (B|K)) \nleq (B|K)|_C(A|H)$. So P2 does not hold either. Still further, $P(A|H)P((B|K)|_C(A|H)) \neq P((A|H) \wedge_S (B|K))$ for some probability P. Thus, P3 does not hold. Finally, we proved that every assessment $(x,y,z) \in [0,1]^3$ on $\{A|H, B|K, (B|K)|_C(A|H)\}$ is coherent. From this it follows that P4 does not hold. We also observed that Import-Export Principle is satisfied.

2.3 The iterated conditional in the trivalent logic of de Finetti.

In this section we analyse the notion of iterated conditional introduced by de Finetti in [32]. After recalling that the notion of conjunction and disjunction of conditionals introduced by de Finetti in [32] coincide with $\wedge_K$ and $\vee_K$ (see Section 1.4), we check the validity of the Import-Export principle and of the properties P1-P4 for this iterated conditional in the corresponding trivalent logic.

Definition 9. *Given any pair of conditional events $A|H$ and $B|K$, de Finetti iterated conditional, denoted by $(B|K)|_{dF}(A|H)$, is defined as*

$$(B|K)|_{dF}(A|H) = B|(AHK). \tag{2.12}$$

C_h	$(A\|H) \wedge_K (B\|K)$	$(B\|K)\|_{dF}(A\|H)$	$((A\|H) \wedge_K (B\|K))\|_{dF}(A\|H)$
$AHBK$	True	True	True
$AH\bar{B}K$	False	False	False
$AH\bar{K} \vee \bar{H}BK \vee \bar{H}\,\bar{K}$	Void	Void	Void
$\bar{A}H \vee \bar{H}\,\bar{B}K$	False	Void	Void

TABLE 2.4: Truth table of $(A|H) \wedge_K (B|K)$, $(B|K)|_{dF}(A|H)$, and $((A|H) \wedge_K (B|K))|_{dF}(A|H)$.

Remark 14 (Import-Export principle for $|_{dF}$)**.** *By applying Definition 9 with $H = \Omega$, it holds that*

$$(B|K)|_{dF}A = B|AK, \tag{2.13}$$

which shows that the Import-Export principle [80] is satisfied by $|_{dF}$. Then, from (2.5), it follows that

$$(B|K)|_C A = (B|K)|_{dF}A = B|AK.$$

2.3.1 Property P1

To check property P1 we observe that from (2.12) it holds that

$$\begin{aligned} &((A|H) \wedge_K (B|K))|_{dF}(A|H) = (AHBK|(HK \vee \bar{A}H \vee \bar{B}K))|_{dF}(A|H) = \\ &AHBK|(AHK \vee AH\bar{B}K) = AHBK|AHK = (B|K)|_{dF}(A|H). \end{aligned} \tag{2.14}$$

Then, property P1 is satisfied by the pair $(\wedge_K, |_{dF})$ (see also Table 2.4).

2.3.2 Property P2

From Table 2.4 we also observe that relation P2 is satisfied by $(\wedge_K, |_{dF})$. Indeed, according to the Equation (1.1), if $(A|H) \wedge_K (B|K)$ is true, then $(B|K)|_{dF}(A|H)$ is true; if $(B|K)|_{dF}(A|H)$ is false, then $(A|H) \wedge_K (B|K)$ is false. Thus, since $(A|H) \wedge_K (B|K) \subseteq (B|K)|_{dF}(A|H)$, from (1.5) it follows that $(A|H) \wedge_K (B|K) \leqslant (B|K)|_{dF}(A|H)$.

2.3.3 Property P3

We consider now the following results regarding the coherence of a probability assessment on $\{A|H, (B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$ (Theorem 9) in order to check the validity of property P3 for the pair $(\wedge_K, |_{dF})$.

Theorem 9. *Let A, B, H, K be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = xy$.*

Proof. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are

$$\begin{aligned} &C_1 = AHBK, C_2 = \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} = \bar{A}H, \\ &C_3 = AH\bar{B}K, C_4 = \bar{H}\,\bar{B}K, C_5 = AH\bar{K}, C_0 = \bar{H}BK \vee \bar{H}\,\bar{K}, \end{aligned}$$

and

$$\begin{aligned} &Q_1 = (1,1,1), Q_2 = (0,y,0), Q_3 = (1,0,0), \\ &Q_4 = (x,y,0), Q_5 = (1,y,z), \mathcal{P} = Q_0 = (x,y,z). \end{aligned}$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

	C_h	$A\|H$	$(B\|K)\|_{dF}(A\|H)$	$(A\|H) \wedge_K (B\|K)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$\bar{A}H$	0	y	0	Q_2
C_3	$AH\bar{B}K$	1	0	0	Q_3
C_4	$\bar{H}\bar{B}K$	x	y	0	Q_4
C_5	$AH\bar{K}$	1	y	z	Q_5
C_0	$\bar{H}BK \vee \bar{H}\bar{K}$	x	y	z	Q_0

TABLE 2.5: Constituents C_h and points Q_h associated with $\mathcal{F} = \{A|H, (B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$, the probability assessment $\mathcal{P} = (x, y, z)$.

$$\begin{cases} \lambda_1 + \lambda_3 + x\lambda_4 + \lambda_5 = x, \\ \lambda_1 + y\lambda_2 + y\lambda_4 + y\lambda_5 = y, \\ \lambda_1 + z\lambda_5 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \\ \lambda_i \geqslant 0,\ i = 1, \dots, 5. \end{cases} \tag{2.15}$$

Lower bound We first prove that the assessment $(x, y, 0)$ is coherent for every $(x, y) \in [0, 1]^2$. We observe that $\mathcal{P} = (x, y, 0) = Q_4$, so a solution of (2.15) is given by $\Lambda = (0, 0, 0, 1, 0)$.
Then, by considering the function ϕ as defined in (1.7), it holds that

$$\begin{aligned} \phi_1(\Lambda) &= \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 = 0, \\ \phi_2(\Lambda) &= \textstyle\sum_{h:C_h \subseteq (AHK)} \lambda_h = \lambda_1 + \lambda_3 = 0, \\ \phi_3(\Lambda) &= \textstyle\sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 > 0. \end{aligned}$$

Let $\mathcal{S}' = \{(0, 0, 0, 1, 0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.15). We have that $M_1' = 0$, $M_2' = 0$, $M_3' = 1$ (as defined in (1.8)). Then $I_0' = \{1, 2\}$ and we set $\mathcal{K} = \mathcal{F}_0' = \{A|B, (B|K)|_{dF}(A|H)\}$ and $\mathcal{V} = \mathcal{P}_0' = (x, y)$. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{P}_0'$ on $\mathcal{F}_0'$ are

$$\begin{aligned} &C_1 = AHBK, C_2 = \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} = \bar{A}H, \\ &C_3 = AH\bar{B}K, C_4 = AH\bar{K}, C_0 = \bar{H}\bar{K} \vee \bar{H}BK \vee \bar{H}\bar{B}K = \bar{H}, \end{aligned}$$

and

$$Q_1 = (1, 1),\ Q_2 = (0, y),\ Q_3 = (1, 0),\ Q_4 = (1, y), \mathcal{P} = Q_0 = (x, y).$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{K}, \mathcal{V})$ becomes

$$\begin{cases} \lambda_1 + \lambda_3 + \lambda_4 = x, \\ \lambda_1 + y\lambda_2 + y\lambda_4 = y, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\ \lambda_i \geqslant 0,\ i = 1, \dots, 4. \end{cases} \tag{2.16}$$

We observe that $(x, y) = (1 - x)Q_2 + xQ_4$ so a solution of (2.16) is $\Lambda = (0, 1 - x, 0, x)$. By considering the function ϕ as defined in (1.7), it holds that

$$\begin{aligned} \phi_1(\Lambda) &= \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\ \phi_2(\Lambda) &= \textstyle\sum_{h:C_h \subseteq (AHK)} \lambda_h = \lambda_1 + \lambda_3 = 0, \end{aligned}$$

Let $\mathcal{S}' = \{(0, 1 - x, 0, x)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.16). We have that $M_1' = 0$, $M_2' = 1$, (as defined in (1.8)). Then, the set I_0' associated with $(\mathcal{K}, \mathcal{V})$

is $I_0' = \{2\}$. We observe that the sub-assessment y on $\{(B|K)|_{dF}(A|H)\}$ is coherent for every $y \in [0,1]$. Then, by Theorem 2, the assessment $(x,y,0)$ on $\mathcal{F}$ is coherent $\forall (x,y) \in [0,1]^2$.

Upper bound We verify that the assessment (x,y,xy) on $\mathcal{F}$ is coherent for every $(x,y) \in [0,1]^2$. Moreover, we show that $z'' = xy$ is the upper bound for $z = P((A|H) \wedge_K (B|K))$ by showing that any assessment (x,y,z) on $\mathcal{F}$ with $(x,y) \in [0,1]^2$ and $z > xy$ is not coherent.
We observe that

$$(x,y,xy) = xyQ_1 + (1-x)Q_2 + x(1-y)Q_3.$$

Then, the vector $\Lambda = (xy, 1-x, x(1-y), 0, 0)$ is a solution of (2.15). Moreover, it holds that

$$\begin{array}{l}
\phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 = 1 > 0,\\
\phi_2(\Lambda) = \sum_{h:C_h \subseteq (AHK)} \lambda_h = \lambda_1 + \lambda_3 = xy + 1 - x,\\
\phi_3(\Lambda) = \sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 > 0.
\end{array}$$

Let $\mathcal{S}' = \{(xy, 1-x, x(1-y), 0, 0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.15). We have that $M_1' = 1$, $M_2' = xy + 1 - x$, $M_3' = 1$ (as defined in (1.8)). We distinguish two cases: (i) $(x \neq 1) \vee (y \neq 0)$, (ii) $(x = 1) \wedge (y = 0)$. In the case (i) we get $M_1' > 0$, $M_2' > 0$, $M_3' > 0$ and hence $I_0' = \emptyset$. By Theorem 2, the assessment (x,y,xy) is coherent $\forall (x,y) \in [0,1]^2$. In the case (ii) we get $M_1' > 0$, $M_2' = 0$, $M_3' > 0$, then $I_0' = \{2\}$. We observe that the sub-assessment $\mathcal{P}_0' = y$ on $\mathcal{F}_0' = \{(B|K)|_{dF}(A|H)\}$ is coherent for every $y \in [0,1]$. Then, by Theorem 2, the assessment (x,y,xy) on $\mathcal{F}$ is coherent $\forall (x,y) \in [0,1]^2$.
We verify that $z'' = xy$ is the upper bound for z, by showing that the assessment (x,y,z) is incoherent when $z > xy$.
Let $z > xy$. We distinguish the following cases: (a) $y \neq 0$; (b) $y = 0$.
Case (a). We observe that the points Q_1, Q_2, Q_3 belong to the plane $\pi : yX + Y - Z = y$, where X, Y, Z are the axes coordinates. We set $f(X,Y,Z) = yX + Y - Z$ and we observe that $f(\mathcal{P}) = f(x,y,z) = xy + y - z$. For each $h = 1,2,3,4,5$, we compute the difference $f(Q_h) - f(\mathcal{P})$. We have

$$\begin{array}{l}
f(Q_1) - f(\mathcal{P}) = f(Q_2) - f(\mathcal{P}) = f(Q_3) - f(\mathcal{P}) = z - xy;\\
f(Q_4) - f(\mathcal{P}) = z; \quad f(Q_5) - f(\mathcal{P}) = y - xy = y(1-x);
\end{array}$$

We consider the sub-cases: (i) $x < 1$; (ii) $x = 1$.
(i) We recall that, by setting the stakes $s_1 = y$, $s_2 = 1$, $s_3 = -1$, it holds that $g_h = f(Q_h) - f(\mathcal{P})$, $h = 1, \dots, 5$, where g_h is the value of the random gain G associated with the constituent $C_h \subseteq \mathcal{H}$. As $x < 1$, it follows that $g_h = f(Q_h) - f(\mathcal{P}) > 0$, $h = 1, \dots, 5$. Thus, as there exist (s_1, s_2, s_3) such that $\min G_{\mathcal{H}} \max G_{\mathcal{H}} > 0$, it follows that the assessment (x,y,z) is not coherent when $z > xy$ and $x < 1$.
(ii). In this case, as $x = 1$, it follows that $\mathcal{P} = (x,y,z) = (1,y,z) = Q_5 \in \mathcal{I}$. Then, the vector $\Lambda = (0,0,0,0,1)$ is a solution of (2.15). Then, by considering the function ϕ as defined in (1.7), it holds that

$$\begin{array}{l}
\phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 = 1 > 0,\\
\phi_2(\Lambda) = \sum_{h:C_h \subseteq (AHK)} \lambda_h = \lambda_1 + \lambda_3 = 0,\\
\phi_3(\Lambda) = \sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0.
\end{array}$$

Then, we get $\mathcal{I}_0 \subseteq \{2,3\}$. and we study the coherence of the sub-assessment $\mathcal{P}_0 = (y,z)$ on $\mathcal{F}_0 = \{(B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$.
The constituents C_h's and the point Q_h's associated with the pair $(\mathcal{P}_0, \mathcal{F}_0)$ are

$$\begin{array}{l} C_1 = AHBK,\ C_2 = \bar{A}H \vee \bar{H}\,\bar{B}K, \\ C_3 = AH\bar{B}K,\ C_0 = \bar{H}\,\bar{K} \vee AH\bar{K} \vee \bar{H}BK, \end{array}$$

with $C_h \subseteq \mathcal{H} = HBK \vee \bar{A}H \vee \bar{B}K$, $h = 1,2,3$. The associated points Q_h's are

$$Q_1 = (1,1),\ Q_2 = (y,0),\ Q_3 = (0,0),\ \mathcal{P} = Q_0 = (x,y,z).$$

As shown in Figure 2.3 the convex hull $\mathcal{I}$ of the points Q_1, Q_2, and Q_3 is the triangle of vertices $(1,1), (y,0)$, and $(0,0)$. We observe that $\mathcal{P}_0 = (y,z) \notin \mathcal{I}$ because $z > y$. Then, the sub-assessment $\mathcal{P}_0$ on $\mathcal{F}_0$ is not coherent. Therefore, by Theorem 2, the assessment $\mathcal{P} = (1,y,z)$ on $\mathcal{F}$ is not coherent too.

Notice that, if $y = 0$, $(0,z) \in \mathcal{I} \iff z = 0$, so if $z > 0$ the assessment is not coherent.

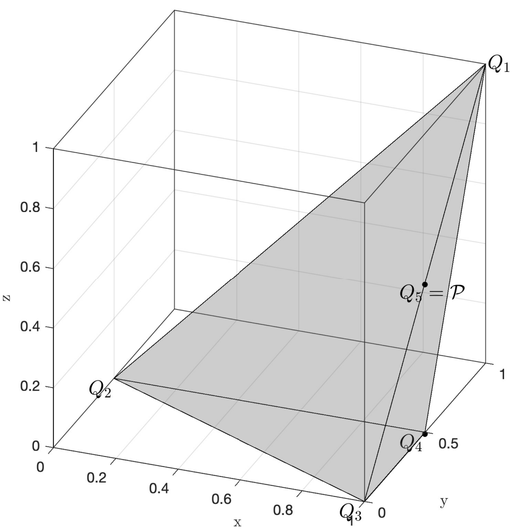

FIGURE 2.2: Convex hull of the points Q_1, Q_2, Q_3, Q_4, Q_5 associated with the pair $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \{A|H, (B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$ and $\mathcal{P} = (x,y,z)$. In the figure the numerical values are: $x = 1, y = 0.5, z = 0.5$. Notice that $Q_5 = \mathcal{P}$.

So in this case we have that the probability assessment is coherent if and only if $z \leqslant xy$.

Case (b). In this case $y = 0$. We have already analyse the situation $(x = 1) \wedge (y = 0)$ in the precedent case, so let's assume $x \neq 1$.
We observe that the points Q_1, Q_2, Q_5 belong to the plane $\pi : zX + (1-z)Y - Z = 0$, where X, Y, Z are the axes coordinates. We set $f(X,Y,Z) = zX + (1-z)Y - Z$ and we observe that $f(\mathcal{P}) = f(x,y,z) = -(1-x)z$. For each $h = 1,2,3,4,5$, we consider

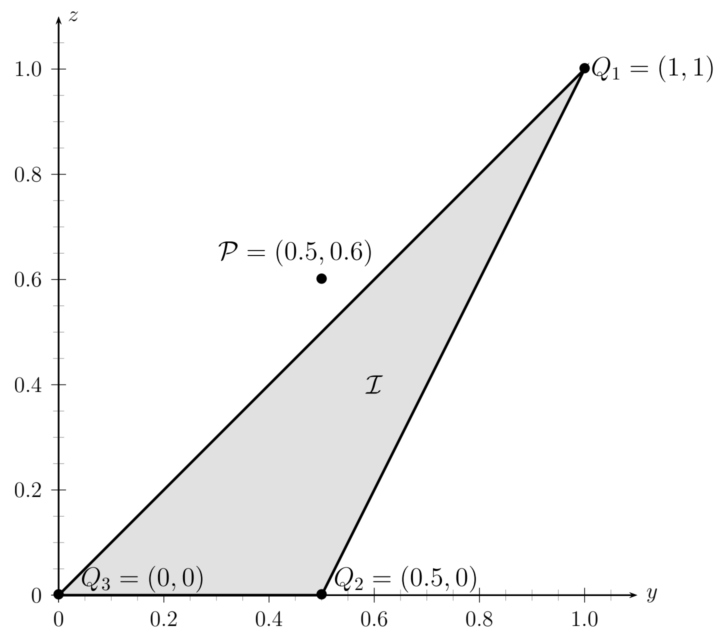

FIGURE 2.3: Convex hull of the points Q_1, Q_2, Q_3 associated with the pair $(\mathcal{P}_0, \mathcal{F}_0)$, where $\mathcal{P}_0 = (y, z)$ and $\mathcal{F}_0 = \{(B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$. In the figure the numerical values are: $y = 0.5$ and $z = 0.6$.

the quantity $f(Q_h) - f(P)$. Then, we obtain

$$\begin{aligned}
f(Q_1) - f(\mathcal{P}) &= z(1-x);\\
f(Q_2) - f(\mathcal{P}) &= z(1-x);\\
f(Q_3) - f(\mathcal{P}) &= z(2-x);\\
f(Q_4) - f(\mathcal{P}) &= z;\\
f(Q_5) - f(\mathcal{P}) &= z(1-x);
\end{aligned}$$

We recall that, by setting the stakes $s_1 = z$, $s_2 = 1 - z$, $s_3 = -1$, it holds that $g_h = f(Q_h) - f(\mathcal{P})$, $h = 1, \dots, 5$, where g_h is the value of the random gain G associated with the constituent $C_h \subseteq \mathcal{H}$. As $x < 1$, it follows that $g_h = f(Q_h) - f(\mathcal{P}) > 0$, $h = 1, \dots, 5$. Thus, as there exist (s_1, s_2, s_3) such that $\min G_{\mathcal{H}} \max G_{\mathcal{H}} > 0$, it follows that the assessment (x, y, z) is not coherent when $z > xy$ and $y = 0$.
We conclude that $z'' = xy$ is the upper bound for z.

□

Remark 15 (Property P3). *From Theorem 9 any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, (B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$, with $(x, y) \in [0, 1]^2$ and $0 \leqslant z \leqslant xy$, is coherent. Then, any assessment which satisfies property P3 is coherent. Moreover, as $z = xy$ is not the unique coherent extension of the conjunction $(A|H) \wedge_K (B|K)$, the quantity $P((A|H) \wedge_K (B|K))$ could not coincide with the product $P((B|K)|_{dF}(A|H))P(A|H)$. For example, if we choose the probability assessment $\mathcal{P} = (1, 1, 0)$, we observe that $\mathcal{P}$ is coherent on $\mathcal{F}$ because $0 \in [z', z''] = [0, 1]$. However, we observe that $\mathcal{P}$ on $\{A, B|A, AB\}$ is not coherent because $P(AB) = 0 \neq P(B|A)P(A)$.*
Then, property P3 is not satisfied by the pair $(\wedge_K, |_{dF})$.

2.3.4 Property P4

To check the validity of property P4 for the iterated conditioning $|_{dF}$ we study the set of all coherent probability assessments on the family $\{A|H, B|K, (B|K)|_{dF}(A|H)\}$

(Theorem 10).

Theorem 10. *Let A, B, H, K be any logically independent events. The probability assessments $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, B|K, (B|K)|_{dF}(A|H)\}$ is coherent for every $(x, y, z) \in [0, 1]^3$.*

Proof. The constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are (see also Table)

$$C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,$$
$$C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\,\bar{B}K, C_0 = \bar{H}\,\bar{K},$$

and

$$Q_1 = (1, 1, 1), Q_2 = (1, 0, 0), Q_3 = (1, y, z), Q_4 = (0, 1, z), Q_5 = (0, 0, z),$$
$$Q_6 = (0, y, z), Q_7 = (x, 1, z), Q_8 = (x, 0, z), \mathcal{P} = Q_0 = (x, y, z).$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

	C_h	$A\|H$	$B\|K$	$(B\|K)\|_{dF}(A\|H)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$AH\bar{B}K$	1	0	0	Q_2
C_3	$AH\bar{K}$	1	y	z	Q_3
C_4	$\bar{A}HBK$	0	1	z	Q_4
C_5	$\bar{A}H\bar{B}K$	0	0	z	Q_5
C_6	$\bar{A}H\bar{K}$	0	y	z	Q_6
C_7	$\bar{H}BK$	x	1	z	Q_7
C_8	$\bar{H}\,\bar{B}K$	x	0	z	Q_8
C_0	$\bar{H}\,\bar{K}$	x	y	z	Q_0

TABLE 2.6: Constituents and points Q_h' associated with $\mathcal{F} = \{A|H, B|K, (B|K)|_{dF}(A|H)\}$, the probability assessment $\mathcal{P} = (x, y, z)$.

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + x\lambda_7 + x\lambda_8 = x, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y, \\ \lambda_1 + z\lambda_3 + z\lambda_4 + z\lambda_5 + z\lambda_6 + z\lambda_7 + z\lambda_8 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1, \\ \lambda_i \geqslant 0, \ i = 1, \dots, 8. \end{cases} \tag{2.17}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_3, Q_6; indeed $(x, y, z) = xQ_3 + (1 - x)Q_6 = x(1, y, z) + (1 - x)(0, y, z)$.
$\mathcal{P}$ also belongs to the segment $\overline{Q_7Q_8}$, $(x, y, z) = yQ_7 + (1 - y)Q_8 = y(x, 1, z) + (1 - y)(x, 0, z)$.
Then

$$(x, y, z) = \frac{x}{2}(1, y, z) + \frac{1-x}{2}(0, y, z) + \frac{y}{2}(x, 1, z) + \frac{1-y}{2}(x, 0, z),$$

so the vector $\Lambda = (0, 0, \frac{x}{2}, 0, 0, \frac{1-x}{2}, \frac{y}{2}, \frac{1-y}{2})$ is a solution of (2.17), with

$$\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5\lambda_6 = \frac{1}{2} > 0,$$
$$\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = \frac{1}{2},$$
$$\phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq AHK} \lambda_h = \lambda_1 + \lambda_2 = 0.$$

Let $\mathcal{S}' = \{(0, 0, \frac{x}{2}, 0, 0, \frac{1-x}{2}, \frac{y}{2}, \frac{1-y}{2})\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.17). We have that $M_1' = \frac{1}{2}$, $M_2' = \frac{1}{2}$, $M_3' = 0$ (1.8). We get $\mathcal{I}_0' = \{3\}$. We observe that the sub-assessment z on $\{(B|K)|_{dF}(A|H)\}$ is coherent for every $z \in [0,1]$. Then, by Theorem 2, the assessment (x,y,z) on $\mathcal{F}$ is coherent $\forall (x,y,z) \in [0,1]^3$. □

Remark 16 (Property P4). *We observe that the probability propagation rule valid for unconditional events (property P4) is no longer valid for de Finetti's iterated conditional. Indeed, from Theorem 10, any probability assessment (x,y,z) on $\mathcal{F} = \{A|H, B|K, (B|K)|_{dF}(A|H)\}$, with $(x,y,z) \in [0,1]^3$ is coherent. For instance, the assessment $(1,1,0)$ is coherent on $\mathcal{F}$ but it is not coherent on $\{A, B, B|A\}$.*

We briefly sum up here the results obtained in this section (see also Table 6.1) for the iterated conditioning $|_{dF}$. We verified that $|_{dF}$ satisfies both properties P1 and P2. Indeed, $(B|K)|_{dF}(A|H) = ((A|H) \wedge_K (B|K))|_{dF}(A|H)$ and $((A|H) \wedge_K (B|K)) \leqslant (B|K)|_{dF}(A|H)$. However, the iterated conditioning $|_{dF}$ does not satisfy property P3, because $P(A|H)P((B|K)|_{dF}(A|H)) \neq P((A|H) \wedge_K (B|K))$ for some probability P. Moreover, we showed that every assessment $(x,y,z) \in [0,1]^3$ on $\{A|H, B|K, (B|K)|_{dF}(A|H)\}$ is coherent and from this it follows that property P4 does not hold. We also observed that Import-Export Principle is satisfied by $|_{dF}$.

2.4 The iterated conditional in the trivalent logic of Farrell.

In this section we consider another definition of iterated conditional as a suitable conditional event which was proposed by Farrell in [37, p. 385] (see also [36]). In his trivalent logic, the author uses $\wedge_K$ and $\vee_K$ as conjunction and disjunction of conditional events (Section 1.4), respectively. We first recall the definition of the iterated conditional, then we check the validity of the Import-Export principle and of the properties P1-P4.

Definition 10. *Given any pair of conditional events $A|H$ and $B|K$, Farrell iterated conditional, here denoted by $(B|K)|_F(A|H)$, is defined as the conditional event*

$$(B|K)|_F(A|H) = AHBK|(AHBK \vee AH\bar{B}K \vee \bar{H}\,\bar{B}K). \tag{2.18}$$

Remark 17 (Import-Export principle for $|_F$). *By applying (2.18) with $H = \Omega$, it holds that*

$$(B|K)|_F A = ABK|(ABK \vee A\bar{B}K) = B|AK.$$

Then, as $(B|K)|_F A = B|AK$, the Import-Export principle is satisfied by $|_F$. Moreover, by recalling (2.5) and (2.13), it follows that

$$(B|K)|_F A = (B|K)|_C A = (B|K)|_{dF} A = B|AK.$$

2.4.1 Property P1

By recalling that $(A|H) \wedge_K (B|K) = AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K)$, from (2.18) it follows that

$$\begin{aligned}((A|H) \wedge_K (B|K))|_F(A|H) = (AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K))|_F(A|H) = \\ AHBK|(AHBK \vee AH\bar{B}K \vee \bar{H}\,\bar{B}K) = (B|K)|_F(A|H).\end{aligned} \tag{2.19}$$

Then, as $((A|H) \wedge_K (B|K))|_F(A|H) = (B|K)|_F(A|H)$, Property P1 is satisfied by the pair $(\wedge_K, |_F)$. This relation can also be obtained by observing that the truth values of $(B|K)|_F(A|H)$ and of $((A|H) \wedge_K (B|K))|_F(A|H)$ in Table 2.7 coincide.

C_h	$(A\|H) \wedge_K (B\|K)$	$(B\|K)\|_F(A\|H)$	$((A\|H) \wedge_K (B\|K))\|_F(A\|H)$
$AHBK$	True	True	True
$AH\bar{B}K \vee \bar{H}\,\bar{B}K$	False	False	False
$AH\bar{K} \vee \bar{H}BK \vee \bar{H}\,\bar{K}$	Void	Void	Void
$\bar{A}H$	False	Void	Void

TABLE 2.7: Truth table of $(A|H) \wedge_K (B|K)$, $(B|K)|_F(A|H)$, and $((A|H) \wedge_K (B|K))|_F(A|H)$.

2.4.2 Property P2

We observe that (see Table 2.7) the iterated conditional $(B|K)|_F(A|H)$ is true when the conjunction $(A|H) \wedge_K (B|K)$ is true; moreover, the conjunction $(A|H) \wedge_K (B|K)$ is false when the iterated conditional $(B|K)|_F(A|H)$ is false. Then, $(A|H) \wedge_K (B|K) \subseteq (B|K)|_F(A|H)$. Thus, it follows from (1.5) that $(A|H) \wedge_K (B|K) \leqslant (B|K)|_F(A|H)$, which means that P2 is satisfied by $(\wedge_K, |_F)$.

2.4.3 Property P3

To check the validity of property P3 for $(B|K)|_F(A|H)$ we study the set of coherent probability assessments on the family $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$.

Theorem 11. *Let A, B, H, K, be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = T_0^H(x, y)$, where*

$$T_0^H(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0 \\ \frac{xy}{x+y-xy}, & \text{if } x \neq 0 \text{ and } y \neq 0, \end{cases}$$

is the Hamacher t-norm with parameter $\lambda = 0$.

Proof. We recall that $\{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\} = \{A|H, AHBK|(AHBK \vee AH\bar{B}K \vee \bar{H}\,\bar{B}K), AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K)\}$. Then, $\mathcal{H}_3 = H \vee (AHBK \vee AH\bar{B}K \vee \bar{H}\ \bar{B}K) \vee (AHBK \vee \bar{A}H \vee \bar{B}K) = H \vee \bar{H}\,\bar{B}K$. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are (see Table 2.8)

$$C_1 = AHBK,\ C_2 = AH\bar{B}K,\ C_3 = AH\bar{K}, C_4 = \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} = \bar{A}H,$$
$$C_5 = \bar{H}\,\bar{B}K,\ C_0 = \bar{H}BK \vee \bar{H}\,\bar{K},$$

and

$$Q_1 = (1,1,1),\ Q_2 = (1,0,0),\ Q_3 = (1,y,z),$$
$$Q_4 = (0,y,0),\ Q_5 = (x,0,0),\ \mathcal{P} = Q_0 = (x,y,z).$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + x\lambda_5 = x, \\ \lambda_1 + y\lambda_3 + y\lambda_4 = y, \\ \lambda_1 + z\lambda_3 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \\ \lambda_i \geqslant 0,\ i = 1, \ldots, 5. \end{cases} \tag{2.20}$$

Lower bound We first prove that the assessment $(x, y, 0)$ is coherent for every $(x, y) \in [0, 1]^2$. We observe that $\mathcal{P} = (x, y, 0) = xQ_3 + (1 - x)Q_4$, so a solution of

	C_h	$A\|H$	$(B\|K)\|_F(A\|H)$	$(A\|H) \wedge_K (B\|K)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$AH\bar{B}K$	1	0	0	Q_2
C_3	$AH\bar{K}$	1	y	z	Q_3
C_4	$\bar{A}H$	0	y	0	Q_4
C_5	$\bar{H}\bar{B}K$	x	0	0	Q_5
C_0	$\bar{H}BK \vee \bar{H}\bar{K}$	x	y	z	Q_0

TABLE 2.8: Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$ and $\mathcal{P} = (x, y, z)$.

(2.20) is given by $\Lambda = (0, 0, x, 1-x, 0)$.
Then, by considering the function ϕ as defined in (1.7), it holds that

$$\begin{aligned}
\phi_1(\Lambda) &= \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 > 0, \\
\phi_2(\Lambda) &= \textstyle\sum_{h:C_h \subseteq (AHBK \vee AH\bar{B}K \vee \bar{H}\bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_5 = 0, \\
\phi_3(\Lambda) &= \textstyle\sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 = 1 - x.
\end{aligned}$$

Let $\mathcal{S}' = \{(0, 0, x, 1-x, 0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.20). We have that $M_1' = 1$, $M_2' = 0$, $M_3' = 1 - x$ (as defined in (1.8)). We distinguish two cases: (i) $x \neq 1$; (ii) $x = 1$.

(i) In this case, $M_1' > 0$, $M_2' = 0$, $M_3' > 0$ and hence $I_0' = \{2\}$. We observe that the sub-assessment $\mathcal{P}_0' = y$ on $\mathcal{F}_0' = \{(B|K)|_F(A|H)\}$ is coherent for every $y \in [0, 1]$. Then, by Theorem 3, the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent $\forall (x, y) \in [0, 1]^2$;

(ii) We have that $M_1' > 0$, $M_2' = M_3' = 0$ and hence $I_0' = \{2, 3\}$. We set $\mathcal{K} = \mathcal{F}_0' = \{(B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$ and $\mathcal{V} = \mathcal{P}_0' = (y, 0)$. By Theorem 3, as (2.20) is solvable and $I_0' = \{2, 3\}$, it is sufficient to check the coherence of the sub-assessment $\mathcal{V}$ on $\mathcal{K}$ in order to check the coherence of $(x, y, 0)$. We have that $\mathcal{H}_2 = (AHBK \vee AH\bar{B}K \vee \bar{H}\bar{B}K) \vee (AHBK \vee \bar{A}H \vee \bar{B}K) = AHBK \vee \bar{A}H \vee \bar{B}K$. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{V}$ on $\mathcal{K}$ are

$$C_1 = AHBK,\ C_2 = AH\bar{B}K \vee \bar{H}\bar{B}K,\ \ C_3 = \bar{A}H, C_0 = AH\bar{K} \vee \bar{H}\bar{K} \vee \bar{H}BK,$$

and

$$Q_1 = (1, 1),\ Q_2 = (0, 0),\ Q_3 = (y, 0),\ \mathcal{P}_0' = Q_0 = (y, 0).$$

We have that $\mathcal{H}_2 = C_1 \vee \cdots \vee C_3 = AHBK \vee \bar{A}H \vee \bar{B}K$. The system (Σ) in (1.6) associated with the pair $(\mathcal{K}, \mathcal{V})$ becomes

$$\begin{cases}
\lambda_1 + y\lambda_3 = y, \\
\lambda_1 = 0, \\
\lambda_1 + \lambda_2 + \lambda_3 = 1, \\
\lambda_i \geqslant 0,\ i = 1, \ldots, 3.
\end{cases} \tag{2.21}$$

We observe that $\mathcal{V} = (y, 0) = Q_3$ so a solution of (2.21 is $\Lambda = (0, 0, 1)$.
By considering the function ϕ as defined in (1.7), it holds that

$$\begin{aligned}
\phi_1(\Lambda) &= \textstyle\sum_{h:C_h \subseteq (AHBK \vee AH\bar{B}K \vee \bar{H}\bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 = 0, \\
\phi_2(\Lambda) &= \textstyle\sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 = 1.
\end{aligned}$$

Let $\mathcal{S}' = \{(0,0,1)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.20). We have that $M_1' = 0$, $M_2' = 1$, (as defined in (1.8)). Then, the set I_0' associated with $(\mathcal{K}, \mathcal{V})$ is $I_0' = 1$. We observe that the sub-assessment y on $\{(B|K)|_{dF}(A|H)\}$ is coherent for every $y \in [0,1]$. Then, by Theorem 3, the sub-assessment $(y,0)$ on $\mathcal{K}$ is coherent for every $y \in [0,1]$ and hence the assessment $(x,y,0)$ on $\mathcal{F}$ is coherent $\forall (x,y) \in [0,1]^2$. Thus, $z' = 0$ is the lower bound of $z = P((A|H) \wedge_K (B|K))$, for every $(x,y) \in [0,1]$ on $\{A|H, (B|K)|_F(A|H)\}$.

Upper bound To study the upper bound z'' of $z = P((A|H) \wedge_K (B|K))$ we distinguish the following cases: (a) $x = 0$; (b) $y = 0$; (c) $x \neq 0$ and $y \neq 0$.
(a). In this case system (2.20) associated to the pair $(\mathcal{F}, (0,y,z))$ becomes

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1 + y\lambda_3 + y\lambda_4 = y, \\ \lambda_1 + z\lambda_3 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \\ \lambda_i \geqslant 0,\ i = 1, \dots, 5. \end{cases} \iff \begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = 0, \\ y\lambda_4 = y \\ z = 0, \\ \lambda_4 + \lambda_5 = 1, \\ \lambda_i \geqslant 0,\ i = 4,5, \end{cases} \tag{2.22}$$

which is solvable only if $z = 0$. We have already verified that $\mathcal{P} = (x,y,0)$ is coherent on $\mathcal{F}$ for all $(x,y) \in [0,1]^2$, hence, in particular, $(0,y,0)$ is coherent. We observe that system (2.22) with $z > 0$ is not solvable, then the assessment $(0,y,z)$, with $z > 0$ is not coherent. Thus, $z'' = 0$ is the upper bound for z when $x = 0$ and $y \in [0,1]$.

(b). We recall that the pair $(\wedge_K, |_F)$ satisfies property P2, that is $z \leqslant y$. As $y = 0$, it follows that $z = 0$. We have already verified that $\mathcal{P} = (x,y,0)$ is coherent on $\mathcal{F}$ for all $(x,y) \in [0,1]^2$, hence, in particular, $(x,0,0)$ is coherent. We also observe that the assessment $(x,0,z)$, with $z > 0$ is not coherent, because the pair $(y = 0, z)$ violates property P2. Thus, $z'' = 0$ is the upper bound for z when $x \in [0,1]$ and $y = 0$.

(c) Fist of all, we verify that the assessment $\mathcal{P} = (x, y, \frac{xy}{x+y-xy})$ on $\mathcal{F}$ is coherent. Then, we verify that $z'' = \frac{xy}{x+y-xy}$ is the upper bound for z, by showing that (x,y,z), with $z > z''$, is incoherent for every $(x,y) \in [0,1]^2$.
We observe that

$$\mathcal{P} = (x, y, \frac{xy}{x+y-xy}) = \frac{xy}{x+y-xy} Q_1 + \frac{y(1-x)}{x+y-xy} Q_4 + \frac{x(1-y)}{x+y-xy} Q_5.$$

Then a solution of (2.20) is $\Lambda = (\frac{xy}{x+y-xy}, 0, 0, \frac{y(1-x)}{x+y-xy}, \frac{x(1-y)}{x+y-xy})$ and it holds that

$$\begin{aligned} \phi_1(\Lambda) &= \textstyle\sum_{h: C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \frac{y}{x+y-xy}, \\ \phi_2(\Lambda) &= \textstyle\sum_{h: C_h \subseteq (AHBK \vee AH\bar{B}K \vee \bar{H}\bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_5 = \frac{x}{x+y-xy}, \\ \phi_3(\Lambda) &= \textstyle\sum_{h: C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 = 1 > 0. \end{aligned}$$

Let $\mathcal{S}' = \{(\frac{xy}{x+y-xy}, 0, 0, \frac{y(1-x)}{x+y-xy}, \frac{x(1-y)}{x+y-xy})\}$ be a subset $\mathcal{S}$ of all solutions of (2.20). We have that $M_1' > 0$, $M_2' > 0$, $M_3' > 0$ (as defined in (1.8)). Then, by Theorem3 the assessment $(x, y, \frac{xy}{x+y-xy})$ is coherent for every $x \neq 0$ and $y \neq 0$.
Now we prove that in this case $z'' = \frac{xy}{x+y-xy}$ is the upper bound for z.
We distinguish two sub-cases: (i) $x = 1$, (ii) $x \neq 1$.
(i) We recall that the pair $(\wedge_K, |_F)$ satisfies property P2 and hence $z \leqslant y$. Then, when

$x = 1$, as $\frac{xy}{x+y-xy} = y$, we have that any assessment $z > \frac{xy}{x+y-xy} = y$ is not coherent. Therefore, $z'' = \frac{xy}{x+y-xy} = y$ is the upper bound for $z = P((A|H) \wedge_K (B|K))$.

(ii) We recall that in this case $x \in (0,1)$ and $y \neq 0$. We observe that the points Q_1, Q_4, Q_5 belong to the plane $\pi : yX + xY - (x+y-xy)Z = xy$, where X, Y, Z are the axes coordinates. We set $f(X,Y,Z) = yX + xY - (x+y-xy)Z - xy$, we choose $\mathcal{P} = (x,y,z)$ with $z > \frac{xy}{x+y-xy}$, i.e. $(x+y-xy)z - xy > 0$, and we observe that $f(\mathcal{P}) = xy - (x+y-xy)z$. For each $h = 1,\dots,5$, we consider the quantity $f(Q_h) - f(\mathcal{P})$. Then, we obtain

$$\begin{aligned}
&f(Q_1) - f(\mathcal{P}) = (x+y-xy)z - xy > 0;\\
&f(Q_2) - f(\mathcal{P}) = y(1-x) + (x+y-xy)z - xy > 0;\\
&f(Q_3) - f(\mathcal{P}) = y - xy = y(1-x) > 0;\\
&f(Q_4) - f(\mathcal{P}) = (x+y-xy)z - xy > 0;\\
&f(Q_5) - f(\mathcal{P}) = (x+y-xy)z - xy > 0.
\end{aligned} \tag{2.23}$$

We recall that, by setting the stakes $s_1 = y, s_2 = x, s_3 = -x-y+xy$, it holds that $g_h = f(Q_h) - f(\mathcal{P})$, $h = 1,\dots,5$, where g_h is the value of the random gain G associated with the constituent $C_h \subseteq \mathcal{H}_3$. From (2.23) we obtain that $g_h = f(Q_h) - f(\mathcal{P}) > 0$, $h = 1,\dots,5$. Thus, as there exist (s_1,s_2,s_3) such that $\min G_{\mathcal{H}} \max G_{\mathcal{H}} > 0$, it follows that the assessment (x,y,z) is not coherent when $z > \frac{xy}{x+y-xy}$.
We conclude that $z'' = \frac{xy}{x+y-xy}$ is the upper bound for z.
Therefore $z'' = T_0^H(x,y)$ is the upper bound for z, for every $(x,y) \in [0,1]^2$. □

Remark 18 (Property P3)**.** *From Theorem 11 it follows that any probability assessment (x,y,z) on $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$, with $(x,y) \in [0,1]^2$ and $z = xy$, is coherent because $xy \in [z', z'']$, where $z' = 0$ and $z'' = T_0^H(x,y)$. Then, any assessment which satisfies property P3 is coherent. Moreover, as $z = xy$ is not the unique coherent extension, the quantity $P((A|H) \wedge_K (B|K))$ could not coincide with the product $P((B|K)|_F(A|H))P(A|H)$. For example, if we choose the probability assessment $\mathcal{P} = (1,1,0)$, we observe that $\mathcal{P}$ is coherent on $\mathcal{F}$, because $0 \in [z', z''] = [0,1]$. However, we observe that the assessment $\mathcal{P} = (1,1,0)$ on $\{A, B|A, AB\}$ is not coherent, because $P(AB) = 0 \neq P(B|A)P(A) = 1$. Then, property P3 is not satisfied by the pair $(\wedge_K, |_F)$.*

2.4.4 Property P4

To check the validity of property P4 for the iterated conditioning $|_F$ we study the set of all coherent probability assessments on the family $\{A|H, B|K, (B|K)|_F(A|H)\}$.

Theorem 12. *Let A, B, H, K be any logically independent events. The probability assessment $\mathcal{P} = (x,y,z)$ on the family of conditional events $\mathcal{F} = \{A|H, B|K, (B|K)|_F(A|H)\}$ is coherent for every $(x,y,z) \in [0,1]^3$.*

Proof. The constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x,y,z)$ on $\mathcal{F}$ are (see also Table2.9)

$$\begin{aligned}
&C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,\\
&C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\,\bar{B}K, C_0 = \bar{H}\,\bar{K},
\end{aligned}$$

and

$$\begin{aligned}
&Q_1 = (1,1,1), Q_2 = (1,0,0), Q_3 = (1,y,z), Q_4 = (0,1,z), Q_5 = (0,0,z),\\
&Q_6 = (0,y,z), Q_7 = (x,1,z), Q_8 = (x,0,0), \mathcal{P} = Q_0 = (x,y,z).
\end{aligned}$$

	C_h	$A\|H$	$B\|K$	$(B\|K)\|_F(A\|H)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$AH\bar{B}K$	1	0	0	Q_2
C_3	$AH\bar{K}$	1	y	z	Q_3
C_4	$\bar{A}HBK$	0	1	z	Q_4
C_5	$\bar{A}H\bar{B}K$	0	0	z	Q_5
C_6	$\bar{A}H\bar{K}$	0	y	z	Q_6
C_7	$\bar{H}BK$	x	1	z	Q_7
C_8	$\bar{H}\bar{B}K$	x	0	0	Q_8
C_0	$\bar{H}\bar{K}$	x	y	z	Q_0

TABLE 2.9: Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, B|K, (B|K)|_F(A|H)\}$ and $\mathcal{P} = (x, y, z)$.

We observe that $C_1 \vee \cdots \vee C_8 = H \vee K = \mathcal{H}_3$. The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + x\lambda_7 + x\lambda_8 = x, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y, \\ \lambda_1 + z\lambda_3 + z\lambda_4 + z\lambda_5 + z\lambda_6 + z\lambda_7 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1, \\ \lambda_i \geqslant 0, \ i = 1, \dots, 8. \end{cases} \tag{2.24}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_3, Q_6; indeed $(x, y, z) = xQ_3 + (1-x)Q_6 = x(1, y, z) + (1-x)(0, y, z)$. Then, the vector $\Lambda = (0, 0, x, 0, 0, 1 - x, 0, 0)$ is a solution of (2.24), with

$$\begin{aligned} \phi_1(\Lambda) &= \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \\ \phi_2(\Lambda) &= \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 0, \\ \phi_3(\Lambda) &= \textstyle\sum_{h:C_h \subseteq [AHBK \vee AH\bar{B}K \vee \bar{H}\bar{B}K]} \lambda_h = \lambda_1 + \lambda_2 + \lambda_8 = 0. \end{aligned}$$

Let $\mathcal{S}' = \{(0, 0, x, 0, 0, 1 - x, 0, 0)\}$ be a subset $\mathcal{S}$ of all solutions of (2.24). We have that $M'_1 > 0$, $M'_2 = 0$, $M'_3 = 0$ (as defined in (1.8)). Then, $I'_0 = \{2, 3\}$. We check the coherence of the sub-assessment $\mathcal{P}'_0 = (y, z)$ on $\mathcal{F}'_0 = \{B|K, (B|K)|_F(A|H)\}$. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{P}'_0$ on $\mathcal{F}'_0$ are

$$\begin{aligned} &C_1 = AHBK, C_2 = AH\bar{B}K \vee \bar{H}\bar{B}K, \\ &C_3 = \bar{A}HBK \vee \bar{H}BK, C_4 = \bar{A}H\bar{B}K, C_0 = \bar{K}, \end{aligned}$$

and

$$Q_1 = (1, 1),\ Q_2 = (0, 0),\ Q_3 = (1, z),\ Q_4 = (0, z), \mathcal{P} = Q_0 = (y, z).$$

We observe that $C_1 \vee \cdots \vee C_4 = K = \mathcal{H}_2$. The system (Σ) in (1.6) associated with the pair $(\mathcal{F}_0, \mathcal{P}_0)$ becomes

$$\begin{cases} \lambda_1 + \lambda_3 = y, \\ \lambda_1 + z\lambda_3 + z\lambda_4 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\ \lambda_i \geqslant 0, \ i = 1, \dots, 4. \end{cases} \tag{2.25}$$

We observe that $(y, z) = yQ_3 + (1 - y)Q_4$ so a solution of (2.25 is $\Lambda_0 = (0, 0, y, 1 - y)$. By considering the function ϕ' it holds that

$$\begin{aligned}\phi_1(\Lambda) &= \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,\\ \phi_2(\Lambda) &= \textstyle\sum_{h:C_h \subseteq [AHBK \vee AH\bar{B}K \vee \bar{H}\bar{B}K]} \lambda_h = \lambda_1 + \lambda_2 = 0.\end{aligned}$$

Let $\mathcal{S}' = \{(0, 0, y, 1 - y)\}$ be a subset $\mathcal{S}$ of all solutions of (2.25). We have that $M_1' > 0$, $M_2' = 0$ (as defined in (1.8)). Then, $I_0' = \{2\}$. We observe that the sub-assessment z on $\{(B|K)|_F(A|H)\}$ is coherent for every $z \in [0, 1]$. Then, by Theorem 3, the assessment (x, y, z) on $\mathcal{F}$ is coherent $\forall (x, y, z) \in [0, 1]^3$. □

Remark 19 (Property P4)**.** *We observe that the probability propagation rule valid for unconditional events (property P4) is no longer valid for Farrell's iterated conditional. Indeed, from Theorem 12, any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, B|K, (B|K)|_F(A|H)\}$, with $(x, y, z) \in [0, 1]^3$ is coherent. For instance, the assessment $(1, 1, 0)$ is coherent on $\mathcal{F}$ but it is not coherent on $\{A, B, B|A\}$.*

We briefly sum up here the results obtained in this section (see also Table 6.1) for the iterated conditioning $|_F$. We verified that the $|_F$ satisfies both properties P1 and P2. Indeed, $(B|K)|_F(A|H) = ((A|H) \wedge_K (B|K))|_F(A|H)$ and $((A|H) \wedge_K (B|K)) \leqslant (B|K)|_F(A|H)$. However, the iterated conditioning $|_F$ does not satisfy property P3, because $P(A|H)P((B|K)|_F(A|H)) \neq P((A|H) \wedge_K (B|K))$ for some probability P. Moreover, we showed that every assessment $(x, y, z) \in [0, 1]^3$ is coherent on $\{A|H, B|K, (B|K)|_F(A|H)\}$ and hence property P4 is not satisfied. Finally, we observed that Import-Export Principle is satisfied by $|_F$.

2.5 Iterated conditionals and compound prevision theorem

We observe that none of the iterated conditioning operations, $|_C$, $|_{dF}$, and $|_F$, studied in the previous sections satisfies the compound probability theorem P3. In this section, we consider iterated conditionals which, among other things, satisfy property P3. We first recall a structure used for defining, in the framework of conditional random quantities, the iterated conditioning $|_{gs}$. Then, we introduce four notions of iterated conditioning which are based on the same structure and on the four conjunctions of trivalent logics recalled in Section 1. For all these new objects we check the validity of the basic properties.

We recall that in [91] (see also [53]), by using the structure

$$\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\overline{\bigcirc}, \tag{2.26}$$

with $\square = B|K$, $\bigcirc = A|H \neq \varnothing$, and $\square \wedge \bigcirc = (B|K) \wedge_{gs} (A|H)$, the iterated conditional $(B|K)|_{gs}(A|H)$ has been defined as the following conditional random quantity

$$(B|K)|_{gs}(A|H) = (A|H) \wedge_{gs} (B|K) + \mu_{gs}\,(\bar{A}|H), \tag{2.27}$$

where $\mu_{gs} = \mathbb{P}[(B|K)|_{gs}(A|H)]$. We underline that when $\square = A, \bigcirc = H$ formula (2.26) reduces to formula (1.4). We also recall that ([53])

$$\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = \mathbb{P}[(B|K)|_{gs}(A|H)]P(A|H). \tag{2.28}$$

Moreover, in [57, Definition 7], by exploiting the structure (2.26), the iterated conditioning $|_{gs}$ has been extended to the case of conjoined conditionals. More precisely, denoting by $\mathcal{C}(\mathcal{F}_1)$ (resp., $\mathcal{C}(\mathcal{F}_2)$) the $\wedge_{gs}$-conjunction of the conditional events in a finite family of conditional events $\mathcal{F}_1$ (resp., $\mathcal{F}_2$), the iterated conditional $\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1)$ has been defined as

$$\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1) = \mathcal{C}(\mathcal{F}_2) \wedge_{gs} \mathcal{C}(\mathcal{F}_1) + \mu(1 - \mathcal{C}(\mathcal{F}_1)) = \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) + \mu(1 - \mathcal{C}(\mathcal{F}_1)), \tag{2.29}$$

where $\mu = \mathbb{P}[\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1)]$. In addition, it holds [57, Equation (8)] that

$$\mathbb{P}[\mathcal{C}(\mathcal{F}_2) \wedge_{gs} \mathcal{C}(\mathcal{F}_1)] = \mathbb{P}[\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1)]\mathbb{P}[\mathcal{C}(\mathcal{F}_1)]. \tag{2.30}$$

Formulas (2.28) and (2.30) generalise the compound probability theorem recalled in (1.13).

We now introduce the different notions of iterated conditioning, beyond $|_{gs}$, obtained with the structure (2.26), by using the trivalent logic conjunctions $\wedge_K$, $\wedge_L$, $\wedge_B$, and $\wedge_S$, recalled in Section 1. Among other things, we will show that a formula like (2.28) still holds for each of these new objects.

Definition 11. *Given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$, for each $i \in \{K, L, B, S, gs\}$, we define the iterated conditional $(B|K)|_i(A|H)$ as*

$$(B|K)|_i(A|H) = (A|H) \wedge_i (B|K) + \mu_i\,(\bar{A}|H), \tag{2.31}$$

where $\mu_i = \mathbb{P}[(B|K)|_i(A|H)]$.

Remark 20. *In a betting framework, by setting $\mathbb{P}[(B|K)|_i(A|H)] = \mu_i$, $i \in \{K, L, B, S, gs\}$, then for every real number s you agree to pay an amount $s\mu_i$ in order to receive the random quantity $s \cdot (B|K)|_i(A|H)$. The associated random gain is*

$$G = s[(B|K)|_i(A|H) - \mu_i] = s[(A|H) \wedge_i (B|K) + \mu_i\,(\bar{A}|H) - \mu_i]. \tag{2.32}$$

We observe that, for each $i \in \{K, L, S, gs\}$, if $\bar{A}|H$ is true, as $(A|H) \wedge_i (B|K) = 0$ (see Table 1.1), it follows that $(B|K)|_i(A|H) = \mu_i$, for every μ_i, and hence $G = s(\mu_i - \mu_i) = 0$. In other words, for $i \in \{K, L, S, gs\}$ the bet on $(B|K)|_i(A|H)$ is called off when the antecedent $A|H$ is false. Concerning $i = B$, if $\bar{A}HK$ is true, as $(A|H) \wedge_B (B|K) = 0$ (see Table 1.1), it follows that $(B|K)|_B(A|H) = \mu_B$ and hence $G = 0$. Then, the bet on $(B|K)|_B(A|H)$ is called off when the antecedent $A|H$ is false and $B|K$ is not void. As we will see in Remark 22, for each iterated conditional $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$, there are other cases where the bet is called off, i.e. $(B|K)|_i(A|H) = \mu_i$.

Now, we check the validity of properties P1–P4, introduced in Section 4.1, for each iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$.

2.5.1 Property P1

In the next result we show that each iterated conditioning $|_i$, $i \in \{K, L, B, S\}$, satisfies property P1.

Theorem 13. *Given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$, it holds that*

$$((A|H) \wedge_i (B|K))|_i(A|H) = (B|K)|_i(A|H), \quad i \in \{K, L, B, S\}. \tag{2.33}$$

Proof. Let $i \in \{K, L, B, S\}$ be given. We set $\mathbb{P}[(B|K)|_i(A|H)] = \mu_i$ and $\mathbb{P}[((A|H) \wedge (B|K))|_i(A|H)] = \nu_i$. By Definition 11, as $(A|H) \wedge_i (A|H) \wedge_i (B|K) = (A|H) \wedge_i (B|K)$, it holds that

$$((A|H) \wedge_i (B|K))|_i(A|H) = (A|H) \wedge_i (B|K) + \nu_i(\bar{A}|H). \tag{2.34}$$

Then, as $(B|K)|_i(A|H) = (A|H) \wedge_i (B|K) + \mu_i(\bar{A}|H)$, in order to prove (2.33) it is enough to verify that $\nu_i = \mu_i$. We observe that $((A|H) \wedge_i (B|K))|_i(A|H) - (B|K)|_i(A|H) = (\nu_i - \mu_i)(\bar{A}|H)$, where $\nu_i - \mu_i = \mathbb{P}[((A|H) \wedge_i (B|K))|_i(A|H) - (B|K)|_i(A|H)]$. By setting $P(A|H) = x$, it holds that

$$(\nu_i - \mu_i)(\bar{A}|H) = \begin{cases} 0, \text{ if } A|H = 1, \\ \nu_i - \mu_i, \text{ if } A|H = 0, \\ (\nu_i - \mu_i)(1-x), \text{ if } A|H = x,\ 0 < x < 1. \end{cases}$$

Notice that, in the betting scheme, $\nu_i - \mu_i$ is the amount to be paid in order to receive the random amount $(\nu_i - \mu_i)(\bar{A}|H)$. Then, by coherence, $\nu_i - \mu_i$ must be a linear convex combination of the possible values of $(\nu_i - \mu_i)(\bar{A}|H)$, by discarding the cases where the bet is called off, that is the cases where you receive back the paid amount $\nu_i - \mu_i$, whatever $\nu_i - \mu_i$ is. In other words, coherence requires that $\nu_i - \mu_i$ must belong to the convex hull of the set $\{0, (\nu_i - \mu_i)(1-x)\}$, that is $\nu_i - \mu_i = \alpha \cdot 0 + (1-\alpha)(\nu_i - \mu_i)(1-x)$, for some $\alpha \in [0,1]$. Then, as $0 < x < 1$, we observe that the previous equality holds if and only if $\nu_i - \mu_i = 0$, that is $\nu_i = \mu_i$. Therefore, equality (2.33) holds. □

We recall that $|_{gs}$ satisfies property P1. Indeed, for the generalised iterated conditional (2.29) it holds that ([57, Theorem 5])

$$\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1) = \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1). \tag{2.35}$$

Then, by applying (2.35) with $\mathcal{F}_1 = \{A|H\}$, $\mathcal{F}_2 = \{B|K\}$, it follows that

$$(B|K)|_{gs}(A|H) = ((A|H) \wedge_{gs} (B|K))|_{gs}(A|H).$$

Thus, property P1 is satisfied by each iterated conditioning $|_i \in \{|_K, |_L, |_B, |_S, |_{gs}\}$.

2.5.2 Property P2

We show that each iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$, satisfies property P2. We observe that, coherence requires $\mu_i \geqslant 0$, $i \in \{K, L, B, S, gs\}$ (see Remark 21 in Section 2.5.3). Then, for each $i \in \{K, L, B, S, gs\}$, as $(A|H) \wedge_i (B|K) \leqslant (A|H) \wedge_i (B|K) + \mu_i(\bar{A}|H)$, from (2.31) it follows that

$$(A|H) \wedge_i (B|K) \leqslant (B|K)|_i(A|H), \tag{2.36}$$

and hence $\mathbb{P}[(A|H) \wedge_i (B|K)] \leqslant \mathbb{P}[(B|K)|_i(A|H)]$.

2.5.3 Property P3

We already recalled in equation (2.28) that the iterated conditioning $|_{gs}$ satisfies P3. Then, by setting $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$, $\mu_{gs} = \mathbb{P}[(B|K)|_{gs}(A|H)]$, and $x = P(A|H)$ it holds that $z_{gs} = x\mu_{gs}$. By exploiting the structure (2.26), we show below that P3 is also valid for $|_K, |_L, |_B$, and $|_S$. Indeed, by the linearity property of a coherent prevision, from (2.31) we obtain that

$$\begin{aligned} \mu_i &= \mathbb{P}[(B|K)|_i(A|H)] = P((B|K) \wedge_i (A|H)) + \mu_i P(\bar{A}|H) = \\ &= P((B|K) \wedge_i (A|H)) + \mu_i\, P(\bar{A}|H) = z_i + \mu_i(1-x)\,, \end{aligned} \tag{2.37}$$

where $x = P(A|H)$ and $z_i = P((A|H) \wedge_i (B|K)), i \in \{K, L, B, S\}$. As $\mu_i = z_i + \mu_i(1-x)$, it follows that

$$z_i = \mu_i x, \;\; i \in \{K, L, B, S\}.$$

Therefore, coherence requires that

$$\mathbb{P}[(A|H) \wedge_i (B|K)] = \mathbb{P}[(B|K)|_i(A|H)]P(A|H), \; i \in \{K, L, B, S, gs\}, \tag{2.38}$$

which states that the compound prevision formula for iterated conditionals (property P3) is valid for each iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$.

Remark 21. *Notice that, as $P(A|H) \geqslant 0$ and $\mathbb{P}[(A|H) \wedge_i (B|K)] \geqslant 0$ for $i \in \{K, L, B, S, gs\}$, from (2.38) it follows that $\mathbb{P}[(B|K)|_i(A|H)] \geqslant 0$, $i \in \{K, L, B, S, gs\}$.*

We set $P(A|H) = x$, $P(B|K) = y$, $P((A|H) \wedge_i (B|K)) = z_i$, $i \in \{K, L, B, S\}$, $\mathbb{P}[(A|H) \wedge_i (B|K)] = z_{gs}$, and $\mathbb{P}[(B|K)|_i(A|H)] = \mu_i$, $i \in \{K, L, B, S, gs\}$. Then, based on Definition 11 and on the compound prevision theorem, for each $|_i$, $i \in \{K, L, B, S, gs\}$, we obtain the random quantities $(B|K)|_i(A|H)$ illustrated below.

- $(|_K)$. We recall that $(A|H) \wedge_K (B|K) = AHBK|(HK \vee \overline{A}H \vee \overline{B}K)$, then we have

$$(B|K)|_K(A|H) = AHBK|(HK \vee \overline{A}H \vee \overline{B}K) + \mu_K(\bar{A}|H) =$$
$$= \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ z_K, & \text{if } AH\bar{K} \text{ is true,} \\ z_K + \mu_K(1-x), & \text{if } \bar{H}BK \vee \bar{H}\,\bar{K} \text{ is true,} \\ \mu_K(1-x), & \text{if } \bar{H}\,\bar{B}K \text{ is true,} \\ \mu_K, & \text{if } \bar{A}H \text{ is true.} \end{cases} \tag{2.39}$$

From (2.38), coherence requires that $z_K = x\mu_K$ and $z_K + \mu_K(1-x) = \mu_K$. Then, we obtain

$$(B|K)|_K(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ x\mu_K, & \text{if } AH\bar{K} \text{ is true,} \\ \mu_K(1-x), & \text{if } \bar{H}\,\bar{B}K \text{ is true,} \\ \mu_K, & \text{if } \bar{A}H \vee \bar{H}BK \vee \bar{H}\,\bar{K} \text{ is true.} \end{cases} \tag{2.40}$$

- $(|_L)$. We recall that $(A|H) \wedge_L (B|K) = AHBK|(AHBK \vee \overline{A}\,H \vee \overline{B}K \vee \overline{H}\,\overline{K})$, then we have

$$(B|K)|_L(A|H) = AHBK|(AHBK \vee \overline{A}\,H \vee \overline{B}K \vee \overline{H}\,\overline{K}) + \mu_L(\bar{A}|H) =$$
$$= \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ z_L, & \text{if } AH\bar{K} \text{ is true,} \\ z_L + \mu_L(1-x), & \text{if } \bar{H}BK \text{ is true,} \\ \mu_L(1-x), & \text{if } \bar{H}\,\bar{B}K \vee \bar{H}\,\bar{K} \text{ is true,} \\ \mu_L, & \text{if } \bar{A}H \text{ is true.} \end{cases} \tag{2.41}$$

From (2.38), coherence requires that $z_L = x\mu_L$ and $z_L + \mu_L(1-x) = \mu_L$. Then, we obtain

$$(B|K)|_L(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \text{ is true}, \\ x\mu_L, & \text{if } AH\bar{K} \text{ is true}, \\ \mu_L(1-x), & \text{if } \bar{H}\,\bar{B}K \vee \bar{H}\,\bar{K} \text{ is true}, \\ \mu_L, & \text{if } \bar{A}H \vee \bar{H}BK \text{ is true}. \end{cases} \tag{2.42}$$

- $(|_B)$. We recall that $(A|H) \wedge_B (B|K) = AHBK|BK$, then we have

$$(B|K)|_B(A|H) = AHBK|HK + \mu_B(\bar{A}|H) =$$
$$= \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \text{ is true}, \\ z_B, & \text{if } AH\bar{K} \text{ is true}, \\ z_B + \mu_B, & \text{if } \bar{A}H\bar{K} \text{ is true}, \\ z_B + \mu_B(1-x), & \text{if } \bar{H} \text{ is true}, \\ \mu_B, & \text{if } \bar{A}HBK \vee \bar{A}H\bar{B}K \text{ is true}. \end{cases} \tag{2.43}$$

From (2.38), coherence requires that $z_B = x\mu_B$ and $z_B + \mu_B(1-x) = \mu_B$. Then, we obtain

$$(B|K)|_B(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \text{ is true}, \\ x\mu_B, & \text{if } AH\bar{K} \text{ is true}, \\ \mu_B(1+x), & \text{if } \bar{A}H\bar{K} \text{ is true}, \\ \mu_B, & \text{if } \bar{H} \vee \bar{A}HBK \vee \bar{A}H\bar{B}K \text{ is true}. \end{cases} \tag{2.44}$$

- $(|_S)$. We recall that $(A|H) \wedge_S (B|K) = ((AH \vee \overline{H}) \wedge (BK \vee \overline{K}))|(H \vee K)$, then we have

$$(B|K)|_S(A|H) = ((AH \vee \overline{H}) \wedge (BK \vee \overline{K}))|(H \vee K) + \mu_S(\bar{A}|H) =$$
$$= \begin{cases} 1, & \text{if } AHB \vee AH\bar{K} \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \text{ is true}, \\ 1 + \mu_S(1-x), & \text{if } \bar{H}BK \text{ is true}, \\ \mu_S(1-x), & \text{if } \bar{H}\,\bar{B}K \text{ is true}, \\ z_S + \mu_S(1-x), & \text{if } \bar{H}\,\bar{K} \text{ is true}, \\ \mu_S & \text{if } \bar{A}H \text{ is true}. \end{cases} \tag{2.45}$$

From (2.38), coherence requires that $z_S = x\mu_S$ and $z_S + \mu_S(1-x) = \mu_S$. Then, we obtain

$$(B|K)|_S(A|H) = \begin{cases} 1, & \text{if } AHB \vee AH\bar{K} \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \text{ is true}, \\ 1 + \mu_S(1-x), & \text{if } \bar{H}BK \text{ is true}, \\ \mu_S(1-x), & \text{if } \bar{H}\,\bar{B}K \text{ is true}, \\ \mu_S & \text{if } \bar{A}H \vee \bar{H}\,\bar{K} \text{ is true}. \end{cases} \tag{2.46}$$

- ($|_{gs}$). We recall that $(A|H) \wedge_{gs} (B|K) = (AHBK + x\bar{H}BK + y\bar{A}HK)|(H \vee K)$, then we have ([53])

$$
(B|K)|_{gs}(A|H) = (AHBK + x\bar{H}BK + yAH\bar{K})|(H \vee K) + \mu_{gs}(\bar{A}|H) =
$$
$$
= \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ x + \mu_{gs}(1-x), & \text{if } \bar{H}BK \text{ is true,} \\ \mu_{gs}(1-x), & \text{if } \bar{H}\,\bar{B}K \text{ is true,} \\ z_{gs} + \mu_{gs}(1-x), & \text{if } \bar{H}\,\bar{K} \text{ is true,} \\ \mu_{gs}, & \text{if } \bar{A}H \text{ is true.} \end{cases} \tag{2.47}
$$

From (2.38), coherence requires that $z_{gs} = x\mu_{gs}$ and $z_{gs} + \mu_{gs}(1-x) = \mu_{gs}$. Then, we recall that

$$
(B|K)|_{gs}(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ x + \mu_{gs}(1-x), & \text{if } \bar{H}BK \text{ is true,} \\ \mu_{gs}(1-x), & \text{if } \bar{H}\,\bar{B}K \text{ is true,} \\ \mu_{gs}, & \text{if } \bar{A}H \vee \bar{H}\,\bar{K} \text{ is true.} \end{cases} \tag{2.48}
$$

In Table 2.10 we summarize the possible values of the iterated conditionals $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$.

	$(B\|K)\|_K(A\|H)$	$(B\|K)\|_L(A\|H)$	$(B\|K)\|_B(A\|H)$	$(B\|K)\|_S(A\|H)$	$(B\|K)\|_{gs}(A\|H)$
$AHBK$	1	1	1	1	1
$AH\bar{B}K$	0	0	0	0	0
$AH\bar{K}$	$x\mu_K$	$x\mu_L$	$x\mu_B$	1	y
$\bar{A}HBK$	μ_K	μ_L	μ_B	μ_S	μ_{gs}
$\bar{A}H\bar{B}K$	μ_K	μ_L	μ_B	μ_S	μ_{gs}
$\bar{A}H\bar{K}$	μ_K	μ_L	$\mu_B(1+x)$	μ_S	μ_{gs}
$\bar{H}BK$	μ_K	μ_L	μ_B	1+ $\mu_S(1-x)$	$x + \mu_{gs}(1-x)$
$\bar{H}\,\bar{B}K$	$\mu_K(1-x)$	$\mu_L(1-x)$	μ_B	$\mu_S(1-x)$	$\mu_{gs}(1-x)$
$\bar{H}\,\bar{K}$	μ_K	$\mu_L(1-x)$	μ_B	μ_S	μ_{gs}

TABLE 2.10: Numerical values of $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$. We denotes $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[(B|K)|_i(A|H)] = \mu_i$, $i \in \{K, L, B, S, gs\}$.

Remark 22. *In addition to the cases considered in Remark 20, based on property P3 (see Table 2.10), for each $i \in \{K, L, B, S, gs\}$, there are other situations where $(B|K)|_i(A|H) = \mu_i$, for every μ_i, and hence the associated bet is called off. For example, the iterated conditional $(B|K)|_K(A|H)$ coincides with μ_K not only when $\bar{A}|H$ is true, but also when $\bar{H}BK$, or $\bar{H}\,\bar{K}$, is true. Moreover, a bet on these iterated conditionals could also be called off for values of x as well. For instance, $(B|K)|_K(A|H)$ reduces to μ_K, when $x = 1$ and $AH\bar{K}$ is true, or when $x = 0$ and $\bar{H}\,\bar{B}K$ is true.*

2.5.4 Property P4

In this section, for each $i \in \{K, L, B, S\}$, we find the set of all coherent assessments on the family $\{A|H, B|K, (A|H) \wedge_i (B|K), (B|K)|_i(A|H)\}$. Moreover, for each $i \in \{K, L, B, S\}$, we determine the interval of coherent extensions on $(B|K)|_i(A|H)$ and we check the validity of property P4. Then, we recall that property P4 is satisfied by $|_{gs}$.

The iterated conditioning $|_K$

Theorem 14. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x, y, z, μ) on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where $\Pi' = \{(x, y, z, \mu) : x \in (0, 1], y \in [0, 1], z \in [z', z''], \mu = \frac{z}{x}\}$ with $z' = 0$, $z'' = min\{x, y\}$, and $\Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0, 1]^2\}$.*

Proof. It is well-known that the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0, 1]^2$. By Table (1.2), the assessment $z = P((A|H) \wedge_K (B|K))$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$ where $z' = 0$ and $z'' = min\{x, y\}$. Assuming $x > 0$, from (2.38), it holds that $\mu = \frac{z}{x}$. Then, every $(x, y, z, \mu) \in \Pi'$ is coherent, i.e., $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x, y, z, \mu) \notin \Pi'$, then the assessment (x, y, z, μ) is not coherent, i.e. $(x, y, z, \mu) \notin \Pi$.
Let us consider now the case $x = 0$, so that $z' = 0$ and $z'' = 0$. We show that the assessment $(0, y, 0, \mu)$ is coherent if and only if $(y, \mu) \in [0, 1]^2$, that is $\Pi'' \subseteq \Pi$.

As $x = 0$, from (2.40), it holds that

$$(B|K)|_K(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \vee AH\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H}BK \vee \bar{H}\,\bar{B}K \vee \bar{H}\,\bar{K} \text{ is true,} \end{cases} \tag{2.49}$$

that is $(B|K)|_K(A|H) = AHBK + \mu(\bar{A} \vee \bar{H})$. Moreover we have that $AHBK + \mu(\bar{A} \vee \bar{H})$ and the conditional event $BK|AH = AHBK + \eta(\bar{A} \vee \bar{H})$, where $\eta = P(BK|AH)$, coincide when AH is true. Then, from Theorem 1, it follows that $\mu = \eta$ and hence $(B|K)|_K(A|H)$ and $BK|AH$ should also coincide when AH is false. Thus, $(B|K)|_K(A|H)$ and $BK|AH$ coincide in all cases. Therefore $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\} = \{A|H, B|K, AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K), BK|AH\}$ and $\mathcal{H}_4 = H \vee K \vee (AHBK \vee \bar{A}H \vee \bar{B}K) \vee AH = H \vee K$. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P} = (0, y, 0, \mu)$ are the following:

$$C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,$$
$$C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\,\bar{B}K, C_0 = \bar{H}\,\bar{K},$$

and

$$Q_1 = (1, 1, 1, 1), Q_2 = (1, 0, 0, 0), Q_3 = (1, y, 0, 0), Q_4 = (0, 1, 0, \mu), Q_5 = (0, 0, 0, \mu),$$
$$Q_6 = (0, y, 0, \mu), Q_7 = (0, 1, 0, \mu), Q_8 = (0, 0, 0, \mu), Q_0 = (0, y, 0, \mu).$$

The constituents contained in $\mathcal{H}_4 = H \vee K$ are $C_1, \ldots, C_8$. The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y, \\ \lambda_1 = 0, \\ \lambda_1 + \mu\lambda_4 + \mu\lambda_5 + \mu\lambda_6 + \mu\lambda_7 + \mu\lambda_8 = \mu \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1, \\ \lambda_i \geqslant 0, \; i = 1, \dots, 8. \end{cases} \tag{2.50}$$

We observe that, as $y \in [0,1]$, $\mathcal{P}$ belongs to the segment with end points Q_4, Q_5 because $\mathcal{P} = yQ_4 + (1-y)Q_5$. Then, $\Lambda = (0,0,0,y,1-y,0,0,0)$ is a solution of (2.50) with

$$\begin{aligned} &\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 > 0, \\ &\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 1 > 0, \\ &\phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 = 1 > 0, \\ &\phi_4(\Lambda) = \textstyle\sum_{h:C_h \subseteq AH} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 = 0. \end{aligned}$$

Let $\mathcal{S}' = \{(0,0,0,y,1-y,0,0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.50). We have that $M_1' = M_2' = M_3' = 1$ and $M_4' = 0$ (as defined in (1.8)). It follows that $I_0' = \{4\}$. As the sub-assessment $\mathcal{P}_0' = \mu$ on $\mathcal{F}_0' = \{BK|AH\}$ is coherent $\forall \mu \in [0,1]$, by Theorem 2, it follows that the assessment $(0,y,0,\mu)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is coherent for every $(y,\mu) \in [0,1]^2$, that is $(0,y,0,\mu) \in \Pi''$. Thus $\Pi'' \subseteq \Pi$. Of course, if $(0,y,z,\mu) \notin \Pi''$ the assessment $(0,y,z,\mu)$ is not coherent and hence $(0,y,z,\mu) \notin \Pi$. Therefore $\Pi = \Pi' \cup \Pi''$. □

Based on Theorem 14, we obtain

Theorem 15. *Let A, B, H, K be any logically independent events. Given any assessment $(x,y) \in [0,1]^2$ on $\{A|H, B|K\}$, for the iterated conditional $(B|K)|_K(A|H)$ the extension $\mu_K = \mathbb{P}((B|K)|_K(A|H))$ is coherent if and only if $\mu_K \in [\mu_K', \mu_K'']$, where*

$$\mu_K' = 0, \quad \mu_K'' = \begin{cases} \min\left\{1, \frac{y}{x}\right\}, & \text{if } x > 0; \\ 1, & \text{if } x = 0. \end{cases} \tag{2.51}$$

Proof. Of course, the assessment (x,y) on $\{A|H, B|K\}$ is coherent. Assume that $x > 0$. We simply write μ instead of μ_K. From Theorem 14, it follows that the set of all coherent assessments (x,y,z,μ) on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is $\Pi' = \{(x,y,z,\mu) : 0 < x \leqslant 1, 0 \leqslant y \leqslant 1, z' \leqslant z \leqslant z'', \mu = \frac{z}{x}\}$, where $z' = 0$ and $z'' = \min\{x,y\}$ (1.2). Then, μ is a coherent extension of (x,y) if and only if $\mu \in [\mu', \mu'']$, where $\mu' = 0$ and $\mu'' = \frac{z''}{x} = \min\left\{1, \frac{y}{x}\right\}$. Assume that $x = 0$. From Theorem 14, it follows that the set of all coherent assessments (x,y,z,μ) on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is $\Pi'' = \{(0,y,0,\mu) : (y,\mu) \in [0,1]^2\}$. Then, μ is a coherent extension to $(B|K)|_K(A|H)$ of $(0,y)$ on $\{A|H, B|K\}$ if and only if $\mu \in [\mu', \mu'']$, where $\mu' = 0$ and $\mu'' = 1$. □

Remark 23. *We notice that the lower and upper bounds μ_K' and μ_K'' for $(B|K)|_K(A|H)$, given in (2.51), do not coincide with the lower and upper bound μ' and μ'' for $B|A$, given in (2.1). In particular, the extension $\mu_K = 0$ to $(B|K)|_K(A|H)$ of $(1,1)$ on $\{A|H, B|K\}$ is coherent because, as $\mu_K' = 0$, the assessment $(1,1,0)$ on $\{A|H, B|K, (B|K)|_K(A|H)\}$ is coherent. However, $\mu = 0$ is not a coherent extension to $B|A$ of $(1,1)$ on $\{A, B\}$, because, as $\mu' = \frac{\max\{1+1-1,0\}}{1} = 1$, the assessment $(1,1,0)$ on $\{A, B, B|A\}$ is not coherent. Therefore, property P4 is not satisfied by $|_K$.*

In the following example we show that the invalidity of property P4 leads to some counterintuitive aspects.

Example 4. *In a random experiment we are supposed to pick a ball from one of two bags, U and V, both containing white balls. We do not know which of the two bags is selected. We set H ="the ball is picked from the bag U" (and hence $\bar{H}$ ="the ball is picked from the bag V") and A ="the drawn ball is white". Of course $P(A|H) = P(A|\bar{H}) = 1$. Based on (2.39) it holds that*

$$(A|\bar{H})|_K(A|H) = (A|\bar{H}) \wedge_K (A|H) + \mu_K(\bar{A}|H) = AH\bar{H}|(H\bar{H} \vee \bar{A}H \vee \bar{A}\,\bar{H}) + \mu_K(\bar{A}|H) = \\ = \emptyset|\bar{A} + \mu_K(\bar{A}|H) = 0 + \mu_K(\bar{A}|H) = 0,$$

because, by coherence, $\mu_K = 0$. Then, $\mathbb{P}[(A|\bar{H})|_K(A|H)] = 0$ is the unique coherent extension of $P(A|H) = P(A|\bar{H}) = 1$. Thus, when using $|_K$, the iterated conditional if the drawn ball is white if picked from U, then it is white if picked from V *has a prevision of 0, even if both conditionals,* the drawn ball is white if picked from U *and* the drawn ball is white if picked from V, *have probability 1.*

The iterated conditioning $|_L$ We obtain results similar to the case $|_K$. Indeed we have

Theorem 16. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessment (x, y, z, μ) on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_L (B|K), (B|K)|_L(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where $\Pi' = \{(x, y, z, \mu) : x \in (0,1], y \in [0,1], z \in [z', z''], \mu = \frac{z}{x}\}$ with $z' = 0$, $z'' = min\{x, y\}$, and $\Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0,1]^2\}$.*

Proof. It is well-known that the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0,1]^2$. By Table (1.2), the assessment $z = P((A|H) \wedge_L (B|K))$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$ where $z' = 0$ and $z'' = min\{x, y\}$. Assuming $x > 0$, from (2.37) it holds that $\mu = \frac{z}{x}$. Then, every $(x, y, z, \mu) \in \Pi'$ is coherent, i.e., $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x, y, z, \mu) \notin \Pi'$, then the assessment (x, y, z, μ) is not coherent, i.e. $(x, y, z, \mu) \notin \Pi$.
Let us consider now the case $x = 0$, so that $z' = 0$ and $z'' = 0$. We show that the assessment $(0, y, 0, \mu)$ is coherent if and only if $(y, \mu) \in [0,1]^2$, that is $\Pi'' \subseteq \Pi$. As $x = 0$, from (2.42), it holds that

$$(B|K)|_L(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \vee AH\bar{K} \text{ is true}, \\ \mu_L, & \text{if } \bar{A}H \vee \bar{H}BK \vee \bar{H}\,\bar{B}K \vee \bar{H}\,\bar{K} \text{ is true}. \end{cases} \tag{2.52}$$

that is $(B|K)|_L(A|H) = BK|AH$ (see proof of Theorem 14). Therefore $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_L (B|K), (B|K)|_L(A|H)\} = \{A|H, B|K, AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K \vee \bar{H}\,\bar{K}), BK|AH\}$ and $\mathcal{H}_4 = H \vee K \vee (AHBK \vee \bar{A}H \vee \bar{B}K \vee \bar{H}\,\bar{K}) \vee AH = \Omega$. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P} = (0, y, 0, \mu)$, are the following (see also Table 2.11):

$$C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K, \\ C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\,\bar{B}K, C_9 = \bar{H}\,\bar{K},$$

and

$$Q_1 = (1,1,1,1), Q_2 = (1,0,0,0), Q_3 = (1,y,0,0), Q_4 = (0,1,0,\mu), Q_5 = (0,0,0,\mu), \\ Q_6 = (0,y,0,\mu), Q_7 = (0,1,0,\mu), Q_8 = (0,0,0,\mu), Q_9 = (0,y,0,\mu).$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{F},\mathcal{P})$ is

	C_h	$A\vert H$	$B\vert K$	$(A\vert H)\wedge_L(B\vert K)$	$(B\vert K)\vert_L(A\vert H)$	Q_h
C_1	$AHBK$	1	1	1	1	Q_1
C_2	$AH\bar{B}K$	1	0	0	0	Q_2
C_3	$AH\bar{K}$	1	y	0	0	Q_3
C_4	$\bar{A}HBK$	0	1	0	μ	Q_4
C_5	$\bar{A}H\bar{B}K$	0	0	0	μ	Q_5
C_6	$\bar{A}H\bar{K}$	0	y	0	μ	Q_6
C_7	$\bar{H}BK$	0	1	0	μ	Q_7
C_8	$\bar{H}\,\bar{B}K$	0	0	0	μ	Q_8
C_9	$\bar{H}\,\bar{K}$	0	y	0	μ	Q_9

TABLE 2.11: Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, B|K, (A|H)\wedge_L (B|K), (B|K)|_L(A|H)\}$ and $\mathcal{P} = (0, y, 0, \mu)$.

$$\begin{cases} \lambda_1+\lambda_2+\lambda_3=0, \\ \lambda_1+y\lambda_3+\lambda_4+y\lambda_6+\lambda_7+y\lambda_9=y, \\ \lambda_1=0, \\ \lambda_1+\mu\lambda_4+\mu\lambda_5+\mu\lambda_6+\mu\lambda_7+\mu\lambda_8+\mu\lambda_9=\mu \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8+\lambda_9=1, \\ \lambda_i\geqslant 0,\ i=1,\dots,9. \end{cases} \tag{2.53}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_4, Q_5; indeed $(0,y,0,\mu) = yQ_4 + (1-y)Q_5 = y(0,1,0,\mu) + (1-y)(0,0,0,\mu)$. Then, the vector $\Lambda = (0,0,0,y,1-y,0,0,0,0)$ is a solution of (2.53), with

$$\begin{aligned}
\phi_1(\Lambda) &= \textstyle\sum_{h:C_h\subseteq H}\lambda_h = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 = 1>0,\\
\phi_2(\Lambda) &= \textstyle\sum_{h:C_h\subseteq K}\lambda_h = \lambda_1+\lambda_2+\lambda_4+\lambda_5 = 1>0,\\
\phi_3(\Lambda) &= \textstyle\sum_{h:C_h\subseteq(AHBK\vee\bar{A}H\vee\bar{B}K\vee\bar{H}\,\bar{K})}\lambda_h = \lambda_1+\lambda_2+\lambda_4+\lambda_5+\lambda_6+\lambda_8+\lambda_9 = 1>0,\\
\phi_4(\Lambda) &= \textstyle\sum_{h:C_h\subseteq AH}\lambda_h = \lambda_1+\lambda_2+\lambda_3 = 0.
\end{aligned}$$

Let $\mathcal{S}' = \{(0,0,0,y,1-y,0,0,0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.53). We have that $M'_1 = M'_2 = M'_3 = 1$ and $M'_4 = 0$ (as defined in (1.8)). It follows that $I'_0 = \{4\}$. As the sub-assessment $\mathcal{P}'_0 = \mu$ on $\mathcal{F}'_0 = \{BK|AH\}$ is coherent $\forall\mu \in [0,1]$, by Theorem 2, it follows that the assessment $(0,y,0,\mu)$ on $\mathcal{F} = \{A|H, B|K, (A|H)\wedge_L(B|K), (B|K)|_L(A|H)\}$ is coherent for every $(y,\mu)\in[0,1]^2$, that is $(0,y,0,\mu)\in\Pi''$. Thus $\Pi''\subseteq\Pi$. Of course, if $(0,y,z,\mu)\notin\Pi''$ the assessment $(0,y,z,\mu)$ is not coherent and hence $(0,y,z,\mu)\notin\Pi$. Therefore $\Pi = \Pi'\cup\Pi''$. □

Based on Theorem 16, we obtain

Theorem 17. *Let A, B, H, K be any logically independent events. Given a coherent assessment (x,y) on $\{A|H, B|K\}$, for the iterated conditional $(B|K)|_L(A|H)$ the extension $\mu_L = \mathbb{P}((B|K)|_L(A|H))$ is coherent if and only if $\mu_L\in[\mu'_L,\mu''_L]$, where*

$$\mu'_L = 0,\quad \mu''_L = \begin{cases} \min\left\{1,\frac{y}{x}\right\}, & \textit{if } x>0; \\ 1, & \textit{if } x=0. \end{cases} \tag{2.54}$$

Proof. The proof is the same as in Theorem 15 where K is replaced by L and Theorem 14 is replaced by Theorem 16. □

Remark 24. *Theorem 17 shows that the interval of coherent extensions* $[\mu_L', \mu_L'']$ *for* $\mu_L = \mathbb{P}[(B|K)|_L(A|H)]$ *does not coincide with the interval* $[\mu', \mu'']$ *of coherent assessment for* $\mu = P(B|A)$ *where* μ' *and* μ'' *are as in (2.1). Thus, property P4 is not satisfied by* $|_L$.

The iterated conditioning $|_B$ We continue by analysing the set of coherent extensions for the iterated conditional $|_B$. We first show that, when we evaluate $\mathbb{P}[(B|K)|_B(A|H)]$, if $P(A|H) = 0$, then coherence requires that $P((A|H) \wedge_B (B|K)) = 0$.

Remark 25. *We recall that, given any* $(x, y) \in [0,1]^2$, *where* $x = P(A|H)$ *and* $y = P(B|K)$, *the interval of coherent extensions on* $z_B = P((A|H) \wedge_B (B|K))$ *is* $[z_B', z_B''] = [0,1]$ *(Table 1.2). In particular it is coherent to assess* $(0, y, z_B)$, *with* $z_B > 0$, *on* $\{A|H, B|K, (A|H) \wedge_B (B|K)\}$. *However, for the object* $(B|K)|_B(A|H)$ *coherence also requires that* $z_B = \mu_B x$ *(see 2.37). Then, coherence requires that* $z_B = 0$ *when we consider the assessment* $(0, y, z_B, \mu_B)$ *on* $\{A|H, B|K, (A|H) \wedge_B (B|K), (B|K)|_B(A|H)\}$.

Theorem 18. *Let* A, B, H, K *be any logically independent events. The set* Π *of all the coherent assessments* (x, y, z, μ) *on the family* $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_B (B|K), (B|K)|_B(A|H)\}$ *is* $\Pi = \Pi' \cup \Pi''$, *where* $\Pi' = \{(x, y, z, \mu) : x \in (0,1], y \in [0,1], z \in [z', z''], \mu = \frac{z}{x}\}$ *with* $z' = 0, z'' = 1$, *and* $\Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0,1]^2\}$.

Proof. Of course the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0,1]^2$. By Table (1.2), the assessment $z = P((A|H) \wedge_B (B|K))$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$ where $z' = 0$ and $z'' = 1$. Assuming $x > 0$, from (2.37) it holds that $\mu = \frac{z}{x}$. Then, every $(x, y, z, \mu) \in \Pi'$ is coherent, i.e., $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x, y, z, \mu) \notin \Pi'$, then the assessment (x, y, z, μ) is not coherent, i.e. $(x, y, z, \mu) \notin \Pi$.

Let us consider now the case $x = 0$. We recall that it is coherent to assess $(0, y, z)$, with $z > 0$, on $\{A|H, B|K, (A|H) \wedge_B (B|K)\}$ (see Table 1.2). However, for the further object $(B|K)|_B(A|H)$ coherence also requires that $z = \mu x$ (see 2.37). Then, coherence requires that $z = 0$ when we consider the assessment $(0, y, z, \mu)$ on $\{A|H, B|K, (A|H) \wedge_B (B|K), (B|K)|_B(A|H)\}$. We show that the assessment $(0, y, 0, \mu)$ is coherent if and only if $(y, \mu) \in [0,1]^2$, that is $\Pi'' \subseteq \Pi$. As $x = 0$, from (2.44), it holds that

$$(B|K)|_B(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \vee AH\bar{K} \text{ is true}, \\ \mu_B, & \text{if } \bar{H} \vee \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} \text{ is true}, \end{cases} \tag{2.55}$$

that is $(B|K)|_B(A|H) = BK|AH$ (see proof Theorem 14). We show that the assessment $\mathcal{P} = (0, y, 0, \mu)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_B (B|K), (B|K)|_B(A|H)\} = \{A|H, B|K, AHBK|HK, BK|AH\}$ is coherent if and only if $(y, \mu) \in [0,1]^2$, that is $\Pi'' \subseteq \Pi$. We observe that $\mathcal{H}_4 = H \vee K \vee HK \vee AH = H \vee K$. The constituents $C_h's$ and the points $Q_h's$ associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P} = (0, y, 0, \mu)$, are the following:

$$C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,$$
$$C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\,\bar{B}K, C_0 = \bar{H}\,\bar{K},$$

and

$$Q_1 = (1,1,1,1), Q_2 = (1,0,0,0), Q_3 = (1,y,0,0), Q_4 = (0,1,0,\mu), Q_5 = (0,0,0,\mu),$$
$$Q_6 = (0,y,0,\mu), Q_7 = (0,1,0,\mu), Q_8 = (0,0,0,\mu), \mathcal{P} = Q_0 = (0,y,0,\mu).$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{F},\mathcal{P})$ is

$$\begin{cases} \lambda_1+\lambda_2+\lambda_3=0,\\ \lambda_1+y\lambda_3+\lambda_4+y\lambda_6+\lambda_7=y,\\ \lambda_1=0,\\ \lambda_1+\mu\lambda_4+\mu\lambda_5+\mu\lambda_6+\mu\lambda_7+\mu\lambda_8=\mu\\ \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8=1,\\ \lambda_i\geqslant 0,\ i=1,\dots,8. \end{cases} \tag{2.56}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_4, Q_5; indeed $(0,y,0,\mu)=yQ_4+(1-y)Q_5=y(0,1,0,\mu)+(1-y)(0,0,0,\mu)$.
Then the vector $\Lambda=(0,0,0,y,1-y,0,0,0)$ is a solution of (2.56), with

$$\begin{aligned} \phi_1(\Lambda) &= \textstyle\sum_{h:C_h\subseteq H}\lambda_h=\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6=1>0,\\ \phi_2(\Lambda) &= \textstyle\sum_{h:C_h\subseteq K}\lambda_h=\lambda_1+\lambda_2+\lambda_4+\lambda_5+\lambda_7+\lambda_8=1>0,\\ \phi_3(\Lambda) &= \textstyle\sum_{h:C_h\subseteq HK}\lambda_h=\lambda_1+\lambda_2+\lambda_4+\lambda_5=1>0,\\ \phi_4(\Lambda) &= \textstyle\sum_{h:C_h\subseteq AH}\lambda_h=\lambda_1+\lambda_2+\lambda_3=0. \end{aligned}$$

Let $\mathcal{S}'=\{(0,0,0,y,1-y,0,0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.56). We have that $M_1'=M_2'=M_3'=1$ and $M_4'=0$ (as defined in (1.8)). It follows that $I_0'=\{4\}$. As the sub-assessment $\mathcal{P}_0'=\mu$ on $\mathcal{F}_0'=\{BK|AH\}$ is coherent $\forall\mu\in[0,1]$, by Theorem 2, it follows that the assessment $(0,y,0,\mu)$ on $\mathcal{F}=\{A|H,B|K,(A|H)\wedge_B(B|K),(B|K)|_B(A|H)\}$ is coherent for every $(y,\mu)\in[0,1]^2$, that is $(0,y,0,\mu)\in\Pi''$. Thus $\Pi''\subseteq\Pi$. Of course, if $(0,y,z,\mu)\notin\Pi''$ the assessment $(0,y,z,\mu)$ is not coherent and hence $(0,y,z,\mu)\notin\Pi$. Therefore $\Pi=\Pi'\cup\Pi''$. □

Theorem 19. *Let A,B,H,K be any logically independent events. Given a coherent assessment (x,y) on $\{A|H,B|K\}$, for the iterated conditional $(B|K)|_B(A|H)$ the extension $\mu_B=\mathbb{P}[(B|K)|_B(A|H)]$ is coherent if and only if $\mu_B\in[\mu_B',\mu_B'']$, where*

$$\mu_B'=0,\quad \mu_B''=\begin{cases}\frac{1}{x}, & \text{if } 0<x<1,\\ 1, & \text{if } x=0\vee x=1.\end{cases} \tag{2.57}$$

Proof. Assume that $x>0$. We simply write μ instead of μ_B. From Theorem 18 it follows that the set of all coherent assessments (x,y,z,μ) on $\mathcal{F}$ is $\Pi'=\{(x,y,z,\mu): 0<x\leqslant 1, 0\leqslant y\leqslant 1, z'\leqslant z\leqslant z'', \mu=\frac{z}{x}\}$, where $z'=0$ and $z''=1$ (see Table 1.2). Then, μ is a coherent extension of (x,y) if and only if $\mu\in[\mu',\mu'']$, where $\mu'=0$ and $\mu''=\frac{z''}{x}=\frac{1}{x}$. Assume that $x=0$. From Theorem 18 it follows that the set of all coherent assessments (x,y,z,μ) on $\mathcal{F}=\{A|H,B|K,(A|H)\wedge_B(B|K),(B|K)|_B(A|H)\}$ is $\Pi''=\{(0,y,0,\mu):(y,\mu)\in[0,1]^2\}$. Then, μ is a coherent extension to $(B|K)|_B(A|H)$ of the assessment $(0,y)$ on $\{A|H,B|K\}$ if and only if $\mu\in[\mu',\mu'']$, where $\mu'=0$ and $\mu''=1$. □

Remark 26. *We notice that the lower and upper bounds μ_B' and μ_B'' for $(B|K)|_B(A|H)$, given in (2.57), do not coincide with the lower and upper bound μ' and μ'' for $B|A$, given in (2.1). In particular, the extension $\mu_B=\frac{1}{x}$ to $(B|K)|_B(A|H)$ of (x,y) on $\{A|H,B|K\}$, with $0<x<1$, is coherent. Indeed, as $\mu_B''=\frac{1}{x}$, from (2.57), the assessment $(x,y,\frac{1}{x})$ on $\{A|H,B|K,(B|K)|_B(A|H)\}$ is coherent. However, we recall that $\mu=\frac{1}{x}$ is not a coherent extension to $B|A$ of (x,y) on $\{A,B\}$, with $0<x<1$. Indeed, as $\frac{1}{x}>1\geqslant\mu''=min\{x,y\}$, from (2.1) the assessment $(x,y,\frac{1}{x})$ on $\{A,B,B|A\}$ is not coherent. Therefore, property P4 is not satisfied by $|_B$.*
We also notice that, when $P(A|H)>0$, as $(A|H)\wedge_B(B|K)=AH|BK$, from (2.38) it holds

that

$$\mathbb{P}[(B|K)|_B(A|H)] = \frac{P((A|H) \wedge_B (B|K))}{P(A|H)} = \frac{P(AH|BK)}{P(A|H)}. \tag{2.58}$$

We point out that the prevision of $(B|K)|_B(A|H)$ given in (2.58) coincides with the probability of the super-conditional event $B_K|A_H$, *denoted by $P^*(B_K|A_H)$, introduced in [6, formula (20)]. As $P^*(B_K|A_H)$ can assume values greater than 1, in [6] it has been called* conditional hyper-probability.

The iterated conditioning $|_S$ We first show that, when we evaluate $\mathbb{P}[(B|K)|_S(A|H)]$, if $P(A|H) = 0$, then coherence requires that $P((A|H) \wedge_S (B|K)) = 0$.

Remark 27. *We recall that, given any $(x, y) \in [0,1]^2$, where $x = P(A|H)$ and $y = P(B|K)$, the interval of coherent extensions on $z_S = P((A|H) \wedge_S (B|K))$ is $[z', z'']$ where (see Table 1.2)*

$$z' = \max\{x + y - 1, 0\} \text{ and } z'' = \begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x,y) \neq (1,1), \\ 1, & \text{if } (x,y) = (1,1). \end{cases}$$

In particular it is coherent to assess $z_S > 0$ when $x = 0$. However, for the object $(B|K)|_S(A|H)$ coherence also requires that $z_S = \mu_S x$ (see 2.37). Then, coherence requires that $z_S = 0$ when we consider the assessment $(0, y, z_S, \mu_S)$ on $\{A|H, B|K, (A|H) \wedge_S (B|K), (B|K)|_S(A|H)\}$.

Theorem 20. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x, y, z, μ) on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_S (B|K), (B|K)|_S(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where*

$$\begin{aligned} &\Pi' = \{(x, y, z, \mu) : x \in (0,1], y \in [0,1], z \in [z', z''], \mu = \tfrac{z}{x}\}, \\ &\text{with } z' = \max\{x + y - 1, 0\}, z'' = \begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x,y) \neq (1,1), \\ 1, & \text{if } (x,y) = (1,1), \end{cases} \end{aligned} \tag{2.59}$$

and

$$\Pi'' = \{(0, y, 0, \mu) : y \in [0,1], \mu \geqslant 0\}. \tag{2.60}$$

Proof. We know that the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0,1]^2$. We also recall that the assessment $z = P((A|H) \wedge_S (B|K))$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$, where (see Table 1.2)

$$z' = \max\{x + y - 1, 0\} \text{ and } z'' = \begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x,y) \neq (1,1), \\ 1, & \text{if } (x,y) = (1,1). \end{cases}$$

Assuming $x > 0$, then it follows from (2.37) that $\mu = \frac{z}{x}$. Thus, as every $(x, y, z, \mu) \in \Pi'$ is coherent, it holds that $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x, y, z, \mu) \notin \Pi'$, then (x, y, z, μ) is not coherent, i.e. $(x, y, z, \mu) \notin \Pi$.
Let us consider now the case $x = 0$. From Remark 27, coherence requires that $z = 0$. We show that the assessment $\mathcal{P} = (0, y, 0, \mu)$ is coherent if and only if $y \in [0,1]$ and $\mu \geqslant 0$, and hence $\Pi'' \subseteq \Pi$. As $x = 0$, by (2.46) it holds that

$$(B|K)|_S(A|H) = \begin{cases} 1, & \text{if } AHBK \vee AH\bar{K} \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \text{ is true}, \\ \mu, & \text{if } \bar{A}H \vee \bar{H}\bar{K} \vee \bar{H}\bar{B}K \text{ is true}, \\ 1 + \mu, & \text{if } \bar{H}BK \text{ is true}, \end{cases} \tag{2.61}$$

that is, when $x = 0$,

$$(B|K)|_S(A|H) = [AHBK + AH\bar{K} + (1+\mu)\bar{H}BK]|(AH \vee \bar{H}BK).$$

Then, we have $\mathcal{F} = \{A|H, B|K, (AH \vee \bar{H}) \wedge (BK \vee \bar{K})|(H \vee K), (AHBK + AH\bar{K} + (1+\mu)\bar{H}BK)|(AH \vee \bar{H}BK)\}$, with $\mathcal{H}_4 = H \vee K \vee (H \vee K) \vee (AH \vee \bar{H}BK) = H \vee K$. Therefore, the constituents and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$ are

$$\begin{array}{l} C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K, \\ C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\,\bar{B}K, C_0 = \bar{H}\,\bar{K}, \end{array}$$

and

$$\begin{array}{l} Q_1 = (1,1,1,1), Q_2 = (1,0,0,0), Q_3 = (1,y,1,1), Q_4 = (0,1,0,\mu), Q_5 = (0,0,0,\mu), \\ Q_6 = (0,y,0,\mu),\ Q_7 = (0,1,1,1+\mu), Q_8 = (0,0,0,\mu), \mathcal{P} = Q_0 = (0,y,0,\mu). \end{array}$$

The system (Σ) in (1.6) associated with the pair $(\mathcal{F}, \mathcal{P})$ is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y, \\ \lambda_1 + \lambda_3 + \lambda_7 = 0, \\ \lambda_1 + \lambda_3 + \mu\lambda_4 + \mu\lambda_5 + \mu\lambda_6 + (1+\mu)\lambda_7 + \mu\lambda_8 = \mu, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1, \\ \lambda_i \geqslant 0,\ i = 1, \ldots, 8. \end{cases} \tag{2.62}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_4, Q_5; indeed $(0,y,0,\mu) = yQ_4 + (1-y)Q_5 = y(0,1,0,\mu) + (1-y)(0,0,0,\mu)$.
Then, the vector $\Lambda = (0,0,0,y,1-y,0,0,0)$ is a solution of (2.62), with

$$\begin{array}{l} \phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 > 0, \\ \phi_2(\Lambda) = \sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 1 > 0, \\ \phi_3(\Lambda) = \sum_{h:C_h \subseteq H \vee K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1 > 0, \\ \phi_4(\Lambda) = \sum_{h:C_h \subseteq (AH \vee \bar{H}BK)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_7 = 0. \end{array}$$

Let $\mathcal{S}' = \{(0,0,0,y,1-y,0,0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (2.62). We have that $M'_1 = M'_2 = M'_3 = 1$ and $M'_4 = 0$ (as defined in (1.8)). It follows that $I'_0 = \{4\}$. We set $\mathcal{F}'_0 = \{(AHBK + AH\bar{K} + (1+\mu)\bar{H}BK)|(AH \vee \bar{H}BK)\}$ and $\mathcal{P}'_0 = \mu$. By exploiting the betting scheme, we show that the assessment μ on the conditional random quantity $(AHBK + AH\bar{K} + (1+\mu)\bar{H}BK)|(AH \vee \bar{H}BK)$ is coherent for every $\mu \geqslant 0$. The random gain for the assessment μ is

$$G = s(AH + \bar{H}BK)(AHBK + AH\bar{K} + (1+\mu)\bar{H}BK - \mu).$$

Without loss of generality we can assume $s = 1$. The constituents in $AH \vee \bar{H}BK$ are $C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{H}BK$ and the corresponding values for the random gain G are $g_1 = 1-\mu$, $g_2 = -\mu$, $g_3 = 1-\mu = g_1$, $g_4 = 1$. Hence the set of values of G restricted to $AH \vee \bar{H}BK$ is $\mathcal{G}_{AH \vee \bar{H}BK} = \{g_1, g_2, g_4\} = \{1-\mu, -\mu, 1\}$. We distinguish two cases: (i) $\mu < 0$; (ii) $\mu \geqslant 0$.
Case (i). We observe that, $1-\mu > 0$, $-\mu > 0$, and hence $\min \mathcal{G}_{AH \vee \bar{H}BK} > 0$. Then the assessment $\mathcal{P}_0$ on $\mathcal{F}_0$ is incoherent.
Case (ii). We observe that, $\min \mathcal{G}_{AH \vee \bar{H}BK} = -\mu \leqslant 0$ and $\max \mathcal{G}_{AH \vee \bar{H}BK} = 1 > 0$. Then, as

$$\min \mathcal{G}_{AH \vee \bar{H}BK} \cdot \max \mathcal{G}_{AH \vee \bar{H}BK} \leqslant 0,$$

the assessment $\mathcal{P}_0$ on $\mathcal{F}_0$ is coherent. Then, as System (2.62) is solvable and $\mathcal{P}_0$ on $\mathcal{F}_0$ is coherent, by Theorem 3 it follows that every assessment $(0, y, 0, \mu)$ on $\mathcal{F}$ is coherent if and only if $(0, y, 0, \mu) \in \Pi'' = \{(0, y, 0, \mu) : y \in [0, 1], \mu \geqslant 0\}$. Thus, $\Pi'' \subseteq \Pi$. Of course, if $(0, y, z, \mu) \notin \Pi''$ the assessment $(0, y, z, \mu)$ is not coherent and hence $(0, y, z, \mu) \notin \Pi$. Therefore $\Pi = \Pi' \cup \Pi''$. $\square$

Remark 28. *From Theorem 20 we observe that, when $x = 0$, the set of all coherent assessments (x, y, z, μ) on $\mathcal{F}$ is Π'' given in (2.60). Then, in this case, $\mu = \mathbb{P}[(B|K)|_S(A|H)] = \mathbb{P}[(AHBK + AH\bar{K} + (1+\mu)\bar{H}BK)|(AH \vee \bar{H}BK)]$ is coherent for every value $\mu \geqslant 0$.*

Based on Theorem 20, when $x > 0$, we obtain the following result on the lower and upper bounds for $\mathbb{P}[(B|K)|_S(A|H)]$.

Theorem 21. *Let A, B, H, K be any logically independent events. Given a coherent assessment (x, y) on $\{A|H, B|K\}$, with $x \neq 0$, for the iterated conditional $(B|K)|_S(A|H)$ the extension $\mu_S = \mathbb{P}[(B|K)|_S(A|H)]$ is coherent if and only if $\mu_S \in [\mu'_S, \mu''_S]$, where*

$$\mu'_S = \max\{\frac{x+y-1}{x}, 0\} \text{ and } \mu''_S = \begin{cases} \frac{x+y-2xy}{x(1-xy)}, & \text{if } (x,y) \neq (1,1); \\ 1, & \text{if } (x,y) = (1,1). \end{cases} \tag{2.63}$$

Proof. We simply write μ instead of μ_S. As $x > 0$, from Theorem 20 it follows that the set of all coherent assessments (x, y, z, μ) on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_S (B|K), (B|K)|_S(A|H)\}$ is the set Π' given in Theorem 20. Then, μ is a coherent extension of (x, y) if and only if $\mu \in [\mu', \mu'']$, where μ' and μ'' are

$$\mu' = \frac{z'}{x} = \max\{\frac{x+y-1}{x}, 0\} \text{ and } \mu'' = \frac{z''}{x} = \begin{cases} \frac{x+y-2xy}{x(1-xy)}, & \text{if } (x,y) \neq (1,1); \\ 1, & \text{if } (x,y) = (1,1). \end{cases}$$

$\square$

Remark 29. *We observe that the lower and upper bounds μ'_S and μ''_S for $(B|K)|_S(A|H)$, given in (2.63), do not coincide with the lower and upper bound μ' and μ'' for $B|A$, given in (2.1). Then, formula P4 is not satisfied by $|_S$. In particular the extension $\mu_S = \frac{1}{x}$ to $(B|K)|_S(A|H)$ of $(x, 1)$ on $\{A|H, B|K\}$, with $0 < x < 1$, is coherent because $\mu''_S = \frac{1-x}{x(1-x)} = \frac{1}{x}$. However, the assessment $\mu = \frac{1}{x}$ is not a coherent extension on $\mu = P(B|A)$ of $P(A) = x$, $P(B) = 1$, because by (2.1) it holds that $\mu'' \leqslant 1 < \frac{1}{x}$. We also notice that in a bet on the iterated conditional$(B|K)|_S(A|H)$, by paying (the coherent assessment) $\mu_S = \frac{1}{x}$, with $x \in (0, 1)$, the amount 1, received when both the conditional events $A|H$ and $B|K$ are true, does not coincide with the maximum value of the random win. In other words, we obtain that the associated random gain $G = (B|K)|_S(A|H) - \mu_S$ is negative even when both the antecedent and the consequent of the iterated conditional are true. For instance, by setting $x = \frac{1}{3}$ and $\mu_S = \frac{1}{x} = 3$, if the event $\bar{H}\,\bar{B}K$ is true, then the iterated conditional $(B|K)|_S(A|H) = \frac{1}{x} - 1 = 3 - 1 = 2 > 1$ and hence $G = 2 - 3 = -1$.*

The iterated conditioning $|_{gs}$ We recall that $|_{gs}$, unlike all the other iterated conditioning, satisfies property P4 ([90, Theorem 4]). Indeed, given a coherent assessment (x, y) on $\{A|H, B|K\}$, under logical independence, for the iterated conditional $(B|K)|_{gs}(A|H)$ the extension $\mu = \mathbb{P}[(B|K)|_{gs}(A|H)]$ is coherent if and only if $\mu \in [\mu'_{gs}, \mu''_{gs}]$, where

$$\mu'_{gs} = \begin{cases} \max\{\frac{x+y-1}{x}, 0\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}, \quad \mu''_{gs} = \begin{cases} \min\{1, \frac{y}{x}\}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

which coincide with μ' and μ'' given in (2.1), respectively. As we can see from Table 2.12, only the iterated conditioning $|_{gs}$ satisfies property P4.

Iterated conditioning	Interval of coherent extensions
$\vert_C$	$[0,1]$
$\vert_{dF}$	$[0,1]$
$\vert_F$	$[0,1]$
$\vert_K$	$\left[0, \begin{cases} \min\{1,\frac{y}{x}\}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}\right]$
$\vert_L$	$\left[0, \begin{cases} \min\{1,\frac{y}{x}\}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}\right]$
$\vert_B$	$\left[0, \begin{cases} \frac{1}{x}, & \text{if } 0 < x < 1, \\ 1, & \text{if } x = 0 \vee x = 1. \end{cases}\right]$
$\vert_S$	$\begin{cases} \left[\max\{\frac{x+y-1}{x},0\}, \begin{cases} \frac{x+y-2xy}{x(1-xy)}, & \text{if } (x,y) \neq (1,1); \\ 1, & \text{if } (x,y) = (1,1). \end{cases}\right] & \text{if } x \neq 0; \\ \mu \geqslant 0, & \text{if } x = 0; \end{cases}$
$\vert_{gs}$	$\left[\begin{cases} \max\{\frac{x+y-1}{x},0\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \begin{cases} \min\{1,\frac{y}{x}\}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}\right]$

TABLE 2.12: Interval $[\mu_i', \mu_i'']$ of coherent extensions of the assessment $(x,y) \in [0,1]^2$ on $\{A|H, B|K\}$ to the iterated conditional $(B|K)|_i(A|H)$, $i \in \{C, dF, F, K, L, B, S, gs\}$, under the assumption that A, H, B, K are logically independent.

2.5.5 Import-Export principle

In this section we show that none of the iterated conditioning $|_K, |_L, |_B, |_S, |_{gs}$ satisfies the Import-Export principle. We recall that the Import-Export principle is valid when $(B|K)|A = B|AK$ (see equation (2.2)). We remind that, in agreement with [1, 71] and differently from [80], for the iterated conditional $(B|K)|_{gs}(A|H)$ the Import-Export principle is not valid. As a consequence, as shown in [53] (see also [91, 90]), Lewis' triviality results ([79]) are avoided by $|_{gs}$. For what concerns the iterated conditioning $|_K, |_L, |_B, |_S$, it holds that $(B|K)|_i A \neq B|AK$, $i \in \{K, L, B, S\}$, because there are some constituents where the two objects may assume different values (see Table 2.13). For instance, when $A\bar{K}$ is true and $z = P(B|AK) \neq 1$, it follows that $(B|K)|_S A = 1 \neq z = B|AK$. Then, none of the iterated conditioning $|_K, |_L, |_B, |_S, |_{gs}$ satisfies the Import-Export principle. However, we observe that Import-Export principle could be satisfied under some suitable logical relations among the events A, B, K. For instance, if $A\bar{K} = \varnothing$, it can be easily proved that $(B|K)|_i A = B|AK$, $i \in \{K, L, S, gs\}$. Thus, when $A\bar{K} = \varnothing$, the Import-Export principle is satisfied by $(B|K)|_K A$, $(B|K)|_L A$, $(B|K)|_S A$, and $(B|K)|_{gs} A$.

	$(B\|K)\|_K A$	$(B\|K)\|_L A$	$(B\|K)\|_B A$	$(B\|K)\|_S A$	$(B\|K)\|_{gs} A$	$B\|AK$
ABK	1	1	1	1	1	1
$A\bar{B}K$	0	0	0	0	0	0
$A\bar{K}$	$x\mu_K$	$x\mu_L$	$x\mu_B$	1	y	z
$\bar{A}BK$	μ_K	μ_L	μ_B	μ_S	μ_{gs}	z
$\bar{A}\,\bar{B}K$	μ_K	μ_L	μ_B	μ_S	μ_{gs}	z
$\bar{A}\,\bar{K}$	μ_K	μ_L	$\mu_B(1+x)$	μ_S	μ_{gs}	z

TABLE 2.13: Numerical values of $(B|K)|_i A$, $i \in \{K, L, B, S, gs\}$ and of the conditional event $B|AK$. We denotes $x = P(A)$, $y = P(B|K)$, $z = P(B|AK)$ and $\mu_i = \mathbb{P}[(B|K)|_i A]$, $i \in \{K, L, B, S, gs\}$. We observe that $(B|K)|_i A \neq B|AK$, $i \in \{K, L, B, S, gs\}$.

2.5.6 Generalised versions of Bayes' rule

In this section, by exploiting property P4, we analyse generalised versions of Bayes' Rule for the iterated conditioning $|_K, |_L, |_B, |_S$, and $|_{gs}$.

From (2.38) it follows that $\mathbb{P}[(B|K) \wedge_i (A|H)] = \mathbb{P}[(B|K)|_i(A|H)]P(A|H) = \mathbb{P}[(A|H)|_i(B|K)]P(B|K)$, $i \in \{K, L, B, S, gs\}$. Then, when $P(A|H) > 0$ it holds that

$$\mathbb{P}[(B|K)|_i(A|H)] = \frac{\mathbb{P}[(A|H)|_i(B|K)]P(B|K)}{P(A|H)},\ i \in \{K, L, B, S, gs\}. \tag{2.64}$$

Formula (2.64) is a generalisation of the following well-known version of Bayes's Rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}, \quad \text{if } P(A) > 0,$$

where the events A, B are replaced by the conditional events $A|H$, $B|K$, respectively, and the conditioning operator $|$ is replaced by the iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$. We also recall that, given two events A and B, it holds that $A = AB \vee A\bar{B}$, and hence $P(A) = P(AB) + P(A\bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(B)$. Then, we obtain the following version of Bayes' Rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}. \tag{2.65}$$

We now check, for each iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$, the validity of the following generalised version of formula (2.65)

$$\mathbb{P}[(B|K)|(A|H)] = \frac{\mathbb{P}[(A|H)|(B|K)]P(B|K)}{\mathbb{P}[(A|H)|(B|K)]P(B|K) + \mathbb{P}[(A|H)|(\bar{B}|K)]P(\bar{B}|K)}. \tag{2.66}$$

Based on (2.38),for each $i \in \{K, L, B, S, gs\}$, it follows that

$$\mathbb{P}[(A|H) \wedge_i (B|K)] + \mathbb{P}[(A|H) \wedge_i (\bar{B}|K)] = \mathbb{P}[(A|H)|_i(B|K)]P(B|K) + \mathbb{P}[(A|H)|_i(\bar{B}|K)]P(\bar{B}|K). \tag{2.67}$$

We recall that when A, B, and $\bar{B}$ in the equality $A = (A \wedge B) \vee (A \wedge \bar{B})$ are replaced by the conditional events $A|H, B|K$, and $\bar{B}|K$, respectively, and the conjunction $\wedge$ is replaced by $\wedge_i$, $i \in \{K, L, B, S\}$, it holds that the corresponding equality is not satisfied. Indeed we have that

$$(A|H) \neq [(A|H) \wedge_i (B|K)] \vee_i [(A|H) \wedge_i (\bar{B}|K)], \quad i \in \{K, L, B, S\}.$$

More precisely, it holds that (see [60, Section 4.1])

- $[(A|H) \wedge_K (B|K)] \vee_K [(A|H) \wedge_K (\bar{B}|K)] = AHK|(AHK \vee \bar{A}H) \neq A|H$;
- $[(A|H) \wedge_L (B|K)] \vee_L [(A|H) \wedge_L (\bar{B}|K)] = AHK|(H \vee \bar{K}) \neq A|H$;
- $[(A|H) \wedge_B (B|K)] \vee_B [(A|H) \wedge_B (\bar{B}|K)] = A|(HK) \neq A|H$;
- $[(A|H) \wedge_S (B|K)] \vee_S [(A|H) \wedge_S (\bar{B}|K)] = (A \vee \bar{H})|(H \vee K) \neq A|H$.

Moreover, for each $i \in \{K, L, B, S\}$, we observe that the conditional probability $P(A|H)$ does not necessarily coincide with the sum

$$P((A|H) \wedge_i (B|K)) + P((A|H) \wedge_i (\bar{B}|K)).$$

Indeed, it can be proved that the assessment $(1,0,0)$ on $\{A|H, (A|H) \wedge_i (B|K), (A|H) \wedge_i (\bar{B}|K)\}$, $i \in \{K, L, B\}$, as well as the assessment $(1,1,1)$ on $\{A|H, (A|H) \wedge_S (B|K), (A|H) \wedge_S (\bar{B}|K)\}$, is coherent. As a consequence, from (2.67), we obtain that the probability $P(A|H)$ does not necessarily coincide with

$$\mathbb{P}[(A|H)|_i(B|K)]P(B|K) + \mathbb{P}[(A|H)|_i(\bar{B}|K)]P(\bar{B}|K).$$

Thus, for each iterated conditioning $|_i$, $i \in \{K, L, B, S\}$, the generalised version of Bayes's Rule given in formula (2.66) is not satisfied. However, we recall that ([50])

$$(A|H) = (A|H) \wedge_{gs} (B|K) + (A|H) \wedge_{gs} (\bar{B}|K).$$

Then, by the linearity property of a coherent prevision and by (2.67), it follows that

$$\begin{aligned} P(A|H) &= \mathbb{P}[(A|H) \wedge_{gs} (B|K)] + \mathbb{P}[(A|H) \wedge_{gs} (\bar{B}|K)] = \\ &= \mathbb{P}[(A|H)|_{gs}(B|K)]P(B|K) + \mathbb{P}[(A|H)|_{gs}(\bar{B}|K)]P(\bar{B}|K). \end{aligned}$$

Hence, when $P(A|H) > 0$, it holds that

$$\mathbb{P}[(B|K)|_{gs}(A|H)] = \frac{\mathbb{P}[(A|H)|_{gs}(B|K)]P(B|K)}{\mathbb{P}[(A|H)|_{gs}(B|K)]P(B|K) + \mathbb{P}[(A|H)|_{gs}(\bar{B}|K)]P(\bar{B}|K)}. \quad (2.68)$$

Therefore, the generalisation of the second version of Bayes' Rule only holds for $|_{gs}$ and does not hold for $|_K, |_L, |_B$, and $|_S$.

2.6 Generalised version of Modus Ponens and two-premise centering

In this section, for selected definitions of the iterated conditioning with possible value in $[0,1]$, based on the probability propagation rules obtained in the previous sections and exploited for checking the validity of property P4 (see Table 2.12), we study the p-validity of the generalisation of two inference rules: Modus Ponens and two-premise centering. We first recall the notions of p-consistency and p-entailment for conditional random quantities, which take values in a finite subset of $[0,1]$ ([90]). These concepts are based on the notions of p-consistency and p-entailment given for conditional events by Adams ([1]) and also studied in the setting of coherence in [46].

Definition 12. *Let $\mathcal{F}_n = \{X_i|H_i\,,\ i = 1, \ldots, n\}$ be a family of n conditional random quantities which take values in a finite subset of $[0,1]$. Then, $\mathcal{F}_n$ is* p-consistent *if and only if, the (prevision) assessment $(\mu_1, \mu_2, \ldots, \mu_n) = (1, 1, \ldots, 1)$ on $\mathcal{F}_n$ is coherent.*

Definition 13. *A p-consistent family $\mathcal{F}_n = \{X_i|H_i\,,\ i = 1,\dots,n\}$* p-entails *a conditional random quantity $X|H$, which takes values in a finite subset of $[0,1]$, denoted by $\mathcal{F}_n \Rightarrow_p X|H$, if and only if for any coherent (prevision) assessment $(\mu_1,\dots,\mu_n,z)$ on $\mathcal{F}_n \cup \{X|H\}$: if $\mu_1 = \cdots = \mu_n = 1$, then $z = 1$.*

We say that the inference from a p-consistent family of premises $\mathcal{F}_n$ to a conclusion $X|H$ is *p-valid* if and only if $\mathcal{F}_n$ p-entails $X|H$.

We recall that as the iterated conditionals $(B|K)|_C(A|H)$, $(B|K)|_{dF}(A|H)$, and $(B|K)|_F(A|H)$ are conditional events, their indicators take values in the interval [0,1]. Moreover, from Table 2.10 we observe that the iterated conditionals $(B|K)|_K(A|H)$, $(B|K)|_L(A|H)$ and $(B|K)|_{gs}(A|H)$ take values in $[0,1]$. On the other hand, the iterated conditionals $(B|K)|_B(A|H)$ and $(B|K)|_S(A|H)$ may take values outside the interval $[0,1]$. Thus, in order to examine the p-validity of generalised inference rules, we will only consider the iterated conditionals $(B|K)|_i(A|H)$, $i \in \{C, dF, F, K, L, gs\}$.

2.6.1 Modus Ponens

We recall that, given two events A and B the Modus Ponens inference, with (p-consistent) premise set $\{A, B|A\}$ and conclusion B, is p-valid ([93], see also [48, 68]), that is

$$\{A, B|A\} \Rightarrow_p B. \tag{2.69}$$

For each $i \in \{C, dF, F, K, L, gs\}$, we will study the p-validity of the generalised version of Modus Ponens obtained when the events A, B are replaced by the conditional events $A|H, B|K$ and the conditional event $B|A$ is replaced by the iterated conditional $(B|K)|_i(A|H)$[2]. An example of this generalisation is (see also [43, 89]):

$\overbrace{\textit{The cup is fragile if made of glass}}^{A|H}$.

If $\overbrace{\textit{the cup is fragile if made of glass}}^{A|H}$, then $\overbrace{\textit{it breaks if dropped}}^{B|K}$.

Therefore, $\overbrace{\textit{the cup breaks if dropped}}^{B|K}$.

Then, we check the p-validity of the generalised version of Modus Ponens where the premise set is $\{A|H, (B|K)|_i(A|H)\}$ and the conclusion is $B|K$, that is

$$P(A|H) = 1,\ \mathbb{P}[(B|K)|_i(A|H)] = 1 \implies P(B|K) = 1. \tag{2.70}$$

It is easy to verify (see Table 2.12) that the family $\{A|H, (B|K)|_i(A|H)\}$ is p-consistent, $i \in \{C, dF, F, K, L, gs\}$. We will show that the generalised version of Modus Ponens is p-valid for $|_i \in \{|_K, |_L, |_{gs}\}$. Indeed, in these cases from Table 2.12 it follows that $\mathbb{P}[(B|K)|_i(A|H)] \leqslant \mu_i'' = \min\{1, \frac{y}{x}\} = y = P(B|K)$, when $P(A|H) = x = 1$. Then, $P(B|K)$ is necessarily equal to 1, when $P(A|H) = 1$ and $\mathbb{P}[(B|K)|_i(A|H)] = 1$ and hence formula (2.70) is satisfied. Thus,

$$\{A|H,\ (B|K)|_i(A|H)\} \Rightarrow_p B|K,\quad i \in \{K, L, gs\}. \tag{2.71}$$

For $i \in \{C, dF, F\}$, as illustrated in Table 2.12, the interval of coherent extensions $[\mu_i', \mu_i'']$ on $(B|K)|_i(A|H)$ of the assessment $(1,0)$ on $\{A|H, B|K\}$ coincides with the unit interval $[0,1]$. In particular the assessment $(1,1,0)$ on $\{A|H, (B|K)|_i(A|H), (B|K)\}$ is

[2] The particular case where $K = \Omega$ and $|_i = |_{gs}$ has been studied in [89].

coherent and hence formula (2.70) is not satisfied for $i \in \{C, dF, F\}$, that is

$$\{A|H,\ (B|K)|_i(A|H)\} \not\Rightarrow_p B|K,\ \ i \in \{C, dF, F\}. \tag{2.72}$$

Concerning the example above, from P(*the cup is fragile if made of glass*) $= 1$ and $\mathbb{P}$[If *the cup is fragile if made of glass*, then *it breaks if dropped*] $= 1$, it follows (as a natural result) that P(*the cup breaks if dropped*) $= 1$ when the iterated conditioning is interpreted as $|_K, |_L, |_{gs}$, because (2.70) is satisfied in these cases. However, the same conclusion does not follow (which is strange) when the iterated conditioning is interpreted as $|_C$, $|_{dF}$, and $|_F$ because (2.70) does not hold in these cases.

2.6.2 Two-premise centering

We recall that the inference of *two-premise centering*, that is inferring *if A then B* from the two separate premises A and B, is p-valid ([90]). Indeed, given two events A and B it holds that

$$\{A, B\} \Rightarrow_p B|A. \tag{2.73}$$

For each $i \in \{C, dF, F, K, L, gs\}$, we will study the p-validity of the generalised version of two-premise centering, where the unconditional events A, B are replaced by the conditional events $A|H$,$B|K$, respectively, and the conditional event $B|A$ is replaced by the iterated conditional $(B|K)|_i(A|H)$. An example of this generalisation is

$\overbrace{\textit{The cup is fragile if made of glass}}^{A|H}$.

$\overbrace{\textit{The cup breaks if dropped}}^{B|K}$.

Therefore, if $\overbrace{\textit{the cup is fragile if made of glass}}^{A|H}$, then it $\overbrace{\textit{breaks if dropped}}^{B|K}$.

For each i, we consider as premise set $\{A|H, B|K\}$, as conclusion the iterated conditional $(B|K)|_i(A|H)$ and we check the p-validity of the generalised version of two-premise centering, that is

$$P(A|H) = 1,\ P(B|K) = 1 \implies \mathbb{P}[(B|K)|_i(A|H)] = 1. \tag{2.74}$$

Of course, as the assessment $(x, y) = (1, 1)$ on $\{A|H, B|K\}$ is coherent, the premise set $\{A|H, B|K\}$ is p-consistent. We recall that generalised two-premise centering is p-valid for $|_{gs}$ ([90, Equation (13)]), that is

$$\{A|H, B|K\} \Rightarrow_p (B|K)|_{gs}(A|H). \tag{2.75}$$

The previous result can be also obtained from Table 2.12, by observing that the lower bound of the coherent extensions on $(B|K)|_{gs}(A|H)$ of the assessment $x = y = 1$ is $\mu_i' = \max\{\frac{x+y-1}{x}, 0\} = 1$. Moreover, for $i \in \{C, dF, F, K, L\}$, from Table 2.12 it follows that that any value in $[0, 1]$ is a coherent extension on $(B|K)|_i(A|H)$ of $P(A|H) = P(B|K) = 1$. Thus, (2.74) is not satisfied by $|_C, |_{dF}, |_F, |_K, |_L$, and hence

$$\{A|H, B|K\} \not\Rightarrow_p (B|K)|_i(A|H), \quad i \in \{C, dF, F, K, L\}. \tag{2.76}$$

Concerning the example above, from $P(\textit{the cup is fragile if made of glass}) = 1$ and $P(\textit{the cup breaks if dropped}) = 1$ it follows that $\mathbb{P}[$If *the cup is fragile if made of glass*, then *it breaks if dropped*$] = 1$, when the iterated conditioning is interpreted as $|_{gs}$, because the generalised two-premise centering is p-valid. However, the same conclusion does not follow when the iterated conditioning is interpreted as $|_C, |_{dF}, |_F, |_K, |_L$, because the generalized two-premise centering is p-invalid in these cases.

Chapter 3

Generalised conjunction and disjunction of two conditional events in the setting of conditional random quantities

The results of this chapter are taken from [20].

The choice of going from definitions of conjunctions in three valued logics to five valued objects is justified by the search for a structure able to generalise the conjunction of two events by also maintaining similar properties. However, one might wonder whether the choice to take the values of the probabilities of the two conditional events when one of them is true and the other undefined is actually too restrictive. In other words, could the object $(A|H) \wedge_{gs} (B|K)$ be just a particular case of a more general definition of conjunction that allows more values in the 'partially' true cases, obtained when a conjunct is true and the other is void ([13]), while also preserving all the logic and probabilistic properties that we would like to have? In this chapter, to answer this question we consider what we called "generalised conjunction" $(A|H) \wedge_{a,b} (B|K)$ that is defined by means of two arbitrary values $a, b \in [0,1]$, which may be related to the probabilities x and y of the two conditional events. In particular, in Section 3.1 we give the definition of generalised conjunction of conditional events $(A|H) \wedge_{a,b} (B|K)$, its negation and the generalise disjunction $(A|H) \vee_{a,b} (B|K)$. Then we introduce the generalised De Morgan's law and we analyse the Sum rule for these new objects by also proving that it is satisfied only for $a = P(A|H)$ and $b = P(B|K)$. In Section 3.2 we present the main results which characterize the set of all coherent prevision assessments, first on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$, and then on the family $\mathcal{F} = \{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$. In Corollary 2 and Corollary 4 we show the relation between the satisfiability of the Fréchet-Hoeffding bounds and the choice of $a = x = P(A|H)$ and $b = y = P(B|K)$. In Section 3.3 both for $\wedge_{a,b}$ and $\vee_{a,b}$ we analyse the case in which H and K are incompatible, i.e $HK = \emptyset$, and then, we consider specific choices for the parameters a and b in which the multi-valued objects reduce to trivalent ones. In particular, we take into account as parameters' values the pairs $(a,b) = (0,0)$, $(a,b) = (1,1)$ and $(a,b) = (p,p)$ with $p = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ for the conjunction and $p = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ for the disjunction, for which we find well-known definitions of conjunctions and disjunctions in three-valued logics, further emphasizing the idea that that our structures provide a more general setting for the study of operations among conditionals events. Then, in Section 3.4 for a more general analysis of the coherence on the generalised conjunction and disjunction, we study the case of interval-valued prevision assessments. We deepen further aspects on the generalised conjunction in Section 3.5, in particular concerning a possible interpretation and the analysis of the inclusion relation $A|H \subseteq B|K$.

3.1 Generalised operations for two conditional events

In this section we first generalise the notion of conjunction between two conditional events recalled in Definition 3 by replacing $x = P(A|H)$ and $y = P(B|K)$ with two arbitrary values a, b in $[0,1]$, then we analyse it in the framework of the betting scheme, and we define its negation. Then, following the same idea we generalise the disjunction in Definition 5 and we study it in a coherent betting framework. Moreover, in order to check if these objects are consistent with classical probabilistic and logical properties, we give a generalisation of De Morgan's laws and the Sum rule for these new random quantities.

Definition 14. *Given four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0,1]$, we define the* generalised conjunction *w.r.t. a and b of the conditional events $A|H$ and $B|K$ as the following conditional random quantity*

$$(A|H) \wedge_{a,b} (B|K) = (AHBK + a\overline{H}BK + bAH\overline{K})|(H \vee K). \tag{3.1}$$

By the linearity of prevision, from (3.1), it follows that

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = P(AHBK|(H \vee K)) + aP(\overline{H}BK|(H \vee K)) + bP(AH\overline{K}|(H \vee K)). \tag{3.2}$$

By exploiting (1.2), by setting $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, it holds that

$$(A|H) \wedge_{a,b} (B|K) = AHBK + a\overline{H}BK + bAH\overline{K} + z(H \vee K). \tag{3.3}$$

In the betting framework, if you assess $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, then you agree to pay z, by receiving the random amount given in (3.3), that is

$$(A|H) \wedge_{a,b} (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ a, & \text{if } \overline{H}BK \text{ is true,} \\ b, & \text{if } AH\overline{K} \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

In other words, you agree to pay z in order to receive:

- 1, if both conditional events $A|H$ and $B|K$ are true;
- 0, if $A|H$ or $B|K$ is false;
- a, if $A|H$ is void and $B|K$ is true ;
- b, if $A|H$ is true and $B|K$ is void;
- z, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void.

We observe that $(A|H) \wedge_{a,b} (B|K)$, when $H \vee K$ is true, assumes values in $[0,1]$. Then, by coherence (see Remark 1) it must be $z \in [0,1]$ and hence $(A|H) \wedge_{a,b} (B|K) \in [0,1]$. Of course, $(A|H) \wedge_{a,b} (B|K) = (B|K) \wedge_{b,a} (A|H)$. We also observe that, if $H = K$, then $\overline{H}BK = AH\overline{K} = \emptyset$, and hence, for each pair (a,b), it holds that

$$(A|H) \wedge_{a,b} (B|H) = AB|H.$$

Remark 30. *When we assess $P(A|H) = x = a$ and $P(B|K) = y = b$, from Definition 3 and Definition 14, and using that $(A|H) \wedge (B|K) = (B|K) \wedge (A|H)$, it holds that*

$$(A|H) \wedge_{x,y} (B|K) = (A|H) \wedge_{gs} (B|K) = (B|K) \wedge_{gs} (A|H) = (B|K) \wedge_{y,x} (A|H),$$

that is both $(A|H) \wedge_{a,b} (B|K)$ and $(B|K) \wedge_{b,a} (A|H)$ reduces to $(A|H) \wedge_{gs} (B|K)$, when $a = x$ and $b = y$. Moreover,

$$\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = \begin{array}{l} P(AHBK|(H \vee K)) + P(A|H)P(\overline{H}BK|(H \vee K)) \\ +P(B|K)P(AH\overline{K}|(H \vee K)). \end{array} \tag{3.4}$$

We also notice that, when $x \leqslant a, y \leqslant b$ it holds that $(A|H) \wedge_{gs} (B|K) - (A|H) \wedge_{a,b} (B|K) = (x-a)(\bar{H}BK|H \vee K) + (y-b)(\bar{K}AH|H \vee K) \leqslant 0$, and hence, by coherence, $\mathbb{P}[(A|H) \wedge_{gs} (B|K)] \leqslant \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

We want now to define the negation for this new structure. Basic probability teaches us that, given an event A, the probability of its negation $\overline{A}$, is $P(\overline{A}) = 1 - P(A)$. The idea is to leave this relationship unchanged while also exploiting the linearity of the prevision. Then we have the following

Definition 15. *Let us consider four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0,1]$. Given the generalised conjunction $(A|H) \wedge_{a,b} (B|K)$ of the conditional events $A|H$ and $B|K$, we define its negation as the following conditional random quantity*

$$\overline{(A|H) \wedge_{a,b} (B|K)} = 1 - (A|H) \wedge_{a,b} (B|K).$$

We observe that

$$\mathbb{P}[\overline{(A|H) \wedge_{a,b} (B|K)}] = 1 - \mathbb{P}[(A|H) \wedge_{a,b} (B|K)].$$

Following the same philosophy used in Definition 14 to generalise to conjunction in Definition 3, considering the disjunction of two conditional events in the framework of conditional random quantities recalled in Definition 5, we define the generalised disjunction as follows

Definition 16. *Given four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0,1]$, we define the* generalised disjunction w.r.t. a and b *of the conditional events $A|H$ and $B|K$ as the following conditional random quantity*

$$(A|H) \vee_{a,b} (B|K) = (AH \vee BK + a\overline{H}\,\overline{B}K + b\overline{A}H\overline{K})|(H \vee K). \tag{3.5}$$

By exploiting (1.2), by setting $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$, it holds that

$$(A|H) \vee_{a,b} (B|K) = (AH \vee BK) + a\overline{H}\,\overline{B}K + b\overline{A}H\overline{K} + w(H \vee K). \tag{3.6}$$

In the betting framework, if you assess $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$, then you agree to pay w, by receiving the random amount

$$(A|H) \vee_{a,b} (B|K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \overline{A}H\overline{B}K \text{ is true,} \\ a, & \text{if } \overline{H}\,\overline{B}K \text{ is true,} \\ b, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\ w, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

In other words, you agree to pay w in order to receive:

- 1, if one of the conditional events $A|H$ or $B|K$ is true;
- 0, if both $A|H$ and $B|K$ are false;
- a, if $A|H$ is void and $B|K$ is false ;
- b, if $A|H$ is false and $B|K$ is void;
- w, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void.

Therefore

$$\begin{aligned}\mathbb{P}[(A|H) \vee_{a,b} (B|K)] = \quad & P((AH \vee BK)|(H \vee K)) + aP(\overline{H}\,\overline{B}K|(H \vee K)) \\ & +bP(\overline{A}H\overline{K}|(H \vee K)).\end{aligned} \tag{3.7}$$

We observe that $(A|H) \vee_{a,b} (B|K)$, when $H \vee K$ is true, assumes values in $[0, 1]$. Then, by coherence (see Remark 1) it must be $w \in [0, 1]$ and hence $(A|H) \vee_{a,b} (B|K) \in [0, 1]$. Of course it holds that $(A|H) \vee_{a,b} (B|K) = (B|K) \vee_{b,a} (A|H)$.

Remark 31. *It can be easily observed that when we take $(a, b) = (x, y)$, it holds that $(A|H) \vee_{a,b} (B|K) = (B|K) \vee_{b,a} (A|H) = (A|H) \vee_{gs} (B|K)$, where the latter disjunction is the one defined in Definition 5.*

3.1.1 Generalised De Morgan's law and sum rule

Two classical relations linking the conjunction and disjunction of events are De Morgan's Laws and the so called Sum Rule. We want now to check if some generalisations of these rules are satisfied also by the new structures $\wedge_{a,b}$ and $\vee_{c,d}$.

We begin by recalling that, given two basic events A and B by De Morgan's law it holds that $A \vee B = \overline{(\overline{A} \wedge \overline{B})}$. On the basis of the previous relation, we seek to find a similar one linking $\wedge_{a,b}$, its negation and generalised disjunction $\vee_{c,d}$, for some $a, b, c, d \in [0, 1]$. Then, also using Definition 15, we consider the following generalised De Morgan law

$$(A|H) \vee_{c,d} (B|K) = \overline{(\bar{A}|H) \wedge_{a,b} (\bar{B}|K)} = 1 - (\bar{A}|H) \wedge_{a,b} (\bar{B}|K). \tag{3.8}$$

By Definition 14, by replacing A with $\bar{A}$ and B with $\bar{B}$, if follows that

$$(\overline{A}|H) \wedge_{a,b} (\overline{B}|K) = (\overline{A}H\overline{B}K + a\overline{H}\,\overline{B}K + b\overline{A}H\overline{K})|(H \vee K).$$

Looking at Table 3.1 and setting $c = 1 - a$ and $d = 1 - b$, we obtain that $(A|H) \vee_{1-a,1-b} (B|K) = 1 - (\bar{A}|H) \wedge_{a,b} (\bar{B}|K)$, when $H \vee K$ is true. Then by Theorem 1, they also coincide when $\overline{H}\ \overline{K}$ is true, that is $w = 1 - z$, where $z = \mathbb{P}[(\overline{A}|H) \wedge_{a,b} (\overline{B}|K)]$. Therefore, formula

$$(A|H) \vee_{a,b} (B|K) = 1 - (\bar{A}|H) \wedge_{1-a,1-b} (\bar{B}|K), \tag{3.9}$$

holds in all cases and hence

$$\mathbb{P}[(A|H) \vee_{a,b} (B|K)] = 1 - \mathbb{P}[(\bar{A}|H) \wedge_{1-a,1-b} (\bar{B}|K)]. \tag{3.10}$$

Another important rule that characterizes the relation between the conjunction and the disjunction of two basic events is the *Sum Rule*. More precisely, given two events A and B it holds that $A + B = AB + A \vee B$ (and that $P(A) + P(B) = P(AB) +$

Const.	$A\|H$	$B\|K$	$(A\|H) \wedge_{a,b} (B\|K)$	$(\bar{A}\|H) \wedge_{a,b} (\bar{B}\|K)$	$1-(\bar{A}\|H) \wedge_{a,b} (\bar{B}\|K)$	$(A\|H) \vee_{c,d} (B\|K)$
$AHBK$	1	1	1	0	1	1
$AH\bar{B}K$	1	0	0	0	1	1
$AH\bar{K}$	1	y	b	0	1	1
$\bar{A}HBK$	0	1	0	0	1	1
$\bar{A}H\bar{B}K$	0	0	0	1	0	0
$\bar{A}H\bar{K}$	0	y	0	b	$1-b$	d
$\bar{H}BK$	x	1	a	0	1	1
$\bar{H}\bar{B}K$	x	0	0	a	$1-a$	c
$\bar{H}\bar{K}$	x	y	$z_{a,b}$	z	$1-z$	w

TABLE 3.1: Numerical values of the events $A|K, B|K$ and of the conditional random quantities $(A|H) \wedge_{a,b} (B|K)$, $(\bar{A}|H) \wedge_{a,b} (\bar{B}|K)$, $1-(\bar{A}|H) \wedge_{a,b} (\bar{B}|K)$ and $(A|H) \vee_{c,d} (B|K)$, where $x = P(A|H)$, $y = P(B|K)$, $z_{a,b} = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, $z = \mathbb{P}[(\bar{A}|H) \wedge_{a,b} (\bar{B}|K)]$, and $w = \mathbb{P}[(A|H) \vee_{c,d} (B|K)]$.

$P(A \vee B)$). We want to generalise this relation to conditional events $A|H$ and $B|K$, that is, we want to verify if, for some values of $a,b,c,d \in [0,1]$, it happens that $A|H + B|K = (A|H) \wedge_{a,b} (B|K) + (A|H) \vee_{c,d} (B|K)$.

We have the following result.

Theorem 22. *Let A, B, H, K be any logically independent events. Let $(x,y) \in [0,1]^2$ be a coherent assessment on $\{A|H, B|K\}$. Given any numbers $a,b,c,d \in [0,1]$ we have that*

$$A|H + B|K = (A|H) \wedge_{a,b} (B|K) + (A|H) \vee_{c,d} (B|K) \iff a = c = x \text{ and } b = d = y. \tag{3.11}$$

Proof. To prove relation (3.11), we compute the values of $A|H + B|K$ and $(A|H) \wedge_{a,b} (B|K) + (A|H) \vee_{c,d} (B|K)$ looking at Table 3.1. If we want that $A|H + B|K = (A|H) \wedge_{a,b} (B|K) + (A|H) \vee_{c,d} (B|K)$ then from the constituent $AH\bar{K}$, it must be $1+y = b+1 \Leftrightarrow y = b$, from $\bar{A}H\bar{K}$, $0+y = 0+d \Leftrightarrow y = d$, from $\bar{H}BK$, $x+1 = a+1 \Leftrightarrow x = a$, and from $\bar{H}\bar{B}K$, we have that $x+0 = 0+c \Leftrightarrow x = b$. Therefore, when $H \vee K$ is true, $A|H + B|K = (A|H) \wedge_{a,b} (B|K) + (A|H) \vee_{c,d} (B|K)$ if and only if $a = c = x$ and $b = d = y$. We observe that $A|H + B|K = (AH + x\bar{H} + BK + y\bar{K})|(H \vee K)$, with $\mathbb{P}[(AH + x\bar{H} + BK + y\bar{K})|(H \vee K)] = x + y$ (see [53, Theorem 2]). Moreover, as both the objects $(A|H) \wedge_{a,b} (B|K)$ and $(A|H) \vee_{c,d} (B|K)$ are conditional random quantities with the same conditioning event $H \vee K$, it holds that $(A|H) \wedge_{a,b} (B|K) + (A|H) \vee_{c,d} (B|K)$ is also a conditional random quantity conditioned to $H \vee K$. Moreover, when $a = c = x$ and $b = d = y$, by Theorem 1, it follows that $(AH + x\bar{H} + BK + y\bar{K})|(H \vee K)$ and $(A|H) \wedge_{a,b} (B|K) + (A|H) \vee_{c,d} (B|K)$, coincide when $H \vee K$ is false. Then eq. (3.11) is verified. □

Remark 32. *From Theorem 22, we have that*

$$A|H + B|K = (A|H) \wedge_{a,b} (B|K) + (A|H) \vee_{c,d} (B|K) \iff (\wedge_{a,b}, \vee_{c,d}) = (\wedge_{gs}, \vee_{gs})$$

where $\wedge_{gs}$ and $\vee_{gs}$ are as in Definition 3 and Definition 5, respectively. Then, the pair $(\wedge_{gs}, \vee_{gs})$ is the only one that satisfies the generalised version of the sum rule and we have that $P(A|H) + P(B|K) = \mathbb{P}[(A|H) \wedge_{gs} (B|K)] + \mathbb{P}[((A|H) \vee_{gs} (B|K)]$.

3.2 Sets of coherent assessments

In this section we study the sets of coherent assessments on the families $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ in Theorem 23, and $\{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$ in Theorem 24. Moreover, both for the generalised conjunction and for the generalised disjunction, we observe that the Fréchet-Hoeffding bounds are satisfied for every $(x, y) \in [0, 1]^2$ only for $(a, b) = (x, y)$, that is for the conjunction and disjunction of conditional events defined in Definition 3 and Definition 5, respectively.

3.2.1 Generalised conjunction and coherence

We begin by analysing the coherent assessments on the generalised conjunction.

Theorem 23. *Let A, B, H, K be any logically independent events. A prevision assessment $\mathcal{P} = (x, y, z)$ on the family of conditional random quantities $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where*

$$z' = \begin{cases} (x + y - 1) \cdot \min\{\frac{a}{x}, \frac{b}{y}, 1\}, & \text{if } x + y - 1 > 0, \\ 0, & \text{otherwise} \end{cases} \tag{3.12}$$

and

$$z'' = \max\{z''_1, z''_2, \min\{z''_3, z''_4\}\}, \tag{3.13}$$

where

$$z''_1 = \min\{x, y\},$$

$$z''_2 = \begin{cases} \dfrac{x(b - ay) + y(a - bx)}{1 - xy}, & \text{if } (x, y) \neq (1, 1), \\ 1, & \text{if } (x, y) = (1, 1), \end{cases}$$

$$z''_3 = \begin{cases} \dfrac{x(1 - a) + y(a - x)}{1 - x}, & \text{if } x \neq 1, \\ 1, & \text{if } x = 1, \end{cases}$$

$$z''_4 = \begin{cases} \dfrac{x(b - y) + y(1 - b)}{1 - y}, & \text{if } y \neq 1, \\ 1, & \text{if } y = 1. \end{cases}$$

Proof. First of all we observe that, by logical independence of A, H, B, K, the assessment (x, y) is coherent for every $(x, y) \in [0, 1]^2$. The constituents C_h's contained in $H \vee K$ and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are $C_1 = AHBK, C_2 = AH\overline{B}K, C_3 = \overline{A}HBK, C_4 = \overline{A}H\overline{B}K, C_5 = AH\overline{K}, C_6 = \overline{H}BK, C_7 = \overline{A}H\overline{K}, C_8 = \overline{H}\,\overline{B}K$ and $Q_1 = (1, 1, 1), Q_2 = (1, 0, 0), Q_3 = (0, 1, 0), Q_4 = (0, 0, 0), Q_5 = (1, y, b), Q_6 = (x, 1, a), Q_7 = (0, y, 0), Q_8 = (x, 0, 0)$. Considering the convex hull $\mathcal{I}$ of $Q_1, \ldots, Q_8$, the coherence of $\mathcal{P}$ requires that the condition $\mathcal{P} \in \mathcal{I}$ be satisfied, that is

$$\mathcal{P} = \textstyle\sum_{h=1}^{8} \lambda_h Q_h, \ \sum_{h=1}^{8} \lambda_h = 1, \ \lambda_h \geqslant 0, \ h = 1, \ldots, 8\,.$$

We observe that $Q_7 = yQ_3 + (1 - y)Q_4$ and $Q_8 = xQ_2 + (1 - x)Q_4$. Then, $\mathcal{I}$ is the convex hull of $Q_1, \ldots, Q_6$. Thus, the condition $\mathcal{P} \in \mathcal{I}$ amounts to the solvability of the following system in the unknowns $\lambda_1, \ldots, \lambda_6$

$$\mathcal{P} = \textstyle\sum_{h=1}^{6} \lambda_h Q_h, \ \sum_{h=1}^{6} \lambda_h = 1, \ \lambda_h \geqslant 0, \ h = 1, \ldots, 6 \tag{3.14}$$

that is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_5 + x\lambda_6 = x, \\ \lambda_1 + \lambda_3 + y\lambda_5 + \lambda_6 = y, \\ \lambda_1 + b\lambda_5 + a\lambda_6 = z, \\ \lambda_1 + \cdots + \lambda_6 = 1,\ \lambda_i \geqslant 0,\ \forall i = 1, \ldots, 6. \end{cases} \tag{3.15}$$

For each solution $\Lambda = (\lambda_1, \ldots, \lambda_6)$ of system (3.15) we have that the functions Φ_j defined in (1.7) are

$$\begin{aligned} \Phi_1(\Lambda) &= \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \\ \Phi_2(\Lambda) &= \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, \\ \Phi_3(\Lambda) &= \textstyle\sum_{h:C_h \subseteq H \vee K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6. \end{aligned} \tag{3.16}$$

For each given $(x, y) \in [0,1]^2$, based on Theorem 4 we determine the lower and upper bounds z', z'' for the coherent extension $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

Lower bound We distinguish two cases: **(A)** $x + y - 1 \leqslant 0$; **(B)** $x + y - 1 > 0$.

(A) We show that $z' = 0$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ by proving that the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent. Let a pair $(x, y) \in [0,1]^2$ be given. We first observe that $\mathcal{P} = (x, y, 0) = xQ_2 + yQ_3 + (1 - x - y)Q_4$. Then, $\mathcal{P} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of $Q_1, \ldots, Q_6$, with a solution of (3.15) given by $\Lambda = (0, x, y, 1 - x - y, 0, 0)$. For the functions Φ_j given in (3.16) it holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. Then, from Remark 4, the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent. Thus, for every $(x, y) \in [0,1]^2$ the assessment $(x, y, 0)$ is coherent and hence $z' = 0$.

(B) As $x + y - 1 > 0$, it holds that $x \neq 0$ and $y \neq 0$. We consider three subcases: $(B.1)$ $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = 1$; $(B.2)$ $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}$; $(B.3)$ $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{b}{y}$.

Case (B.1). We observe that $a \geqslant x$ and $b \geqslant y$. We first prove that the assessment $(x, y, x + y - 1)$ is coherent. Then, we show that $z' = x + y - 1$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$. As $\mathcal{P} = (x, y, x + y - 1) = (x + y - 1)Q_1 + (1 - y)Q_2 + (1 - x)Q_3$, it follows that $\mathcal{P} \in \mathcal{I}$. Then, a solution of system (3.15) is given by $\Lambda = (x + y - 1, 1 - y, 1 - x, 0, 0, 0)$. From (3.16) it holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. Then, from Remark 4, the assessment $(x, y, x + y - 1)$ on $\mathcal{F}$ is coherent.

In order to prove that $z' = x + y - 1$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, we verify that the assessment $\mathcal{P} = (x, y, z)$, with $(x, y) \in [0,1]^2$ and $z < z' = x + y - 1$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_1, Q_2, Q_3 belong to the plane $\pi : X + Y - Z = 1$. We set $f(X, Y, Z) = X + Y - Z - 1$ and we obtain $f(Q_1) = f(Q_2) = f(Q_3) = 0$, $f(Q_4) = -1 < 0$, $f(Q_5) = y - b \leqslant 0$, $f(Q_6) = x - a \leqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z < x + y - 1$, it holds that

$$f(\mathcal{P}) = x + y - 1 - z > 0 \geqslant f(Q_h),\ h = 1, \ldots, 6,$$

and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ linear convex combination of $Q_1, \ldots, Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Thus, the lower bound for z is $z' = x + y - 1$, for every $(x, y) \in [0,1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = 1$.

Case (B.2). We notice that $a \leqslant x$ and $\frac{a}{x} \leqslant \frac{b}{y}$.
We show that in this case

$$z' = (x+y-1)\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}(x+y-1).$$

We first prove that $(x, y, \frac{a}{x}(x+y-1))$ is coherent. Indeed, we observe that $\mathcal{P} = (x, y, \frac{a}{x}(x+y-1)) = (1-y)Q_2 + \frac{(1-x)(1-y)}{x}Q_3 + \frac{x+y-1}{x}Q_6$. Then, $\mathcal{P} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of $Q_1, \ldots, Q_6$, with a solution of (3.15) given by $\Lambda = (0, 1-y, \frac{(1-x)(1-y)}{x}, 0, 0, \frac{x+y-1}{x})$. By recalling (3.16), it holds that $\Phi_1(\Lambda) = \frac{1-y}{x}$, $\Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. We distinguish two cases: (i) $y \neq 1$, (ii) $y = 1$. In the case (i) we get $\Phi_j(\Lambda) > 0$, $j = 1, 2, 3$ and hence by Remark 4 it follows that the assessment $(x, y, \frac{a}{x}(x+y-1))$ is coherent. In case (ii), as $\Phi_1(\Lambda) = 0$, it holds that $\mathcal{I}_0 \subseteq \{1\}$, with the sub-assessment $\mathcal{P}_0 = x$ on $\mathcal{F}_0 = \{A|H\}$ coherent because $x \in [0, 1]$. Then, by Theorem 2, the assessment $(x, y, \frac{a}{x}(x+y-1)) = (x, 1, a)$ on $\mathcal{F}$ is coherent. Thus, in this sub-case the assessment $(x, y, \frac{a}{x}(x+y-1))$ on $\mathcal{F}$ is coherent for every $(x, y) \in [0, 1]^2$. In order to prove that $\frac{a}{x}(x+y-1)$ is the lower bound z' for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, we verify that (x, y, z), with $(x, y) \in [0, 1]^2$ and $z < \frac{a}{x}(x+y-1)$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_2, Q_3, Q_6 belong to the plane $\pi : aX + aY - xZ = a$. We set $f(X, Y, Z) = a(X+Y-1) - xZ$ and we obtain $f(Q_2) = f(Q_3) = f(Q_6) = 0$, $f(Q_1) = a - x \leqslant 0$, $f(Q_4) = -a \leqslant 0$, $f(Q_5) = ay - bx \leqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z < \frac{a}{x}(x+y-1)$, it holds that $f(\mathcal{P}) = f(x, y, z) = a(x+y-1) - xz > 0 \geqslant f(Q_h)$, $h = 1, \ldots, 6$, and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ linear convex combination of $Q_1, \ldots, Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Thus, in this sub-case the lower bound for $z = \mathbb{P}((A|H) \wedge_{a,b} (B|K))$ is $z' = \frac{a}{x}(x+y-1)$, for every $(x, y) \in [0, 1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}$.

Case (B.3). We notice that $b \leqslant y$ and $\frac{a}{x} \geqslant \frac{b}{y}$. We prove that $(x, y, \frac{b}{y}(x+y-1))$ is coherent and that $z' = \frac{b}{y}(x+y-1)$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.
We observe that $\mathcal{P} = (x, y, \frac{b}{y}(x+y-1)) = \frac{(1-x)(1-y)}{y}Q_2 + (1-x)Q_3 + \frac{x+y-1}{y}Q_5$. Then, $\mathcal{P} \in \mathcal{I}$, with a solution of (3.15) given by $\Lambda = (0, \frac{(1-x)(1-y)}{y}, 1-x, 0, \frac{x+y-1}{y}, 0)$.It holds that $\Phi_1(\Lambda) = \Phi_3(\Lambda) = 1$, $\Phi_2(\Lambda) = \frac{1-x}{y}$. We distinguish two cases: (i) $x \neq 1$, (ii) $x = 1$. In the case (i) we get $\Phi_j(\Lambda) > 0$, $j = 1, 2, 3$, and hence by Remark 4, the assessment $(x, y, \frac{b}{y}(x+y-1))$ is coherent. In the case (ii) we get $\mathcal{I}_0 \subseteq \{2\}$, with the sub-assessment $\mathcal{P}_0 = y$ on $\mathcal{F}_0 = \{B|K\}$ coherent because $y \in [0, 1]$. Then, by Theorem 2, the assessment $(x, y, \frac{b}{y}(x+y-1)) = (1, y, b)$ on $\mathcal{F}$ is coherent. Thus, the assessment $(x, y, \frac{b}{y}(x+y-1))$ on $\mathcal{F}$ is coherent for every $(x, y) \in [0, 1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{b}{y}$. In order to prove that $\frac{b}{y}(x+y-1)$ is the lower bound z' for $z = \mathbb{P}((A|H) \wedge_{a,b} (B|K))$, we verify that (x, y, z), with $(x, y) \in [0, 1]^2$ and $z < \frac{b}{y}(x+y-1)$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_2, Q_3, Q_5 belong to the plane $\pi : bX + bY - yZ = b$. We set $f(X, Y, Z) = b(X+Y-1) - yZ$ and we obtain $f(Q_2) = f(Q_3) = f(Q_5) = 0$, $f(Q_1) = b - y < 0$, $f(Q_4) = -b \leqslant 0$,

$f(Q_5) = bx - ay \leqslant y$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z < \frac{b}{y}(x + y - 1)$, it holds that $f(\mathcal{P}) = f(x, y, z) = b(x + y - 1) - yz > 0 \geqslant f(Q_h)$, $h = 1, \ldots, 6$, and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ linear convex combination of $Q_1, \ldots, Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Thus, the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ is $z' = \frac{b}{y}(x + y - 1)$.

Therefore, for every $(x, y) \in [0, 1]^2$ the value z' given in formula (3.12) is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

Upper bound We distinguish two cases: **(C)** $a(1 - y) + b(1 - x) + xy - 1 > 0$; **(D)** $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$.

(C) We first prove that
$$\max\{z_1'', z_2'', \min\{z_3'', z_4''\}\} = z_2'', \tag{3.17}$$
where z_i'', $i = 1, 2, 3, 4$, are given in (3.13). Then, we prove that (x, y, z_2'') is coherent. Finally we prove that z_2'' is the upper bound, by showing that (x, y, z) with $z > z_2''$ is not coherent. We observe that to be in case **(C)** we must have $a > x$ and $b > y$. Indeed, if it were $a \leqslant x$, as
$$\begin{aligned} &a(1 - y) + b(1 - x) + xy - 1 \leqslant x(1 - y) + b(1 - x) + xy - 1 \\ &= b(1 - x) + x - 1 = (1 - x)(b - 1) \leqslant 0, \end{aligned}$$
we would be in case **(D)**. Likewise, if it were $b \leqslant y$, as
$$\begin{aligned} &a(1 - y) + b(1 - x) + xy - 1 \leqslant a(1 - y) + y(1 - x) + xy - 1 \\ &= a(1 - y) + y - 1 = (1 - y)(a - 1) \leqslant 0, \end{aligned}$$
we would be in case **(D)**. As $a > x$ and $b > y$, it holds that $x \neq 1$ and $y \neq 1$. Moreover,
$$a - bx > x - bx = x(1 - b) \geqslant 0, \quad b - ay > y(1 - a) \geqslant 0. \tag{3.18}$$
We observe that
$$\begin{aligned} z_2'' - z_3'' &= \frac{x(b - ay) + y(a - bx)}{1 - xy} - \frac{x(1 - a) + y(a - x)}{1 - x} \\ &= \frac{x(1 - y)[a(1 - y) + b(1 - x) + xy - 1]}{(1 - xy)(1 - x)} \geqslant 0 \end{aligned} \tag{3.19}$$
and
$$\begin{aligned} z_2'' - z_4'' &= \frac{x(b - ay) + y(a - bx)}{1 - xy} - \frac{x(b - y) + y(1 - b)}{1 - y} \\ &= \frac{y(1 - x)[a(1 - y) + b(1 - x) + xy - 1]}{(1 - xy)(1 - y)} \geqslant 0. \end{aligned} \tag{3.20}$$
Then $z_2'' \geqslant z_3''$ and $z_2'' \geqslant z_4''$. Concerning the relation between z_2'' and z_1'' we distinguish two cases: (i) $x \leqslant y$; (ii) $x \geqslant y$.
(i). In this case $z_1'' = \min\{x, y\} = x$. As $a(1 - y) + b(1 - x) > 1 - xy$, it holds

that $x(a(1-y)+b(1-x)) \geqslant x(1-xy)$. Then,

$$\begin{aligned} z_2'' &= \frac{x(b-ay)+y(a-bx)}{(1-xy)} \geqslant \frac{x(b-ay)+x(a-bx)}{1-xy} = \\ &= \frac{x(a(1-y)+b(1-x))}{1-xy} \geqslant \frac{x(1-xy)}{1-xy} = x = z_1''. \end{aligned}$$

(ii) In this case $z_1'' = \min\{x,y\} = y$. As $a(1-y)+b(1-x) > 1-xy$, it holds that $y(a(1-y)+b(1-x)) \geqslant y(1-xy)$. Then,

$$\begin{aligned} z_2'' &= \frac{x(b-ay)+y(a-bx)}{(1-xy)} \geqslant \frac{y(b-ay)+y(a-bx)}{1-xy} \\ &= \frac{y(a(1-y)+b(1-x))}{1-xy} \geqslant \frac{y(1-xy)}{1-xy} = y = z_1''. \end{aligned}$$

Thus, formula (3.17) holds.

We now prove that $(x,y,z_2'') = (x,y,\frac{x(b-ay)+y(a-bx)}{1-xy})$ is a coherent assessment on $\mathcal{F}$. We observe that $\mathcal{P} = (x,y,z_2'') = \frac{(1-x)(1-y)}{1-xy}Q_4 + \frac{x(1-y)}{1-xy}Q_5 + \frac{y(1-x)}{1-xy}Q_6$. Then, $\mathcal{P} \in \mathcal{I}$ with a solution of (3.15) given by $\Lambda = (0,0,0,\frac{(1-x)(1-y)}{1-xy},\frac{x(1-y)}{1-xy},\frac{y(1-x)}{1-xy})$. From (3.16) it follows

$$\Phi_1(\Lambda) = \frac{1-y}{1-xy},\ \Phi_2(\Lambda) = \frac{1-x}{1-xy},\ \Phi_3(\Lambda) = 1.$$

As $x < 1$ and $y < 1$, it holds that $\Phi_j(\Lambda) > 0$, $j = 1,2,3$. Then, by Remark 4 the assessment (x,y,z'') on $\mathcal{F}$ is coherent.

In order to prove that z_2'' is the upper bound z'' for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, we verify that (x,y,z), with $z > z''$, is not coherent because $(x,y,z) \notin \mathcal{I}$. We observe that the points Q_4, Q_5, Q_6 belong to the plane $\pi: -(b-ay)X-(a-bx)Y+(1-xy)Z = 0$. We set $f(X,Y,Z) = -(b-ay)X-(a-bx)Y+(1-xy)Z$. It holds that $f(Q_4) = f(Q_5) = f(Q_6) = 0$, $f(Q_1) = -[a(1-y)+b(1-x)+xy-1] < 0$, $f(Q_2) = -(b-ay) < 0$, $f(Q_3) = -(a-bx) < 0$. Then, for every $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ is a linear convex combination of $Q_1,\ldots,Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^{6}\lambda_h Q_h) = \sum_{h=1}^{6}\lambda_h f(Q_h) \leqslant 0$. Then, by considering $\mathcal{P} = (x,y,z)$, with $z > z_2''$, it holds that $z(1-xy) > x(b-ay)+y(a-bx)$ and hence

$$f(\mathcal{P}) = -(b-ay)x-(a-bx)y+(1-xy)z > 0.$$

Then, $\mathcal{P} = (x,y,z) \notin \mathcal{I}$. Thus, in case **(C)** $z'' = \max\{z_1'', z_2'', \min\{z_3'', z_4''\}\} = z_2''$.

(D) First of all we observe that $z_2'' \leqslant \min\{z_3'', z_4''\}$. Indeed, if $x \neq 1$ and $y \neq 1$, from (3.19) and (3.20) it holds that $z_2'' \leqslant z_3''$ and $z_2'' \leqslant z_4''$. If $x = 1$, $y \neq 1$, of course $z_2'' \leqslant z_3'' = 1$ and from (3.20) it holds that $z_2'' \leqslant z_4''$. If $y = 1$, $x \neq 1$, of course $z_2'' \leqslant z_4'' = 1$ and from (3.19) it holds that $z_2'' \leqslant z_3''$. Finally, if $x = y = 1$, then $z_2'' = z_3'' = z_4'' = 1$. Then, $\max\{z_1'', z_2'', \min\{z_3'', z_4''\}\} = \max\{z_1'', \min\{z_3'', z_4''\}\}$. Moreover, as $\frac{x(1-a)+y(a-x)}{1-x} - \frac{x(b-y)+y(1-b)}{1-y} = \frac{(y-x)[a(1-y)+b(1-x)+xy-1]}{(1-y)(1-x)}$, we obtain

$$z_3'' - z_4'' = \begin{cases} \frac{(y-x)[a(1-y)+b(1-x)+xy-1]}{(1-y)(1-x)}, \text{ if } x \neq 1 \text{ and } y \neq 1, \\ 1-b, \text{ if } x = 1 \text{ and } y \neq 1, \\ a-1, \text{ if } x \neq 1 \text{ and } y = 1, \\ 0, \text{ if } x = y = 1. \end{cases} \tag{3.21}$$

We consider the following subcases: *(D.1)* $x \leqslant y$; *(D.2)* $y < x$;

Case (D.1). From (3.21) it holds that $\min\{z''_3, z''_4\} = z''_3$. Then, $\max\{z''_1, \min\{z''_3, z''_4\}\} = \max\{x, z''_3\}$. We distinguish two cases: (i) $a \leqslant x$; (ii) $a > x$.
(i). We observe that, if $x = 1$ (and hence $y = 1$) we have that $\max\{1, z''_3\} = 1 = z''_1$. If $x \neq 1$, as

$$\begin{aligned} z''_3 - z''_1 = \frac{x(1-a) + y(a-x)}{1-x} - x = \frac{x(1-a) + y(a-x) - x(1-x)}{1-x} = \\ = \frac{a(y-x) - x(y-x)}{1-x} = \frac{(y-x)(a-x)}{1-x} \end{aligned} \tag{3.22}$$

it holds that $z''_3 \leqslant z''_1 = x$. Then $\max\{z''_1, z''_3\} = z''_1 = x$.
Now we show that the assessment $\mathcal{P} = (x, y, z''_1) = (x, y, x)$ is coherent and that the upper bound z'' for z is $z''_1 = x$. We observe that $\mathcal{P} = (x, y, x) = xQ_1 + (y-x)Q_3 + (1-y)Q_4$, then, $\mathcal{P} \in \mathcal{I}$ and a solution of (3.15) is given by $\Lambda = (x, 0, y-x, 1-y, 0, 0)$. It holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1$, then by Remark 4 the probability assessment (x, y, x) on $\mathcal{F}$ is coherent. In order to prove that $z'' = z''_1 = x$ is the upper bound for z, we verify that $\mathcal{P} = (x, y, z)$, with $z > z'' = x$, is not coherent because $\mathcal{P} \notin \mathcal{I}$. We observe that the points Q_1, Q_3, Q_4 belong to the plane $\pi : X - Z = 0$. We set $f(X, Y, Z) = X - Z$ and we obtain $f(Q_1) = f(Q_3) = f(Q_4) = 0$, $f(Q_2) = 1 > 0$, $f(Q_5) = 1 - b \geqslant 0$, $f(Q_6) = x - a \geqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$ it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z > x$, it holds that $f(\mathcal{P}) = f(x, y, z) = x - z < 0$, and hence $\mathcal{P} \notin \mathcal{I}$. Thus, $z'' = z''_1 = x$.
(ii) As $x < a$, $x \neq 1$. In this case, from (3.22) it holds that

$$\max\{z''_1, z''_3\} = z''_3 = \frac{x(1-a) + y(a-x)}{1-x}.$$

We prove that $(x, y, z''_3) = (x, y, \frac{x(1-a)+y(a-x)}{1-x})$ is a coherent assessment on $\mathcal{F} = \{A|H, B|K, (A|H)|_{a,b}(B|K)\}$ and that $z'' = z''_3$. We observe that $\mathcal{P} = (x, y, \frac{x(1-a)+y(a-x)}{1-x}) = \frac{x(1-y)}{1-x}Q_1 + (1-y)Q_4 + \frac{y-x}{1-x}Q_6$, hence $\mathcal{P} \in \mathcal{I}$. A solution of (3.15) is given by $\Lambda = (\frac{x(1-y)}{1-x}, 0, 0, 1-y, 0, \frac{y-x}{1-x})$ and it holds that $\Phi_1(\Lambda) = \frac{1-y}{1-x}$, $\Phi_2(\Lambda) = \Phi_3(\Lambda) = 1$. We distinguish two cases: $y \neq 1$ and $y = 1$. If $y \neq 1$ we get $\Phi_1(\Lambda) > 0$ and $\Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$; then $\mathcal{I}_0 = \emptyset$ and by Remark 4 the assessment (x, y, z''_3) is coherent. If $y = 1$ we get $\mathcal{I}_0 \subseteq \{1\}$, with the sub-assessment $\mathcal{P}_0 = x$ on $\mathcal{F}_0 = \{A|H\}$ coherent because $x \in [0, 1]$. Then, by Remark 4, the assessment (x, y, z''_3) on $\mathcal{F}$ is coherent. In order to prove that $z'' = z''_3$, we verify that (x, y, z), with $z > z''_3$ is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_1, Q_4, Q_6 belong to the plane $\pi : (a-1)X + (x-a)Y + (1-x)Z = 0$. We set $f(X, Y, Z) = (a-1)X + (x-a)Y + (1-x)Z$ and we obtain $f(Q_1) = f(Q_4) = f(Q_6) = 0$, $f(Q_2) = a - 1 \leqslant 0$, $f(Q_3) = x - a < 0$, $f(Q_5) = a - 1 + y(x-a) + b(1-x) = a(1-y) + b(1-x) + xy - 1 \leqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$ it follows that $f(\mathcal{Q}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z > \frac{x(1-a)+y(a-x)}{1-x}$, i.e. $z(1-x) > x(1-a) + y(a-x)$, it holds that $f(\mathcal{P}) = f(x, y, z) = -[(1-a)x + (a-x)y] + (1-x)z > 0$ and hence $\mathcal{P} \notin \mathcal{I}$. Thus, the upper bound for $z = P((A|H) \wedge_{a,b} (B|K))$ is

$z'' = z_3'' = \frac{x(1-a)+y(a-x)}{1-x}$.

Case (D.2) From (3.21) it holds that $\min\{z_3'', z_4''\} = z_4''$. Then, $\max\{z_1'', \min\{z_3'', z_4''\}\} = \max\{y, z_4''\}$. We distinguish two cases: (j) $b \leqslant y$; (jj) $b > y$.
(j). As $y < x$, it holds that $y \neq 1$. We observe that

$$z_4'' - z_1'' = \frac{y(1-b)+x(b-y)}{1-y} - y = \frac{(x-y)(b-y)}{1-y} \tag{3.23}$$

and hence, as $y < x$ and $b \leqslant y$, $z_4'' \leqslant z_1''$. Then $\max\{z_1'', z_4''\} = z_1'' = y$. We show that $\mathcal{P} = (x, y, z_1'') = (x, y, y)$ is coherent and that the upper bound z'' for z is $z_1'' = y$. We observe that $\mathcal{P} = (x, y, y) = yQ_1 + (x-y)Q_2 + (1-x)Q_4$, then, $\mathcal{P} \in \mathcal{I}$ and a solution of (3.15) is given by $\Lambda = (y, x-y, 0, 1-x, 0, 0)$. It holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1$, then by Remark 4 the probability assessment (x, y, y) on $\mathcal{F}$ is coherent. In order to prove that $z'' = z_1'' = y$ is the upper bound for z, we verify that $\mathcal{P} = (x, y, z)$, with $z > z'' = y$, is not coherent because $\mathcal{P} \notin \mathcal{I}$. We observe that the points Q_1, Q_2, Q_4 belong to the plane $\pi : Y - Z = 0$. We set $f(X, Y, Z) = Y - Z$ and we obtain $f(Q_1) = f(Q_2) = f(Q_4) = 0$, $f(Q_3) = 1 > 0$, $f(Q_5) = y - b \geqslant 0$, $f(Q_6) = 1 - a \geqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$ it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z > y$, it holds that $f(\mathcal{P}) = f(x, y, z) = y - z < 0$, and hence $\mathcal{P} \notin \mathcal{I}$. Thus, $z'' = z_1'' = y$.
(jj) In this case, from (3.23) it holds that

$$\max\{z_1'', z_4''\} = z_4'' = \frac{x(b-y)+y(1-b)}{1-y}.$$

We prove that $(x, y, z_4'') = (x, y, \frac{x(b-y)+y(1-b)}{1-y})$ is a coherent assessment on $\mathcal{F}$ and that $z'' = z_4''$. We observe that $\mathcal{P} = (x, y, \frac{y(1-b)+x(b-y)}{1-y}) = \frac{y(1-x)}{1-y}Q_1 + (1-x)Q_4 + \frac{x-y}{1-y}Q_5$. Then, $\mathcal{P} \in \mathcal{I}$ with a solution of (3.15) given by $\Lambda = (\frac{y(1-x)}{1-y}, 0, 0, 1-x, \frac{x-y}{1-y}, 0)$. Based on (3.16) it holds that $\Phi_1(\Lambda) = \Phi_3(\Lambda) = 1 > 0$, and $\Phi_2(\Lambda) = \frac{1-x}{1-y}$. We distinguish two cases: $x \neq 1$; $x = 1$. If $x \neq 1$, we get $\Phi_j(\Lambda) > 0$, $j = 1, 2, 3$. Then $\mathcal{I}_0 = \emptyset$ and by Remark 4 the assessment (x, y, z_4'') on $\mathcal{F}$ is coherent. If $x = 1$, we get $\mathcal{I}_0 \subseteq \{2\}$, with the sub-assessment $\mathcal{P}_0 = y$ on $\mathcal{F}_0 = \{B|K\}$ coherent because $y \in [0, 1]$. Then, by Theorem 2, the assessment (x, y, z_4'') on $\mathcal{F}$ is coherent. In order to prove that $z'' = z_4'' = \frac{y(1-b)+x(b-y)}{1-y}$ is the upper bound for $z = P((A|H) \wedge_{a,b} (B|K))$, we verify that (x, y, z), with $(x, y) \in [0, 1]^2$ and $z > z_4''$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_1, Q_4, Q_5 belong to the plane $\pi : (y-b)X + (b-1)Y + (1-y)Z = 0$. We set $f(X, Y, Z) = (y-b)X + (b-1)Y + (1-y)Z$ and we obtain $f(Q_1) = f(Q_4) = f(Q_5) = 0$, $f(Q_2) = y - b < 0$, $f(Q_3) = b - 1 \leqslant 0$, $f(Q_6) = x(y-b) + (b-1) + a(1-y) = a(1-y) + b(1-x) + xy - 1 \leqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Moreover, by considering $\mathcal{P} = (x, y, z)$, with $z > \frac{x(b-y)+y(1-b)}{1-y}$, i.e. $z(1-y) > x(b-y) + y(1-b)$, it holds that $f(\mathcal{P}) = f(x, y, z) = -[(b-y)x + (1-b)y] + (1-y)z > 0$ and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Thus, the upper bound for $z = P((A|H) \wedge_{a,b} (B|K))$ is $z'' = z_4'' = \frac{x(b-y)+y(1-b)}{1-y}$.

Case	z'
(A): $x+y-1 \leqslant 0$	0
(B): $x+y-1>0$	
Sub-cases:	
(B.1):$a \geqslant x$ and $b \geqslant y$	$x+y-1$
(B.2):$a < x$ and $\frac{a}{x} \leqslant \frac{b}{y}$	$\frac{a}{x}(x+y-1)$
(B.3):$b < y$ and $\frac{b}{y} \leqslant \frac{a}{x}$	$\frac{b}{y}(x+y-1)$

TABLE 3.2: Values of the lower bound z' of $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0,1]$.

Case	z''
(C) $a(1-y)+b(1-x) > 1-xy$	$z_2'' = \frac{x(b-ay)+y(a-bx)}{1-xy}$
(D) $a(1-y)+b(1-x) \leqslant 1-xy$	
Sub-cases:	
$x \leqslant y$ and $a \leqslant x$	$z_1'' = x$
$x \leqslant y$ and $a > x$	$z_3'' = \frac{x(1-a)+y(a-x)}{1-x}$
$y < x$ and $b \leqslant y$	$z_1'' = y$
$y < x$ and $b > y$	$z_4'' = \frac{x(b-y)+y(1-b)}{1-y}$

TABLE 3.3: Values of the upper bound z'' of $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0,1]$.

Therefore, for every $(x,y) \in [0,1]^2$ the value z'' given in formula (3.13) is the upper bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

□

A summary of the different values of the lower and upper bounds z' and z'' for the prevision of $(A|H) \wedge_{a,b} (B|K)$ are given in Table 3.2 and in Table 3.3, respectively.

Moreover, from Theorem 23 it follows that

Corollary 1. *Let A, B, H, K be any logically independent events. Then, the set $\Pi_{a,b}$ of all coherent prevision assessments (x,y,z) on $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$, is*

$$\Pi_{a,b} = \{(x,y,z) : (x,y) \in [0,1]^2, z \in [z', z'']\}, \tag{3.24}$$

where z' and z'' are defined in (3.12) and (3.13), respectively.

We recall that, under logical independence, the set of all coherent assessment (x,y,z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$ is the tetrahedron $\mathcal{T}$ of points $(1,1,1), (1,0,0), (0,1,0), (0,0,0)$, that is ([53]) $\mathcal{T} = \{(x,y,z) : (x,y) \in [0,1]^2, z \in [\max\{x+y-1,0\}, \min\{x,y\}]\}$. As, from (3.12) and (3.13), $z' \leqslant \max\{x+y-1,0\}$ and $z'' \geqslant \min\{x,y\}$ for every $(a,b) \in [0,1]^2$, it holds that (see Figure 3.1)

$$\mathcal{T} \subseteq \Pi_{a,b} \;\; \forall (a,b) \in [0,1]^2. \tag{3.25}$$

From Theorem 23, we also have that the Fréchet-Hoeffding bounds are not satisfied by $(A|H) \wedge_{a,b} (B|K)$ for every $(x,y) \in [0,1]^2$, when a and b are arbitrarily chosen. For example, if we take $a > x$ and $b > y$, then from Equation (3.13), the upper bound of the coherent assessments on $(A|H) \wedge_{a,b} (B|K)$ is $z_2'' = \frac{x(b-ay)+y(a-bx)}{1-xy} \neq \min\{x,y\}$.

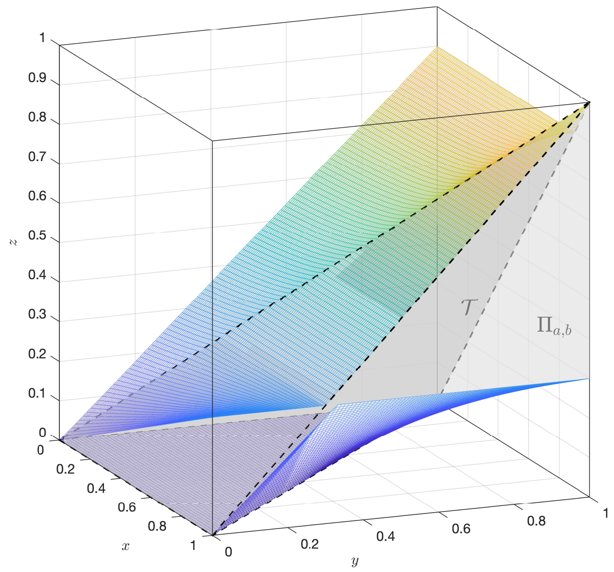

FIGURE 3.1: The gray region enclosed between the two surfaces represents the set $\Pi_{a,b}$ of all coherent prevision assessments (x, y, z) on $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$, with $a = 0.9$ and $b = 0.3$. The tetrahedron $\mathcal{T}$ is the set of all coherent prevision assessments (x, y, z) on $\{A|H, B|K, (A|H) \wedge_{gs} (B|K)\}$. It can be observed that $\mathcal{T} \subset \Pi_{a,b}$.

In the following corollary we show that the only case the Fréchet-Hoeffding bounds for the conjunction are satisfied for every $(x, y) \in [0, 1]^2$ is for $a = x$ and $b = y$, that is the only conjunction of two conditional events always coherent with these bounds is $(A|H) \wedge_{gs} (B|K)$, explicitly depending on the probabilities of the conditionals.

Corollary 2. *Let A, B, H and K be any logically independent events. The generalised conjunction $(A|H) \wedge_{a,b} (B|K)$ satisfies the Fréchet-Hoeffding bounds for every coherent assessment $(x, y) \in [0, 1]^2$ on $\{A|H, B|K\}$, if and only if $a = x$ and $b = y$.*

Proof. If $a = x$ and $b = y$, from Remark 30 we have that $(A|H) \wedge_{a,b} (B|K) = (A|H) \wedge_{gs} (B|K)$ and this object satisfies the Fréchet-Hoeffding bounds for every $(x, y) \in [0, 1]^2$.

Let us suppose now that $(A|H) \wedge_{a,b} (B|K)$ satisfies the Fréchet-Hoeffding bounds for every $(x, y) \in [0, 1]^2$ but $a \neq x$. We observe from Table 3.3 that to have the Fréchet-Hoeffding upper bound $z'' = \min\{x, y\}$ we need to have $a(1-y) + b(1-x) \leqslant 1 - xy$ and $[(x \leqslant y \quad \wedge \quad a \leqslant x) \vee (x > y \quad \wedge \quad b \leqslant y)]$. Let us consider any coherent assessment (x, y) on $\{A|H, B|K\}$ such that $a(1-y) + b(1-y) \leqslant 1 - xy$, $x \leqslant y$ and $x + y - 1 > 0$. Then, $z'' = \min\{x, y\} = x$ and $a < x$, because we supposed $a \neq x$. Having that $x + y - 1 > 0$, the Fréchet-Hoeffding lower bound for these assessments would be $z' = x + y - 1$. Nevertheless, being $\frac{a}{x} < 1$, we have that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} \neq 1$ and hence $z' = (x + y - 1)\min\{\frac{a}{x}, \frac{b}{y}, 1\} \neq x + y - 1$ so we arrive to a contradiction. Then it must be $a = x$. A similar contradiction arises if we suppose $b \neq y$, then for the Fréchet-Hoeffding bounds to be satisfied for every coherent assessment (x, y) on $\{A|H, B|K\}$ it is needed that $(a, b) = (x, y)$. $\square$

In other words Corollary 2 amounts to say that $\Pi_{a,b} = \mathcal{T}$ if and only if $a = x$ and $b = y$, for every assessment $(x, y) \in [0, 1]^2$ on $\{A|H, B|K\}$.

3.2.2 Generalised disjunction and coherence

In the next result we compute the lower and upper bounds for the prevision of the generalised disjunction.

Theorem 24. *Let A, B, H, K be any logically independent events. A prevision assessment $\mathcal{P} = (x, y, w)$ on the family of conditional random quantities $\mathcal{F} = \{A|H, (B|K), (A|H) \vee_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $w \in [w', w'']$, where*

$$w' = \min\{w'_1, w'_2, \max\{w'_3, w'_4\}\}, \tag{3.26}$$

with

$$w'_1 = \max\{x, y\},$$
$$w'_2 = \begin{cases} \dfrac{ax + by - (a + b - 1)xy}{x + y - xy}, \text{ if } (x, y) \neq (0, 0), \\ 0, \text{ if } (x, y) = (0, 0), \end{cases}$$

$$w'_3 = \begin{cases} \dfrac{a(x - y) + xy}{x}, \text{ if } x \neq 0, \\ 0, \text{ if } x = 0, \end{cases}$$

$$w'_4 = \begin{cases} \dfrac{b(y - x) + xy}{y}, \text{ if } y \neq 0, \\ 0, \text{ if } y = 0, \end{cases}$$

and

$$w'' = \begin{cases} 1 - (1 - x - y) \cdot \min\{\frac{1-a}{1-x}, \frac{1-b}{1-y}, 1\}, & \text{if } x + y < 1, \\ 1, & \text{otherwise.} \end{cases} \tag{3.27}$$

Proof. Of course the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0, 1]^2$. For each given $(x, y) \in [0, 1]^2$, based on Theorem 4, we determine the interval of the coherent extensions on $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$. We observe that the lower and upper bounds, w' and w'', for w can be easily obtained from those of the conjunction $(\bar{A}|H) \wedge_{1-a,1-b} (\bar{B}|K)$ by exploiting formula (3.9). Indeed, since $\mathbb{P}[(A|H) \vee_{a,b} (B|K)] = 1 - \mathbb{P}[(\bar{A}|H) \wedge_{1-a,1-b} (\bar{B}|K)]$, it holds that $w' = 1 - \eta''$ and $w'' = 1 - \eta'$, where η' and η'' are the lower and upper bounds for $\mathbb{P}[(\bar{A}|H) \wedge_{1-a,1-b} (\bar{B}|K)]$, respectively.

Lower bound Let us consider first the lower bound w'. For each tuple $(x, y, a, b) \in [0, 1]^4$ we denote by $z''(x, y, a, b)$ and by $z''_i(x, y, a, b)$, $i = 1, 2, 3, 4$ the values of z'' and z''_i, $i = 1, 2, 3, 4$, obtained by applying formula (3.13). We observe that $\eta'' = z''(1 - x, 1 - y, 1 - a, 1 - b)$ and hence $w' = 1 - z''(1 - x, 1 - y, 1 - a, 1 - b)$. Then, as $1 - \max\{x_1, \ldots, x_n\} = \min\{1 - x_1, \ldots, 1 - x_n\}$, it holds that

$$\begin{aligned} w' &= 1 - z''(1 - x, 1 - y, 1 - a, 1 - b) \\ &= 1 - \max\{z''_1(1 - x, 1 - y, 1 - a, 1 - b), z''_2(1 - x, 1 - y, 1 - a, 1 - b), \\ &\quad \min\{z''_3(1 - x, 1 - y, 1 - a, 1 - b), z''_4(1 - x, 1 - y, 1 - a, 1 - b)\}\} = \\ &= \min\{1 - z''_1(1 - x, 1 - y, 1 - a, 1 - b), 1 - z''_2(1 - x, 1 - y, 1 - a, 1 - b), \\ &\quad \max\{1 - z''_3(1 - x, 1 - y, 1 - a, 1 - b), 1 - z''_4(1 - x, 1 - y, 1 - a, 1 - b)\}\}, \\ &= \min\{w'_1, w'_2, \max\{w'_3, w'_4\}\}, \end{aligned}$$

where

$$w'_1 = 1 - z''_1(1-x, 1-y, 1-a, 1-b) = 1 - \min\{1-x, 1-y\} = \max\{x, y\},$$

$$w'_2 = 1 - z''_2(1-x, 1-y, 1-a, 1-b) =$$

$$= 1 - \begin{cases} \dfrac{(1-x)[1-b-(1-a)(1-y)] + (1-y)[1-a-(1-b)(1-x)]}{1-(1-x)(1-y)}, \text{ if } (1-x, 1-y) \neq (1,1), \\ 1, \text{ if } (1-x, 1-y) = (1,1), \end{cases} =$$

$$= \begin{cases} \dfrac{ax + by - (a+b-1)xy}{x+y-xy}, \text{ if } (x,y) \neq (0,0), \\ 0, \text{ if } (x,y) = (0,0), \end{cases}$$

$$w'_3 = 1 - z''_3(1-x, 1-y, 1-a, 1-b) =$$

$$= 1 - \begin{cases} \dfrac{(1-x)a + (1-y)(1-a-1+x)}{x}, \text{if } 1-x \neq 1, \\ 1, \text{ if } 1-x = 1, \end{cases} = \begin{cases} \dfrac{a(x-y)+xy}{x}, \text{if } x \neq 0, \\ 0, \text{ if } x = 0, \end{cases}$$

$$w'_4 = 1 - z''_4(1-x, 1-y, 1-a, 1-b) =$$

$$= 1 - \begin{cases} \dfrac{(1-x)(1-b-1+y) + (1-y)b}{y}, \text{if } 1-y \neq 1, \\ 1, \text{ if } 1-y = 1, \end{cases} = \begin{cases} \dfrac{b(y-x)+xy}{y}, \text{ if } y \neq 0, \\ 0, \text{ if } y = 0. \end{cases}$$

Therefore, formula (3.26) holds.

Upper bound Now, let us consider the upper bound w''. For each tuple $(x, y, a, b) \in [0,1]^4$ we denote by $z'(x, y, a, b)$ the value of z' obtained by applying formula (3.12). We observe that the lower bound η' for $\mathbb{P}[(\bar{A}|H) \wedge_{1-a,1-b} (\bar{B}|K)]$ coincides with $z'(1-x, 1-y, 1-a, 1-b)$. Then

$$\begin{aligned} w'' &= 1 - \eta' = 1 - z'(1-x, 1-y, 1-a, 1-b) = \\ &= 1 - \begin{cases} (1-x+1-y-1)\cdot\min\{\frac{1-a}{1-x}, \frac{1-b}{1-y}, 1\}, & \text{if } 1-x+1-y-1 > 0, \\ 0, & \text{otherwise}, \end{cases} = \\ &= \begin{cases} 1-(1-x-y)\cdot\min\{\frac{1-a}{1-x}, \frac{1-b}{1-y}, 1\}, & \text{if } x+y < 1, \\ 1, & \text{otherwise}. \end{cases} \end{aligned}$$

Therefore, formula (3.27) holds. □

A summary of the different values of the lower bound w' for the prevision of $(A|H) \vee_{a,b} (B|K)$ can be found in Table 3.4. They are obtained from Table 3.3 by exploiting the relation $w' = 1 - z''(1-x, 1-y, 1-a, 1-b)$. Likewise, a summary of the different values of the upper bound w'' for the prevision of $(A|H) \vee_{a,b} (B|K)$ is given in Table 3.5. These values are obtained from Table 3.2 by exploiting the relation $w'' = 1 - z'(1-x, 1-y, 1-a, 1-b)$.

Moreover, from Theorem 24 it follows that

Case	w'
(C)	
$ax+by-xy<0$	$w'_2 = \frac{ax+by-(a+b-1)xy}{x+y-xy}$
(D)	
$ax+by-xy \geqslant 0$	
Sub-cases:	
$x \geqslant y$ and $a \geqslant x$	$w'_1 = x$
$x \geqslant y$ and $a < x$	$w'_3 = \frac{a(x-y)+xy}{x}$
$y > x$ and $b \geqslant y$	$w'_1 = y$
$y > x$ and $b < y$	$w'_4 = \frac{b(y-x)+xy}{y}$

TABLE 3.4: Values of the lower bound w' of $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0,1]$.

Case	w''
(A): $x+y \geqslant 1$	1
(B): $x+y<1$	
Sub-cases:	
(B.1):$a \leqslant x$ and $b \leqslant y$	$x+y$
(B.2):$a > x$ and $\frac{1-a}{1-x} \leqslant \frac{1-b}{1-y}$	$\frac{a(1-x)+(1-a)y}{1-x}$
(B.3):$b > y$ and $\frac{1-b}{1-y} \leqslant \frac{1-a}{1-x}$	$\frac{b(1-y)+(1-b)x}{1-y}$

TABLE 3.5: Values of the upper bound w'' of $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0,1]$.

Corollary 3. *Let A, B, H, K be any logically independent events. Then, the set $\Pi'_{a,b}$ of all coherent prevision assessments (x,y,w) on $\mathcal{F} = \{A|H, (B|K), (A|H) \vee_{a,b} (B|K)\}$, is*

$$\Pi'_{a,b} = \{(x,y,w) : (x,y) \in [0,1]^2, w \in [w', w'']\}, \tag{3.28}$$

where w' and w'' are defined in eq. (3.26) and in eq. (3.27), respectively.

We recall that, under logical independence, the set of all coherent assessment (x,y,w) on $\{A|H, B|K, (A|H) \vee_{gs} (B|K)\}$ is the tetrahedron $\mathcal{T}'$ of points $(1,1,1), (1,0,1), (0,1,1), (0,0,0)$, that is $\mathcal{T}' = \{(x,y,w) : (x,y) \in [0,1]^2, w \in [\max\{x,y\}, \min\{x+y,1\}]\}$. As, from (3.26) and (3.27), $w' \leqslant \max\{x,y\}$ and $w'' \geqslant \min\{x+y,1\}$ for every $(a,b) \in [0,1]^2$, it holds that (see Figure 3.2)

$$\mathcal{T}' \subseteq \Pi'_{a,b} \;\; \forall (a,b) \in [0,1]^2.$$

From Theorem 24, we have that in general the disjunction $(A|H) \vee_{a,b} (B|K)$ does not satisfy the Fréchet-Hoeffding bounds for every $(x,y) \in [0,1]^2$, when a and b are arbitrary chosen. For example, if we take $0 \neq b = x < y$, then from Equation (3.26) the lower bound of the coherent assessments on $(A|H) \vee_{a,b} (B|K)$ is $w'_4 = \frac{a(x-y)+xy}{x} \neq \max\{x,y\}$. In the following corollary we show that the only case the Fréchet-Hoeffding bounds are satisfied for every $(x,y) \in [0,1]^2$ is for $a = x$ and $b = y$, that is the only disjunction of two conditional events always coherent with these bounds is $(A|H) \vee_{gs} (B|K)$, explicitly depending on the probabilities of the conditionals.

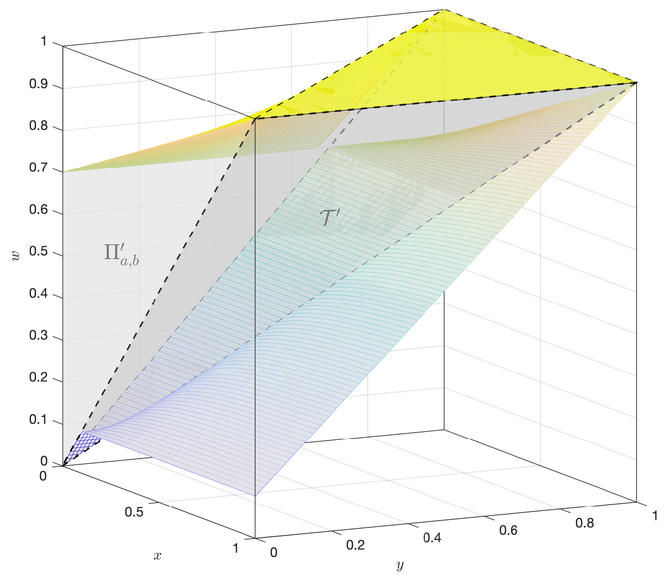

FIGURE 3.2: The gray region enclosed between the two surfaces represents the set $\Pi'_{a,b}$ of all coherent prevision assessments (x, y, w) on $\{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$, with $a = 0.1$ and $b = 0.7$. The tetrahedron $\mathcal{T}'$ of points $(1,1,1), (1,0,1), (0,1,1), (0,0,0)$ is the set of all coherent prevision assessments (x, y, w) on $\{A|H, B|K, (A|H) \vee_{gs} (B|K)\}$. It can be observed that $\mathcal{T}' \subset \Pi'_{a,b}$.

Corollary 4. *Let A, B, H and K be any logically independent events. The generalised disjunction $(A|H) \vee_{a,b} (B|K)$ satisfies the Fréchet-Hoeffding bounds for every coherent assessment $(x, y) \in [0,1]^2$ on $\{A|H, B|K\}$ if and only if $a = x$ and $b = y$.*

Proof. If $a = x$ and $b = y$, from Remark 31 we have that $(A|H) \vee_{a,b} (B|K) = (A|H) \vee_{gs} (B|K)$ and this definition of disjunction satisfies the Fréchet-Hoeffding bounds for every $(x, y) \in [0,1]^2$.

Let us suppose now that $(A|H) \vee_{a,b} (B|K)$ satisfies the Fréchet-Hoeffding bounds for every $(x, y) \in [0,1]^2$ but $a \neq x$. Consider the set of coherent assessments on $\{A|H, B|K\}$ such that $x \geqslant y$ and $x + y < 1$. Then, it holds that $w' = \max\{x, y\} = x$ and $w'' = \min\{x + y, 1\} = x + y$. From Table 3.5 to have such a lower bound we need to have $ax + by - xy \geqslant 0$ and $a \geqslant x$. Then it must be $a > x$ using that $a \neq x$. However, to have that upper bound we need to have $a < x$, and hence we have a contradiction. Then it must be $a = x$. A similar contradiction arises if we suppose $b \neq y$. Then for the Fréchet-Hoeffding bounds to be satisfied for any coherent assessment (x, y) on $\{A|H, B|K\}$ the only choice for a and b is $(a, b) = (x, y)$. $\square$

By Corollary 4 it follows that $\Pi'_{a,b} = \mathcal{T}'$ if and only if $a = x$ and $b = y$, for every assessment $(x, y) \in [0,1]^2$ on $\{A|H, B|K\}$.

We conclude this section by underlining again that if we want to extend the operation of conjunction and disjunction from basic events A, B to conditionals $A|H, B|K$ in a way that preserve similar properties, a suitable choice is to consider the structures $(A|H) \wedge_{gs} (B|K)$ (Definition 3) and $(A|H) \vee_{gs} (B|K)$ (Definition 5).

3.3 Some particular cases

In this section, first we consider some particular cases of the generalised conjunction (Definition 14) and the generalised disjunction (Definition 16) obtained when the conditioning events H and K are incompatible. Then we analyse these structures for some given values of a and b in which the generalised objects reduce to conditional events. More precisely, considering $a, b \in \{0, 1, p\}$, with $p = z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ in the case of the conjunction, and $p = w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ in the case of the disjunction, we analyse the particular cases $(a, b) = (0, 0)$, $(a, b) = (1, 1)$, and $(a, b) = (p, p)$.

3.3.1 The case H and K incompatible

We analyse the generalised conjunction in the particular case where the conditioning events H and K are incompatible, i.e. $HK = \emptyset$.

Theorem 25. *Let $A|H$, $B|K$, be two conditional events with A, H, B logically independent, A, B, K logically independent, and $HK = \emptyset$. A prevision assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where*

$$z' = \min\{ay, bx\}, \quad z'' = \max\{ay, bx\}. \tag{3.29}$$

Proof. The constituents and the points Q_h's associated with the pair $(\mathcal{F}, \mathcal{P})$ are $C_1 = AH\overline{K}$, $C_2 = \overline{H}BK$, $C_3 = \overline{A}H\overline{K}$, $C_4 = \overline{H}\,\overline{B}K$, $C_0 = \overline{H}\,\overline{K}$, and $Q_1 = (1, y, b)$, $Q_2 = (x, 1, a)$, $Q_3 = (0, y, 0), Q_4 = (x, 0, 0)$, $Q_0 = \mathcal{P} = (x, y, z)$. We observe that $\mathcal{H}_n = H \vee K$ and that $\mathcal{I}$ is the convex hull of points $Q_1, \ldots Q_4$. For each given assessment (x, y) based on Theorem 4 we determine the lower and upper bounds z', z'' for the coherent extension $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$. We distinguish two cases: $(i) ay \leqslant bx$, $(ii) bx < ay$. *Case* (i). We show that the assessment (x, y, z) on $\mathcal{F}$ is coherent if and only $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = ay$ and $z'' = bx$.

Lower bound First we prove that (x, y, ay), is coherent, and then that $z' = ay$ is the lower bound for z. We observe that $\mathcal{P} = (x, y, ay) = yQ_2 + (1 - y)Q_4$. Then, $\mathcal{P} \in \mathcal{I}$ with a solution of (3.15) given by $\Lambda = (0, y, 0, 1 - y)$. It follows that

$$\begin{aligned}
\Phi_1(\Lambda) &= \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_3 = 0,\\
\Phi_2(\Lambda) &= \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_2 + \lambda_4 = 1,\\
\Phi_3(\Lambda) &= \textstyle\sum_{h:C_h \subseteq H \vee K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1.
\end{aligned}$$

As $\Phi_1(\Lambda) = 0$, it holds that $\mathcal{I}_0 \subseteq \{1\}$, with the sub-assessment $\mathcal{P}_0 = x$ on $\mathcal{F}_0 = \{A|H\}$ coherent because $x \in [0, 1]$. Then, by Theorem 2 the assessment $\mathcal{P} = (x, y, ay)$ on $\mathcal{F}$ is coherent. Now we show that $z' = ay$ is the lower bound. Of course, if $x = 0$, as $ay \leqslant bx = 0$, it holds that $ay = 0$. In this case we have that $z' = 0$ is the lower bound because $(A|H) \wedge_{a,b} (B|K) \in [0, 1]$. We assume now that $x \neq 0$. We observe that Q_2, Q_3, Q_4 belong to the plane $\pi : ayX + axY - xZ - axy = 0$. By setting $f(X, Y, Z) = ayX + axY - xZ - axy$, it holds that

$$f(Q_1) = ay - bx \leqslant 0, \quad f(Q_2) = f(Q_3) = f(Q_4) = 0.$$

Then, for every $\mathcal{P} \in \mathcal{I}$ it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{4} \lambda_h Q_h) = \sum_{h=1}^{4} \lambda_h f(Q_h) \leqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z < ay$, it holds that $f(\mathcal{P}) = f(x, y, z) = x(ay - z) > 0$, and hence $\mathcal{P} \notin \mathcal{I}$. Thus, as (x, y, ay) is coherent and (x, y, z), with $z < ay$, is not coherent, it follows that $z' = ay$ is the lower bound for $(A|H) \wedge_{a,b} (B|K)$.

Upper bound We show that the assessment $\mathcal{P} = (x, y, bx)$ on $\mathcal{F}$ is coherent. We observe that $(x, y, bx) = xQ_1 + (1 - x)Q_3$. Then, $\mathcal{P} \in \mathcal{I}$ with a solution of (3.15) given by $\Lambda = (x, 0, 1-x, 0)$. It follows that

$$\Phi_1(\Lambda) = \Phi_3(\Lambda) = 1,\ \Phi_2(\Lambda) = 0.$$

As $\Phi_2(\Lambda) = 0$, it holds that $\mathcal{I}_0 \subseteq \{2\}$, with the sub-assessment $\mathcal{P}_0 = y$ on $\mathcal{F}_0 = \{B|K\}$ coherent because $y \in [0,1]$. Then, by Theorem 2 the assessment $\mathcal{P} = (x, y, bx)$ on $\mathcal{F}$ is coherent. We show that $z'' = bx$ is the upper bound for z. We distinguish two cases: $y > 0$ and $y = 0$. Let us suppose for now that $y > 0$. We observe that Q_1, Q_3, Q_4 belong to the plane $\pi : byX + bxY - yZ = bxy$. Considering the function $f(X, Y, Z) = byX + bxY - yZ - bxy$, it holds that

$$f(Q_1) = f(Q_3) = f(Q_4) = 0,\ f(Q_2) = bx - ay \geqslant 0.$$

Then, for every $\mathcal{P} \in \mathcal{I}$ it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z > bx$, it holds that $f(\mathcal{P}) = f(x, y, z) = y(bx - z) < 0$ and hence $\mathcal{P} \notin \mathcal{I}$. Thus, the assessment $\mathcal{P} = (x, y, z)$ with $z > bx$ is not coherent and hence $z'' = bx$.
If $y = 0$, we observe that Q_1, Q_2, Q_3 belong to the plane $\pi : -bX + (bx - a)Y + Z = 0$. We set $f(X, Y, Z) = -bX + (bx - a)Y + Z$ and it holds that $f(Q_1) = f(Q_2) = f(Q_3) = 0$, $f(Q_4) = -bx \leqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$ it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Then, by considering $\mathcal{P} = (x, 0, z)$, with $z > bx$, it holds that $f(\mathcal{P}) = f(x, 0, z) = -bx + z > 0$, and hence $\mathcal{P} \notin \mathcal{I}$. Thus, the assessment $\mathcal{P} = (x, 0, z)$, with $z > bx$, is not coherent and hence $z'' = bx$.

Case (ii). The proof can be obtained in a way similar to the proof in *Case* (i), when switching x with y and a with b. □

We observe that the bounds obtained in Theorem 25 (under the logical relation $HK = \emptyset$) are more restrictive than the bounds obtained in Theorem 23 (under the assumption of logical independence of A, H, B, K). That is, $[\min\{ay, bx\}, \max\{ay, bx\}] \subseteq [z', z'']$, where z' and z'' are given in (3.12) and (3.13), respectively. Of course, when $a < 1$ and $b < 1$, it holds that $z'' < 1$. This is in agreement to the fact that $(A|H) \wedge_{a,b} (B|K)$ can never be 1 in this case.

Remark 33. *The result of Theorem 25 can be also obtained in a different way by exploiting the linearity of prevision. Indeed, when $HK = \emptyset$, it holds that $AHBK = \emptyset$, $\overline{H}BK = BK$ and $AH\overline{K} = AH$ and hence*

$$(A|H) \wedge_{a,b} (B|K) = (aBK + bAH)|(H \vee K). \tag{3.30}$$

We observe that $P(BK|(H \vee K)) = P(B|K)P(K|(H \vee K))$ and $P[AH|(H \vee K)] = P(A|H)P[H|(H \vee K)]$. Then, from (3.30), by the linearity of prevision, it follows that

$$\begin{aligned} z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)] &= aP(\overline{H}BK|(H \vee K)) + bP(\overline{K}AH|(H \vee K)) = \\ &= aP(B|K)P(K|(H \vee K)) + bP(A|H)P(H|(H \vee K)). \end{aligned} \tag{3.31}$$

By setting $x = P(A|H)$, $y = P(B|K)$, $\alpha = P(H|(H \vee K)) = 1 - P(K|(H \vee K))$, formula (3.31) becomes

$$z = ay(1 - \alpha) + bx\alpha. \tag{3.32}$$

It can be proved[1] *that the assessment (x, y, α) on $\{A|H, B|K, H|(H \vee K)\}$, with $HK = \emptyset$, is coherent for every $(x, y, \alpha) \in [0,1]^3$. Then, from (3.32) we obtain that $z \in$*

[1] For proving that (x, y, α) on $\{A|H, B|K, H|(H \vee K)\}$, with $HK = \emptyset$, is coherent for every $(x, y, \alpha) \in [0,1]^3$ it is sufficient to check that each of the eight vertices of the unit cube is coherent ([47]).

$[\min\{ay, bx\}, \max\{ay, bx\}]$, *because* $\alpha \in [0,1]$.

Remark 34. *When* $a = x$ *and* $b = y$*, from (3.29) we obtain that* $z' = z'' = ay = bx = xy$*. That is, when* $HK = \emptyset$*, it holds that*

$$\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = \mathbb{P}[(A|H) \wedge_{gs} (B|K)] = P(A|H)P(B|K),$$

which is in agreement with the result given in [55] (see also [52]).

In the next theorem we analyse the generalised disjunction (Definition 16) when the conditioning events H and K are incompatible, i.e. $HK = \emptyset$. From Theorem 25, and by relation (3.9), it follows

Theorem 26. *Let* $A|H$, $B|K$ *be two conditional events with* A, H, B *logically independent,* A, B, K *logically independent, and* $HK = \emptyset$*. A prevision assessment* $\mathcal{P} = (x, y, w)$ *on* $\mathcal{F} = \{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$ *is coherent if and only if* $(x, y) \in [0,1]^2$ *and* $w \in [w', w'']$*, where*

$$w' = \min\{a + y - ay,\ b + x - bx\}, \quad w'' = \max\{a + y - ay,\ b + x - bx\}. \tag{3.33}$$

Proof. From Equation (3.10), it holds that $w' = 1 - z''(1-x, 1-y, 1-a, 1-b)$, where $z''(1-x, 1-y, 1-a, 1-b)$ is the upper bound of the generalised conjunction valued in $1-x, 1-y, 1-a$, and $1-b$. Then

$$\begin{aligned} w' &= 1 - z''(1-x,1-y,1-a,1-b) = 1 - \max\{(1-a)(1-y),\ (1-b)(1-x)\} = \\ &= \min\{1-(1-a)(1-y),\ 1-(1-b)(1-x)\} = \min\{a+y-ay,\ b+x-bx\}. \end{aligned}$$

In the same way, from (3.10), it follows that $w'' = 1 - z'(1-x, 1-y, 1-a, 1-b)$, where $z'(1-x, 1-y, 1-a, 1-b)$ is the lower bound of the generalised conjunction valued in $1-x, 1-y, 1-a$, and $1-b$. Then

$$\begin{aligned} w'' &= 1 - z'(1-x,1-y,1-a,1-b) = 1 - \min\{(1-a)(1-y),\ (1-b)(1-x)\} = \\ &= \max\{1-(1-a)(1-y),\ 1-(1-b)(1-x)\} = \max\{a+y-ay,\ b+x-bx\}. \end{aligned}$$

□

Remark 35. *We observe that, when* $a = x$ *and* $b = y$*, from (3.33) we obtain that* $w' = w'' = a + y - ay = b + x - bx = x + y - xy$*. That is, when* $HK = \emptyset$*, it holds that*

$$\mathbb{P}[(A|H) \vee_{x,y} (B|K)] = \mathbb{P}[(A|H) \vee_{gs} (B|K)] = P(A|H) + P(B|K) - P(A|H)P(B|K).$$

3.3.2 The case $a = b = 0$

In the case where $a = b = 0$ the generalised conjunction reduces to a trivalent object, in particular it corresponds to the forth conjunction proposed by Walker in [94]. Indeed,

$$(A|H) \wedge_{0,0} (B|K) = AHBK|(H \vee K). \tag{3.34}$$

Based on Theorem 23, under logical independence, it follows that the probability assessment (x, y, z) on $\{A|H, B|K, (A|H) \wedge_{0,0} (B|K)\} = \{A|H, B|K, AHBK|(H \vee K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = min\{x, y\}$.

For what concerns the generalised disjunction $(A|H) \vee_{a,b} (B|K)$, when $a = b = 0$ it reduces to a trivalent object which coincides with the quasi disjunction (see Section 1.4). Indeed,

$$(A|H) \vee_{0,0} (B|K) = (AH \vee BK)|(H \vee K) = (A|H) \vee_S (B|K).$$

Then, when $H \vee K$ is true it holds that $(A|H) \vee_S (B|K) \leqslant (A|H) \vee_{a,b} (B|K)$. By applying [56, Theorem 6], it holds that $P((A|H) \vee_S (B|K)) \leqslant \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ and hence

$$(A|H) \vee_S (B|K) \leqslant (A|H) \vee_{a,b} (B|K)$$

in all cases. Moreover, based on Theorem 24 it follows that (x, y, w) on $\{A|H, B|K, (A|H) \vee_{0,0} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $w \in [w', w'']$, where

$$w' = w_2' = \begin{cases} \dfrac{xy}{x+y-xy}, \text{ if } (x,y) \neq (0,0), \\ 0, \text{ if } (x,y) = (0,0), \end{cases}$$

because $ax + by - xy = -xy < 0$, and

$$\begin{aligned} w'' &= \begin{cases} 1 - (1-x-y) \cdot \min\{\frac{1}{1-x}, \frac{1}{1-y}, 1\}, & \text{if } x+y < 1, \\ 1, & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1 - (1-x-y) = x+y, & \text{if } x+y < 1, \\ 1, & \text{otherwise.} \end{cases} = \min\{x+y, 1\}, \end{aligned}$$

because $\frac{1}{1-x} \geqslant 1$ and $\frac{1}{1-y} \geqslant 1$.

Then, the lower and upper bounds w' and w'' coincide with w_S' and w_S'' given in Table 1.3. We observe that w_S' is the Hamacher t-norm with the parameter $\lambda = 0$. Thus, differently from the unconditional case where $P(A \vee B) = 1$ is the only coherent extension of $(1, 0)$ on $\{A, B\}$, for the probability of the quasi disjunction $(A|H) \vee_S (B|K)$, any value $w_S \in [0, 1]$ is a coherent extension of $(1, 0)$ on $\{A|H, B|K\}$ because $w_S' = 0$ and $w_S'' = 1$. From a probabilistic point of view this is not desirable because it allows to coherently assess probability 0 for the quasi disjunction even if one of the two conjuncts has probability 1 and the other one has probability zero. A similar comment can be also done in the particular case where H and K are incompatible. Indeed, when $HK = \varnothing$, by instantiating equation (3.33) with $a = b = 0$, the lower and upper bounds for $(A|H) \vee_S (B|K)$ become

$$w' = \min\{x, y\}, \quad w'' = \max\{x, y\}. \tag{3.35}$$

3.3.3 The case $a = b = 1$

It is interesting to notice that quasi conjunction is a particular case of the generalised conjunction $(A|H) \wedge_{a,b} (B|K)$ where $a = 1$ and $b = 1$. Indeed, looking at the Sobocinski conjunction in terms of conditional random quantity, it holds that

$$(A|H) \wedge_S (B|K) = (AHBK + \overline{H}BK + AH\overline{K})|(H \vee K). \tag{3.36}$$

Then, we have that

$$(A|H) \wedge_{1,1} (B|K) = (AHBK + \overline{H}BK + AH\overline{K})|(H \vee K) = (A|H) \wedge_S (B|K).$$

When $H \vee K$ is true it holds that $(A|H) \wedge_S (B|K) \geqslant (A|H) \wedge_{a,b} (B|K)$. By applying [56, Theorem 6], it holds that $P[(A|H) \wedge_S (B|K)] \geqslant \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ and hence

$$(A|H) \wedge_S (B|K) \geqslant (A|H) \wedge_{a,b} (B|K)$$

in all cases. Moreover, based on Theorem 23 it follows that (x, y, z) on $\{A|H, B|K, (A|H) \wedge_{1,1} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where

$$z' = \begin{cases} (x+y-1), & \text{if } x+y-1>0, \\ 0, & \text{otherwise} \end{cases} = \max\{x+y-1,0\},$$

and

$$z'' = \begin{cases} \dfrac{x(b-ay)+y(a-bx)}{1-xy}, \text{ if } (x,y) \neq (1,1), \\ 1, \text{ if } (x,y)=(1,1), \end{cases} = \begin{cases} \dfrac{x+y-2xy}{1-xy}, \text{ if } (x,y) \neq (1,1), \\ 1, \text{ if } (x,y)=(1,1), \end{cases}$$

because in this case it holds that $a(1-y)+b(1-x)+xy-1 = 1-y+1-x+xy-1 = 1-y-x+xy = 1-y-x(1-y) = (1-y)(1-x) \geqslant 0$. Then, the lower and upper bounds z' and z'' coincide with z'_S and z''_S given in Table 1.2. We recall that z''_S is the Hamacher t-conorm with the parameter $\lambda = 0$. Then, for every $(x,y) \in [0,1]^2$, as $\max(x,y)$ is the smallest t-conorm, it holds that

$$z''_S \geqslant \max\{x,y\} \geqslant \min\{x,y\},$$

and hence the quasi conjunction do not preserve the bounds in (1.15). Thus, differently from the unconditional case where $P(AB) = 0$ is the only coherent extension of the assessment $(1,0)$ on $\{A,B\}$, for the probability of the quasi conjunction $(A|H) \wedge_S (B|K)$, any value $z_S \in [0,1]$ is a coherent extension of $(1,0)$ on $\{A|H, B|K\}$ because $z'_S = 0$ and $z''_S = 1$. This is not desirable from a probabilistic point of view because it allows to coherently assess probability 1 for the quasi conjunction even if one of the two conjuncts has probability 1 and the other one has probability zero. A similar comment can be also done in the particular case where H and K are incompatible. Indeed, when $HK = \varnothing$, by instantiating equation (3.29) with $a=b=1$, the lower and upper bounds for $(A|H) \wedge_S (B|K)$ are

$$z' = \min\{x,y\}, \quad z'' = \max\{x,y\}. \tag{3.37}$$

Then, as $(A|H) \wedge_{1,1} (B|K) = (\bar{H}BK \vee AH\bar{K})|(H \vee K))$, it follows that, given two conditional events $A|H$ and $B|K$, with $HK = \varnothing$, a probability assessment $\mathcal{P} = (x,y,z)$ on $\mathcal{F} = \{A|H, B|K, (\bar{H}BK \vee AH\bar{K})|(H \vee K))\}$ is coherent if and only if $(x,y) \in [0,1]^2$ and $z \in [\min\{x,y\}, \max\{x,y\}]$.

In the case where $a=b=1$ the generalised disjunction reduces to a trivalent object, in particular it corresponds to the forth disjunction proposed by Walker in [94]. Indeed,

$$(A|H) \vee_{1,1} (B|K) = ((AH \vee BK) + \bar{H}\,\bar{B}K + \bar{A}H\bar{K})|(H \vee K).$$

Based on Theorem 24, under logical independence, it follows that (x,y,w) on $\{A|H, B|K, (A|H) \vee_{1,1} (B|K)\}$ is coherent if and only if $(x,y) \in [0,1]^2$ and $w \in [w', w'']$, where $w' = \max\{x,y\}$ and $w'' = 1$.

3.3.4 The case $a = b = p$

We analyse now the particular case for the generalised conjunction in which $a = b = z = \mathbb{P}[(A|H) \wedge_{z,z} (B|K)]$. From Equation (3.3), setting $a=b=z$ we have

$$(A|H) \wedge_{z,z} (B|K) = AHBK + z\overline{H}BK + zAH\overline{K} + z(H \vee K) = AHBK|(AHBK \vee \overline{A}H \vee \overline{B}K).$$

Moreover, for what concerns the generalised disjunction, by setting $a = b = w = \mathbb{P}[(A|H) \vee_{w,w} (B|K)]$ in Equation (3.6), we have that

$$'.$$

Then for these choices of parameters the conditioning events change and the generalised conjunction and disjunction reduce, respectively, to the trivalent conjunction and disjunction of Kleene-de Finetti ([24, 61]). The lower and upper bound for the coherent assessments on these objects are well-known and they are $z' = 0$ and $z'' = \min\{x, y\}$ for the conjunction, and $w' = \max\{x, y\}$ and $w'' = 1$ for the disjunction.

It worth notice that with the choices $(a, b) = (0, 0)$, $(a, b) = (1, 1)$, and $(a, b) = (p, p)$, we obtain three of the four associative conjunctions and disjunctions proposed by Walker ([94]) among the nine that were reasonable for the author to consider in a trivalent setting.

3.4 Interval-valued prevision assessments

In this section, to deepen the study on the generalised conjunction $\wedge_{a,b}$ and disjunction $\vee_{a,b}$, we analyse the case of interval-valued probability assessments. For both structures, given the imprecise assessment $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ on $\{A|H, B|K\}$, we study the coherent extensions to $(A|H) \wedge_{a,b} (B|K)$ (Theorem 27) and $(A|H) \vee_{a,b} (B|K)$ (Theorem 29), respectively. Moreover, we consider the case $HK = \varnothing$ both for the generalised conjunction (Theorem 28) and disjunction (Theorem 30).

We recall that under logical independence any assessment $(x, y) \in [0, 1]^2$ on $\{A|H, B|K\}$ is coherent. Based on Theorem 23, we denote by $[z'(x, y), z''(x, y)]$ the interval of coherent prevision extensions of (x, y) to $(A|H) \wedge_{a,b} (B|K)$. Given the interval-valued assessment $[x_1, x_2] \times [y_1, y_2] \subseteq [0, 1]^2$ on $\{A|H, B|K\}$, we set $[z^*, z^{**}]$ the interval of coherent extensions z on $(A|H) \wedge_{a,b} (B|K)$, where

$$z^* = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(x, y)$$

and

$$z^{**} = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(x, y).$$

The assessment $([x_1, x_2] \times [y_1, y_2] \times [z^*, z^{**}])$ on $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$ is coherent w.r.t. Definition 7. Moreover, any assessment $[x_1, x_2] \times [y_1, y_2] \times [\alpha, \beta]$, with $[\alpha, \beta] \supset [z^*, z^{**}]$, is not coherent. Then, the interval $[z^*, z^{**}]$ is the least committal extension, that is the natural extension ([95], see also [84]), of the interval-valued assessment $[x_1, x_2] \times [y_1, y_2]$, which is equivalent to the lower probability assessment $(x_1, 1 - x_2, y_1, 1 - y_2)$ on $\{A|H, \overline{A}|H, B|K, \overline{B}|K\}$. In what follows we compute the lower and upper bounds, z^* and z^{**}, by also considering the case where $HK = \varnothing$.

Theorem 27. *Let A, B, H, K be any logically independent events and let $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued assessment on $\{A|H, B|K\}$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \wedge_{a,b} (B|K)$ is the interval $[z^*, z^{**}] = [z'(x_1, y_1), z''(x_2, y_2)]$, where $z'(x, y)$ and $z''(x, y)$ are defined in formula (3.12) and formula (3.13), respectively.*

Proof. We first want to observe that $z^* = z'(x_1, y_1)$ and that $z^{**} = z''(x_2, y_2)$, where $z'(x, y)$ and $z''(x, y)$ are given in Theorem 23. The proof is straightforward by observing that

$$z^* = z'(x_1, y_1) = \min_{(x,y)\in\mathcal{A}} z'(x, y),$$

where $z'(x, y)$ is given in (3.12), and that

$$z^{**} = z''(x_2, y_2) = \max_{(x,y)\in\mathcal{A}} z''(x, y),$$

where $z''(x, y)$ is given in (3.13).
We now show for the lower bound $z^* = z'(x_1, y_1)$ that $z'(x_1, y_1) = \min_{(x,y)\in\mathcal{A}} z'(x, y)$ and for the upper bound $z^{**} = z''(x_2, y_2)$ that $z''(x_2, y_2) = \max_{(x,y)\in\mathcal{A}} z''(x, y)$.

Lower bound We distinguish the following cases: **(A)**: $x_1 + y_1 - 1 \leqslant 0$; **(B)**: $x_1 + y_1 - 1 > 0$.
Case **(A)**. If $x_1 + y_1 - 1 \leqslant 0$ then $z'(x_1, y_1) = 0 = \min_{(x,y)\in\mathcal{A}} z'(x, y)$ and hence $z^* = z'(x_1, y_1) = 0$.
Case **(B)**. We consider three subcases: (*B.1*): $\min\{\frac{a}{x_1}, \frac{b}{y_1}, 1\} = 1$; (*B.2*): $\min\{\frac{a}{x_1}, \frac{b}{y_1}, 1\} = \frac{a}{x_1}$; (*B.3*): $\min\{\frac{a}{x_1}, \frac{b}{y_1}, 1\} = \frac{b}{y_1}$;
Case (B.1). If $x_1 + y_1 - 1 > 0$ and $a \geqslant x_1$ and $b \geqslant y_1$, then $z'(x_1, y_1) = x_1 + y_1 - 1$. In this case $z'(x_1, y_1) = \min_{(x,y)\in\mathcal{A}} z'(x, y)$. Indeed, as $x \geqslant x_1$ and $y \geqslant y_1$, it holds that $x + y - 1 \geqslant x_1 + y_1 - 1 > 0$, from Table 3.2, we can have the following three cases: (i) $z'(x, y) = x + y - 1$, (ii) $z'(x, y) = \frac{a}{x}(x + y - 1)$, (iii) $z'(x, y) = \frac{b}{y}(x + y - 1)$.
(i). Of course, $z'(x, y) = x + y - 1 \geqslant x_1 + y_1 - 1 = z'(x_1, y_1)$.
(ii). We observe that $\frac{a}{x}(x + y - 1)$ is non-decreasing in the arguments x and y. Then, as $a \geqslant x_1$, it holds that

$$z'(x, y) = \tfrac{a}{x}(x + y - 1) \geqslant \tfrac{a}{x}(x + y_1 - 1) \geqslant \tfrac{a}{x_1}(x_1 + y_1 - 1) \geqslant (x_1 + y_1 - 1) = z'(x_1, y_1).$$

(iii). Similar to the reasoning exploited in (ii) it can be easily proved that $z'(x, y) = \frac{b}{y}(x + y - 1) \geqslant (x_1 + y_1 - 1) = z'(x_1, y_1)$.
Case (B.2).
In this case $a < x_1$ and $\frac{a}{x_1} \leqslant \frac{b}{y_1}$. We show that $z'(x_1, y_1) = \frac{a}{x_1}(x_1 + y_1 - 1) = \min_{(x,y)\in\mathcal{A}} z'(x, y)$. As $x \geqslant x_1 > a$ and $y \geqslant y_1$, it holds that $x + y - 1 \geqslant x_1 + y_1 - 1 > 0$, and $a < x$. Then from Table 3.2, we can have the following cases: (i) $z'(x, y) = \frac{a}{x}(x + y - 1)$, (ii) $z'(x, y) = \frac{b}{y}(x + y - 1)$.
(i). As $\frac{a}{x}(x + y - 1)$ is non-decreasing in the arguments x and y it holds that $z'(x_1, y_1) = \frac{a}{x_1}(x_1 + y_1 - 1) \leqslant \frac{a}{x}(x + y - 1) = z'(x, y)$.
(ii). We observe that $\frac{b}{y}(x + y - 1)$ is non-decreasing in the arguments x and y. Then, as $\frac{b}{y_1} \geqslant \frac{a}{x_1}$, it holds that

$$\tfrac{b}{y}(x + y - 1) \geqslant \tfrac{b}{y_1}(x_1 + y_1 - 1) \geqslant \tfrac{a}{x_1}(x_1 + y_1 - 1) = z'(x_1, y_1).$$

Case (B.3). By applying a reasoning similar to the case *(B.2)* it can be shown that $z'(x_1, y_1) = \frac{b}{y_1}(x_1 + y_1 - 1) = \min_{(x,y)\in\mathcal{A}} z'(x, y)$.

Upper bound We distinguish two cases: **(C)** $a(1 - y_2) + b(1 - x_2) + x_2 y_2 - 1 > 0$; **(D)** $a(1 - y_2) + b(1 - x_2) + x_2 y_2 - 1 \leqslant 0$.

Case (**C**). From Table 3.3 it holds that

$$z^{**} = z''(x_2, y_2) = \frac{x_2(b - ay_2) + y_2(a - bx_2)}{1 - x_2 y_2}. \tag{3.38}$$

We show that $z^{**} = z''(x_2, y_2) \geqslant z''(x, y)$ for every $(x, y) \in \mathcal{A}$. Let be given $(x, y) \in \mathcal{A}$. We distinguish the following cases: (i) $a(1-y) + b(1-x) + xy - 1 > 0$; (ii) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a \leqslant x$; (iii) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a > x$ (iv) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $y < x$, $b \leqslant y$; (v) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $y < x$, $b > y$.
(i) We have that $z''(x, y) = z''_2(x, y)$. From Equation (3.18), as $b - ay \geqslant 0$, it holds that

$$z''_2(x, y) - z''_2(x_2, y) = -\tfrac{(x_2 - x)\,(1-y)\,(b - a\,y)}{(1 - x\,y)\,(1 - x_2\,y)} \leqslant 0, \tag{3.39}$$

that is $z''_2(x, y) \leqslant z''_2(x_2, y)$. Moreover, as $a - bx_2 \geqslant 0$ (see Equation (3.18)), it holds that

$$z''_2(x_2, y) - z''_2(x_2, y_2) = -\tfrac{(y_2 - y)\,(1-x_2)\,(a - b\,x_2)}{(1 - x_2\,y)\,(1 - x_2\,y_2)} \leqslant 0. \tag{3.40}$$

Then, $z''(x, y) = z''_2(x, y) \leqslant z''_2(x_2, y) \leqslant z''_2(x_2, y_2) = z^{**}$.
(ii) We have that $z''(x, y) = z''_1(x, y) = x$. Moreover, by recalling (3.13), it holds that

$$z''(x, y) = x = \min\{x, y\} \leqslant \min\{x_2, y_2\} == z''_1(x_2, y_2) \leqslant z''_2(x_2, y_2).$$

Then, $z''(x, y) \leqslant z''_2(x_2, y_2)$.
(iii) We have that $z''(x, y) = z''_3(x, y)$. We observe that

$$z''_3(x, y) - z''_3(x_2, y) = -\tfrac{(x_2 - x)\,(1-a)\,(1-y)}{(1-x)\,(1-x_2)} \leqslant 0 \tag{3.41}$$

and that

$$z''_3(x_2, y) - z''_3(x_2, y_2) = -\tfrac{(a - x_2)\,(y_2 - y)}{1 - x_2} \leqslant 0 \tag{3.42}$$

because $a > x_2$. Then, by recalling (3.13), it holds that $z''(x, y) = z''_3(x, y) \leqslant z''_3(x_2, y) \leqslant z''_3(x_2, y_2) \leqslant z''_2(x_2, y_2) = z^{**}$.
(iv) We have that $z''(x, y) = z''_1(x, y) = y$ and, as in case (ii), it can be shown that $z''(x, y) \leqslant z''_2(x_2, y_2) = z^{**}$.

(v) We have that $z''(x, y) = z''_4(x, y)$. As in case (iii), it can be shown that $z''(x, y) = z''_4(x, y) \leqslant z''_4(x_2, y) \leqslant z''_4(x_2, y_2) \leqslant z''_2(x_2, y_2) = z^{**}$.

Case (**D**). From Table 3.3 we can distinguish the following situations: *(D.1)* $x_2 \leqslant y_2$ and $a \leqslant x_2$; *(D.2)* $x_2 \leqslant y_2$ and $a > x_2$; *(D.3)* $y_2 < x_2$ and $b \leqslant y_2$; *(D.4)* $y_2 < x_2$ and $b > y_2$.
(D.1) It holds that

$$z^{**} = z''(x_2, y_2) = z''_1(x_2, y_2) = x_2.$$

We show that $z^{**} = z''(x_2, y_2) \geqslant z''(x, y)$ for every $(x, y) \in \mathcal{A}$. Let $(x, y) \in \mathcal{A}$ be given. We distinguish the following cases: (i) $a(1-y) + b(1-x) + xy - 1 > 0$; (ii) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a \leqslant x$; (iii) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a > x$ (iv) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $y < x$, $b \leqslant y$; (v) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $y < x$, $b > y$.

(i) We have that $z''(x,y) = z''_2(x,y)$. Moreover,

$$z''_2(x,y) - a = -\tfrac{(1-y)\,(a-b\,x)}{1-x\,y} \leqslant 0$$

because $a - bx > 0$. Then,

$$z''_2(x,y) \leqslant a \leqslant x_2 = z''(x,y) = z^{**}.$$

(ii) We have that $z''(x,y) = z''_1(x,y) = x \leqslant x_2 = z''(x,y) = z^{**}$.
(iii) We have that $z''(x,y) = z''_3(x,y)$. We observe that

$$z''_3(x,y) - a = -\tfrac{(a-x)\,(1-y)}{1-x} \leqslant 0$$

because $a > x$. Then

$$z''_3(x,y) \leqslant a \leqslant x_2 = z''(x,y) = z^{**}.$$

(iv) We have that $z''(x,y) = z''_1(x,y) = y < x \leqslant x_2 = z''_1(x_2,y_2) = z^{**}$.

(v) We have that $z''(x,y) = z''_4(x,y)$. As $y < x$, we obtain

$$z''_4(x,y) - x = -\tfrac{(x-y)\,(1-b)}{1-y} \leqslant 0$$

Then

$$z''_4(x,y) \leqslant x \leqslant x_2 = z''(x,y) = z^{**}.$$

(D.2) It holds that

$$z^{**} = z''(x_2,y_2) = z''_3(x_2,y_2) = \frac{x_2(1-a) + y_2(a-x_2)}{1-x_2}.$$

We show that $z^{**} = z''(x_2,y_2) = \frac{x_2(1-a)+y_2(a-x_2)}{1-x_2} \geqslant z''(x,y)$ for every $(x,y) \in \mathcal{A}$. Let $(x,y) \in \mathcal{A}$ be given . We distinguish the following cases: (i) $a(1-y) + b(1-x) + xy - 1 > 0$; (ii) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a \leqslant x$; (iii) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a > x$ (iv) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $y < x$, $b \leqslant y$; (v) $a(1-y) + b(1-x) + xy - 1 \leqslant 0$, $y < x$, $b > y$.
(i) We have that $z''(x,y) = z''_2(x,y)$. As $b - ay > 0$ and $a > x_2 \geqslant bx_2$, it follows from (3.39) and (3.40) that $z''_2(x,y) \leqslant z''_2(x_2,y_2)$. By recalling from Theorem 23 that in case *(D.2)* $z''_2(x_2,y_2) \leqslant z''_3(x_2,y_2)$, we obtain that $z''_2(x,y) \leqslant z''_3(x_2,y_2) = z^{**}$.
(ii) This subcase is not satisfied by any $(x,y) \in \mathcal{A}$. Indeed, we are considering $a > x_2$, then in this case $a > x_2 \geqslant x$ and hence it is not possible to have $a \leqslant x$.

(iii) We have that $z''(x,y) = z''_3(x,y)$. From relations (3.41) and (3.42) it holds that

$$z''_3(x,y) \leqslant z''_3(x_2,y_2) = z^{**}.$$

(iv) We have that, from (3.13), $z''(x,y) = z''_1(x,y) = y < x \leqslant x_2 = z''_1(x_2,y_2) \leqslant z''_3(x_2,y_2) = z^{**}$.

(v) We have that $z''(x,y) = z''_4(x,y)$. As $y < x$, we obtain

$$z''_4(x,y) - x = -\tfrac{(x-y)\,(1-b)}{1-y} \leqslant 0$$

Then, by (3.13),

$$z''_4(x,y) \leqslant x \leqslant x_2 = z''_1(x_2,y_2) \leqslant z''_3(x_2,y_2) = z''(x,y) = z^{**}.$$

(D.3) In this case, by following a reasoning similar to the case *(D.1)*, it can be proved that $z^{**} = z''(x_2,y_2) = z''_1(x_2,y_2) = y_2$.

(D.4) In a way similar to the one in case *(D.2)*, it can be proved that $z^{**} = z''(x_2,y_2) = z''_4(x_2,y_2) = \frac{x_2(b-y)+y_2(1-b)}{1-y_2}$. □

We now generalise Theorem 25 for interval-valued prevision assessments.

Theorem 28. *Let $\mathcal{A} = ([x_1,x_2] \times [y_1,y_2])$ be an interval-valued probability assessment on $\{A|H, B|K\}$ with $HK = \emptyset$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \wedge_{a,b} (B|K)$ is the interval $[z^*, z^{**}]$, where*

$$z^* = \min\{ay_1, bx_1\}, \quad z^{**} = \max\{ay_2, bx_2\}. \tag{3.43}$$

Proof. The proof is straightforward by recalling that every assessment $(x,y) \in [0,1]^2$ on $\{A|H, B|K\}$ is coherent when $HK = \emptyset$ and by observing that both the lower and upper bounds z' and z'' given in Theorem 25 are non-decreasing functions in the arguments x and y. □

Using the results obtained in Theorem 27, we can analyse the imprecise probabilities case also for the generalised disjunction $(A|H) \vee_{a,b} (B|K)$.

Theorem 29. *Let A, B, H, K be any logically independent events and let $\mathcal{A} = ([x_1,x_2] \times [y_1,y_2])$ be an interval-valued assessment on $\{A|H, B|K\}$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \vee_{a,b} (B|K)$ is the interval $[w^*, w^{**}] = [w'(x_1,y_1), w''(x_2,y_2)]$, where $w'(x,y)$ and $w''(x,y)$ are defined in formula (3.26) and formula (3.27), respectively.*

Proof. Given the interval-valued assessment $[x_1,x_2] \times [y_1,y_2] \subseteq [0,1]^2$ on $\{A|H, B|K\}$, we set $[w^*, w^{**}]$ the interval of coherent extensions of w on $(A|H) \vee_{a,b} (B|K)$, with

$$w^* = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w'(x,y)$$

and

$$w^{**} = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w''(x,y),$$

where $w'(x,y)$ and $w''(x,y)$ are defined in formula (3.26) and formula (3.27), respectively. We recall that from Theorem 27, the lower and upper bounds for the generalised conjunction, $(A|H) \wedge_{a,b} (B|K)$, in the imprecise case where $(x,y) \in [x_1,x_2] \times [y_1,y_2]$, are

$$z^* = z'(x_1,y_1) = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(x,y), \quad z^{**} = z''(x_2,y_2) = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(x,y).$$

We observe that, as $(1-x, 1-y) \in [1-x_2, 1-x_1] \times [1-y_2, 1-y_1]$, then

$$z'(1-x_2, 1-y_2) = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(1-x, 1-y),$$

and

$$z''(1-x_1, 1-y_1) = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(1-x, 1-y).$$

From (3.10), we have that

$$w'(x,y) = 1 - z''(1-x, 1-y, 1-a, 1-b), \quad w''(x,y) = 1 - z'(1-x, 1-y, 1-a, 1-b),$$

and hence

$$\begin{aligned} w^* &= \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w'(x,y) = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} (1 - z''(1-x, 1-y, 1-a, 1-b)) = \\ &= 1 - \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(1-x, 1-y) = 1 - z''(1-x_1, 1-y_1) = w'(x_1, y_1) \end{aligned}$$

$$\begin{aligned} w^{**} &= \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w''(x,y) = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} (1 - z'(1-x, 1-y, 1-a, 1-b)) = \\ &= 1 - \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(1-x, 1-y) = 1 - z'(1-x_2, 1-y_2) = w''(x_2, y_2). \end{aligned}$$

□

Moreover, considering the interval of coherent assessments obtained in the precise case (Theorem 26), we also find the interval of coherent prevision extensions for H and K incompatible.

Theorem 30. *Let $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued probability assessment on $\{A|H, B|K\}$ with $HK = \emptyset$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \vee_{a,b} (B|K)$ is the interval $[w^*, w^{**}]$, where*

$$w^* = \min\{a + y_1 - ay_1, b + x_1 - bx_1\}, \quad w^{**} = \max\{a + y_2 - ay_2, b + x_2 - bx_2\}. \tag{3.44}$$

Proof. From Theorem 26 we have that

$$w'(x,y) = \min\{a + y - ay,\ b + x - bx\}, \quad w''(x,y) = \max\{a + y - ay,\ b + x - bx\}.$$

We recall that

$$w^* = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w'(x,y)$$

and

$$w^{**} = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w''(x,y),$$

then, by observing that $b + x_1(1-b) \leqslant b + x(1-b) \leqslant b + x_2(1-b)$ and $a + y_1(1-a) \leqslant a + y(1-a) \leqslant a + y_2(1-a)$, it holds that

$$w^* = \min\{a + y_1 - ay_1, b + x_1 - bx_1\}, \quad w^{**} = \max\{a + y_2 - ay_2, b + x_2 - bx_2\}.$$

□

3.5 Further aspects on $(A|H) \wedge_{a,b} (B|K)$

In this section we deepen two further aspects of the generalised conjunction $(A|H) \wedge_{a,b} (B|K)$. We first give a subjective interpretation of $(A|H) \wedge_{a,b} (B|K)$ when we consider two individuals O and O'. Then, we examine $(A|H) \wedge_{a,b} (B|K)$, when $A|H \subseteq B|K$.

Let us consider two individuals O and O'. Suppose that O' asserts $P'(A|H) = a$ and $P'(B|K) = b$. Then, based on Remark 30, for O' the conjunction $(A|H) \wedge_{a,b} (B|K)$ coincides with its conjunction $(AHBK + P'(A|H)\overline{H}BK + P'(B|K)AH\overline{K})|(H \vee K)$, which we denote by $(A|H) \wedge'_{gs} (B|K)$. Thus, by coherence, $\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)]$ satisfies the

Fréchet-Hoeffding bounds, that is:

$$\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)] = \mathbb{P}'[(A|H) \wedge'_{gs} (B|K)] \in [\max\{a+b-1,0\}, \min\{a,b\}].$$

Now, suppose that O asserts $P(A|H) = x$ and $P(B|K) = y$. Then, under logical independence, the lower and upper bounds z' and z'' on $(A|H) \wedge_{a,b} (B|K)$ computed in Theorem 23, for the individual O, represent the lower and upper bounds for the coherent extension $\mathbb{P}[(A|H) \wedge'_{gs} (B|K)]$ of the assessment (x,y) on $\{A|H, B|K\}$. Thus, according to eq. (3.25), it generally holds that

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = \mathbb{P}[(A|H) \wedge'_{gs} (B|K)] \in [z', z''] \supseteq [\max\{x+y-1,0\}, \min\{x,y\}],$$

while

$$\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = \mathbb{P}[(A|H) \wedge_{gs} (B|K)] \in [\max\{x+y-1,0\}, \min\{x,y\}].$$

A real world application of this interpretation could be given in the framework of a joint-bet in soccer betting (see [41, 62]). The conditional events can be interpreted as $A|H$ = *The outcome of the first match is home win, given that the first match is valid*, and $B|K$ = *The outcome of the second match is draw, given that the second match is valid*. Suppose the bettor assesses $x = P(A|H)$, $y = P(B|K)$. However these are his personal evaluations, and when he bets on these matches he has to deal with the bookmaker ones. That is, bookmaker probabilities may be $a = P'(A|H) = \frac{1}{Q_1}$ and $b = P'(B|K) = \frac{1}{Q_2}$, where Q_1 and Q_2 are the assigned decimal odds on the two single bets. In a single bet the stake will be refunded if the match is not valid. A double bet on $A|H$ and $B|K$ in a linked series of the two single bets where the return of one bet is stacked to the other bet. Then, the bettor actually bets on the joint-bet $(A|H) \wedge_{a,b} (B|K)$, in the context where he has his own probabilities but the object is determined by those of the bookmaker. Moreover the bookmaker usually considers the two bets as "independent" and for him $(A|H) \wedge_{a,b} (B|K) = (A|H) \wedge'_{gs} (B|K)$ with $\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)] = ab = \frac{1}{Q_1 Q_2}$, while the bettor could coherently assess $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)] \in [z', z'']$ as in Theorem 23. In particular, the bettor, paying ab, might feel he has an advantage when $ab < z'$, where z' is the lower bound in (3.12). This condition can only occur when $0 < ab < (x+y-1)\min\{\frac{a}{x}, \frac{b}{y}\}$. Indeed, if $x+y-1 \leqslant 0$ then $z' = 0 \leqslant ab$, and if $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = 1$ then $z' = x+y-1 \leqslant a+b-1 \leqslant ab$. Moreover, $0 < ab < (x+y-1)\min\{\frac{a}{x}, \frac{b}{y}\}$ also requires that $0 < a < x$ and $0 < b < y$. Indeed, if $\frac{a}{x} \leqslant \frac{b}{y}$, then it must be $b < \frac{x+y-1}{x} \leqslant \frac{xy}{x} \leqslant y$. Similarly, if $\frac{b}{y} < \frac{a}{x}$, then it must be $a < \frac{x+y-1}{y} \leqslant \frac{xy}{y} \leqslant x$. As an example, consider the case where $x = 0.8$, $y = 0.9$, $a = 0.7$, and $b = 0.8$. In this case, $ab = 0.56 < z' = (0.8+0.9-1)\frac{7}{8} = 0.6125$.

Inclusion relation We recall that, when $A|H \subseteq B|K$, it holds that $(A|H) \wedge_{gs} (B|K) = A|H$ (see, e.g., [58, formula (16)]) and hence $\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = P(A|H)$. These relations are not preserved by $(A|H) \wedge_{a,b} (B|K)$. Indeed, if $A|H \subseteq B|K$, as $AHBK = AH$, $\overline{K}AH = \emptyset$, and $\overline{H}BK = \overline{H}K$, from (3.1) it holds that

$$(A|H) \wedge_{a,b} (B|K) = (AH + a\overline{H}K)|(H \vee K) \neq A|H, \tag{3.45}$$

because, when $\overline{H}K$ is true, it follows that $(A|H) \wedge_{a,b} (B|K) = a$, while $A|H = x$, where $x = P(A|H)$. Moreover, it holds that

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = P(A|H)P(H|H \vee K) + aP(\overline{H}K|H \vee K) = \alpha x + (1-\alpha)a,$$

where $x = P(A|H)$, $\alpha = P(H|H \vee K)$. Thus, when $A|H \subseteq B|K$, it follows that

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] \in [\min\{x, a\}, \max\{x, a\}],$$

whereas

$$\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = P(A|H) = x.$$

In particular when $a = b = 1$, that is $(A|H) \wedge_{a,b} (B|K) = (A|H) \wedge_S (B|K)$, from (3.45) it follows that $(A|H) \wedge_{a,b} (B|K) = (AH \vee \overline{H}K)|(H \vee K) \supseteq (A|H)$ (see [59, Remark 4]) and $P((AH \vee \overline{H}K)|(H \vee K)) \in [x, 1]$.

Analogous observations from this section can be applied to $(A|H) \vee_{a,b} (B|K)$.

Chapter 4

Compound conditionals and fuzzy sets

The results of this chapter are mostly taken from [18]. To help the readability and simplify the notation, in the following we use the symbols $\wedge$ to refer to $\wedge_{gs}$.
In this chapter, starting from the probabilistic interpretation of the membership function of a fuzzy set as a suitable conditional probability given in [26, 28], we propose a slightly different definition of a fuzzy set. Then, we introduce suitable set operations for fuzzy sets. In particular, we consider the complement and the intersection of fuzzy sets, the latter defined exploiting the conjunction of conditional events as random quantities. We also analyse the choice of a t-norm as membership of the intersection in the context of coherent conditional probabilities.

4.1 Fuzzy sets and random quantities

Let X be a random quantity with range of values C_X, for each $x \in C_X$, we denote by A_x the event $(X = x)$. Obviously, the family $\{A_x\}_{x \in C_X}$ is a partition (possibly not finite) of the sure event Ω. In this paper we only consider finite sets C_X. Let φ be any property related to the random quantity X. Let us consider an experiment of selecting an agent randomly from a given population. The selected agent, for a given $x \in C_X$, must state whether the property φ holds for X or not. Let E_φ be the following event: E_φ = "They claim that X has the property φ". Then, we consider the conditional event $E_\varphi|A_x$ = "They claim that X has the property φ, knowing that $X = x$" and a conditional probability $P(E_\varphi|A_x)$, seen as a real function $\mu_\varphi(x) = P(E_\varphi|A_x)$ defined on C_X.

Remark 36. *We recall that, every function $\mu_\varphi : C_X \to [0, 1]$, such that $\mu_\varphi(x) = 0$ if $A_x E_\varphi = \emptyset$, and $\mu_\varphi(x) = 1$ if $A_x \subseteq E_\varphi$, is a coherent conditional probability [26, Theorem 2]. In particular, for every $x \in C_X$, if E_φ and A_x are logically independent, as $A_x E_\varphi \neq \emptyset$ and $A_x \nsubseteq E_\varphi$, it follows that every $\mu_\varphi(x) \in [0, 1]$, $x \in C_X$, is a coherent conditional probability (see also [47, Proposition 12]).*

To look at $P(E_\varphi|A_x) = \mu_\varphi(x)$ in the betting framework, we consider a conditional bet that You place on the assertion of the agent randomly chosen from the given population. More precisely, by choosing $P(E_\varphi|A_x) = \mu_\varphi(x)$, for every $s \in \mathbb{R}$, You are willing to pay the amount $s\mu_\varphi(x)$ to receive the random amount $s \cdot (E_\varphi|A_x) = s \cdot (E_\varphi A_x + \mu_\varphi(x)\bar{A}_x)$. In other words, for every $s \in \mathbb{R}$, You pay the amount $s\mu_\varphi(x)$ in order to receive the amount: s, if They claim that X has property φ and $X = x$; 0, if They claim that X has property $\bar{\varphi}$ and $X = x$; $s\mu_\varphi(x)$, that is the paid amount, if $X \neq x$. Similarly to [26, Definition 2] the following probabilistic interpretation of a fuzzy set can be given.

Definition 17 (Fuzzy set)**.** *Let X be a random quantity with set of possible values C_X and a property φ on X. Given a coherent conditional probability P on the family $\mathcal{F}_\varphi = \{E_\varphi|A_x, x \in C_X\}$, we define a* fuzzy set *E^*_φ in $\mathcal{F}_\varphi$, with membership μ_φ, the set*

$$E^*_\varphi = \{(E_\varphi|A_x, \mu_\varphi(x)) : x \in C_X\}, \text{ where } \mu_\varphi(x) = P(E_\varphi|A_x). \tag{4.1}$$

In this setting, for every φ, the fuzzy *emptyset*, $\varnothing^*$, is the fuzzy set where the membership function is $\mu_\varnothing(x) = 0$, for every $x \in C_X$. Analogously, the fuzzy sample set Ω^* is the fuzzy set where the membership function is $\mu_\Omega(x) = 1$ for every $x \in C_X$. Of course, the fuzzy set $\varnothing^*$ (resp., Ω^*) can only be defined if the assessment $\mu_\varphi(x) = 0$ (resp., $\mu_\Omega(x) = 1$), $x \in C_X$, is coherent. Based on Remark 36, under the assumption that, for every $x \in C_X$, the events E_φ and A_x are logically independent, it follows that the fuzzy sets $\varnothing^*$ and Ω^* are properly defined. In our framework, the negation of the event E_φ, denoted by $\bar{E}_\varphi$, is the event "They claim that it is not true that X has the property φ". The sentence "it is not true that X has the property φ" will be simply denoted as "X has the property $\bar{\varphi}$". Then, $\bar{E}_\varphi = E_{\bar{\varphi}} =$ "They claim that X has the property $\bar{\varphi}$". Thus, the negation $\bar{E}_\varphi|A_x$ of the conditional event $E_\varphi|A_x$ coincides with the conditional event $E_{\bar{\varphi}}|A_x$="They claim that X has the property $\bar{\varphi}$, knowing that $X = x$".

Definition 18 (Complement of a fuzzy set)**.** *Given a fuzzy set E^*_φ in $\mathcal{F}_X$, the complement of E^*_φ, denoted by $\bar{E}^*_\varphi$, is the following fuzzy set in $\mathcal{F}_{\bar{\varphi}}$: $\bar{E}^*_\varphi = E^*_{\bar{\varphi}} = \{(E_{\bar{\varphi}}|A_x, \mu_{\bar{\varphi}}(x)) : x \in C_X\}$, where $\mu_{\bar{\varphi}}(x) = P(E_{\bar{\varphi}}|A_x)$.*

In Definition 18 coherence requires that $\mu_{\bar{\varphi}}(x) = P(E_{\bar{\varphi}}|A_x) = P(\bar{E}_\varphi|A_x) = 1 - P(E_\varphi|A_x) = 1 - \mu_\varphi(x)$, for every $x \in C_X$. Therefore,

$$\bar{E}^*_\varphi = \{(\bar{E}_\varphi|A_x, 1 - \mu_\varphi(x)) : x \in C_X\}. \tag{4.2}$$

To illustrate the definition of these fuzzy sets we give the following example.

Example 5. *Let X denote the height (in cm) of a certain individual (let us choose Andy), with $X \in C_X = \{140, 141, \dots, 220\}$. Let φ be the property "tall". We consider the following two events: E_φ = "They claim that Andy is* tall*", A_x= "Andy's height is x cm", $x \in C_X$. Let us consider the fuzzy set E^*_φ on C_X, where, for every $x \in C_X$, $\mu_\varphi(x) = P(E_\varphi|A_x)$ is your degree of belief on the conditional event $E_\varphi|A_x$, that is on the conditional sentence "They claim that Andy is* tall *knowing that his height is x cm". By Definition 18, it holds that the complement $\bar{E}^*_\varphi$ of E^*_φ is the set of the pairs $(E_{\bar{\varphi}}|A_x, \mu_{\bar{\varphi}}(x)) = (\bar{E}_\varphi|A_x, 1 - \mu_\varphi(x))$, for every $x \in C_X$, where $E_{\bar{\varphi}}|A_x$ is the conditional event "They claim that Andy is* not tall*, knowing that his height is x cm".*

4.2 Intersection of fuzzy sets

In this section, to continue with the formalization of set operations among fuzzy sets defined as in Definition 17, we consider a new definition of intersection of fuzzy sets (Definition 19). Moreover, we analyse the cases of the intersection of a fuzzy set with it self and with its complement.

Based on in Definition 3, given two fuzzy sets E^*_φ and E^*_ψ, we define their intersection $E^*_{\varphi\wedge\psi}$ as follows.

Definition 19 (Intersection)**.** *Let X, Y be two random quantities and φ, ψ two properties on X and Y, respectively. Moreover, let $E^*_\varphi = \{(E_\varphi|A_x, \mu_\varphi(x)) : x \in C_X\}$ and $E^*_\psi = \{(E_\psi|A_y, \mu_\psi(y)) : y \in C_Y\}$ be two fuzzy sets on $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$, respectively. The intersection*

$E^*_{\varphi\wedge\psi}$ of E^*_φ and E^*_ψ is the following fuzzy set on $\mathcal{F}_{\varphi\wedge\psi} = \{(E_\varphi|A_x) \wedge (E_\psi|A_y), (x,y) \in C_X \times C_Y\}$:

$$E^*_{\varphi\wedge\psi} = E^*_\varphi \wedge E^*_\psi = \{((E_\varphi|A_x) \wedge (E_\psi|A_y), \mu_{\varphi\wedge\psi}(x,y)) : (x,y) \in C_X \times C_Y\},$$

where $\mu_{\varphi\wedge\psi}(x,y) = \mathbb{P}[(E_\varphi|A_x) \wedge (E_\psi|A_y)]$.

For each pair $(x,y) \in C_X \times C_Y$, the conjunction $(E_\varphi|A_x) \wedge (E_\psi|A_y)$ in Definition 19 can be interpreted as the compound conditional "They claim that: X has the property φ, knowing that $X = x$, and Y has the property ψ, knowing that $Y = y$".

Remark 37. *In [26, 28], the authors consider, under suitable assumptions of conditional independence, as conjunction of two conditional events $E_\varphi|A_x$ and $E_\psi|A_y$ the conditional event $(E_\varphi \wedge E_\psi)|(A_x \wedge A_y)$ (Bochvar conjunction), which can be interpreted as the following conditional "They claim that: X has the property φ and Y has the property ψ, knowing that $X = x$ and $Y = y$". Notice that in general the conditional event $(E_\varphi \wedge E_\psi)|(A_x \wedge A_y)$ does not coincide with the conjuntion $(E_\varphi|A_x) \wedge (E_\psi|A_y)$, which is instead a conditional random quantity. Then, the intersection $E^*_\varphi \wedge E^*_\psi$ introduced in Definition 19 differs from the one given in [28, Definition 21, p.229]. In particular in [28], under the assumption that $P(E_\varphi|(A_x \wedge A_y)) = P(E_\varphi|A_x)$ and $P(E_\psi|(A_x \wedge A_y)) = P(E_\psi|A_y)$, for every $(x,y) \in C_X \times C_Y$, the membership function of the intersection, $\mu'_{\varphi\wedge\psi}$, is interpreted as the probability $P((E_\varphi \wedge E_\psi)|(A_x \wedge A_y))$, for every $(x,y) \in C_X \times C_Y$. Under these hypotheses, it follows that $\mu'_{\varphi\wedge\psi}(x,y)$ satisfies the Fréchet-Hoeffding bounds (eq. (1.15)), that is: $\max\{\mu_\varphi(x) + \mu_\psi(y) - 1, 0\} \leqslant \mu'_{\varphi\wedge\psi}(x,y) \leqslant \min\{\mu_\varphi(x), \mu_\psi(y)\}$. The assumptions of conditional independence recalled above are essential to have these bounds on the probability of the conjunction. Otherwise, for every $(x,y) \in C_X \times C_Y$, given the assessment $(\mu_\varphi(x), \mu_\psi(y)) \in [0,1]^2$ on $\{E_\varphi|A_x, E_\psi|A_y\}$, every $\mu'_{\varphi\wedge\psi}(x,y) \in [0,1]$ is a coherent extension on $(E_\varphi \wedge E_\psi)|(A_x \wedge A_y)$ [88]. We observe that, using the conjunction in Definition 3, that is $(E_\varphi|A_x) \wedge (E_\psi|A_y)$, the Fréchet-Hoeffding bounds (1.15) are satisfied for every $(\mu_\varphi(x), \mu_\psi(y)) \in [0,1]^2$. Nevertheless, sometimes the assumption of conditional independence may not be reasonable, as shown in the following example.*

Example 6. *We set X and E^*_φ as in Example 5. Let Y denote the weight (in kg) of Andy, with $Y \in C_Y = \{40, 41, \dots, 120\}$. Let ψ be the property "fit". We consider the events E_ψ = "They claim that Andy is* fit*", A'_y= "Andy's weight is y kg", $y \in C_Y$. By Remark 36, every $\mu_\psi(y) \in [0,1]$ is a coherent assessment for $E_\psi|A'_y$. We observe that it seems natural not to consider E_ψ stochastically independent from A_x conditionally to A'_y, that is, it seems reasonable to assume that $P(E_\psi|A_xA'_y)$ could not coincide with $P(E_\psi|A'_y)$. Indeed, the information on Andy's height could modify the belief on claiming that Andy is fit only knowing his weight. By Definition 19, for instance, the conditional random quantity $(E_\varphi|A_{180}) \wedge (E_\psi|A'_{75})$ can be interpreted as the conjoint sentence "They claim that: Andy is tall, knowing that his height is* 180 cm, and *Andy is fit, knowing that his weight is* 75 *kg". Suppose to assign the following values to the membership functions: $\mu_\varphi(180) = P(E_\varphi|A_{180}) = 0.85$ and $\mu_\psi(75) = P(E_\psi|A'_{75}) = 0.8$. By Theorem 5, it holds that $\mu_{\varphi\wedge\psi}(180,75) = \mathbb{P}[(E_\varphi|A_{180}) \wedge (E_\psi|A'_{75})]$ is coherent if and only if $\mu_{\varphi\wedge\psi}(180,75) \in [\max\{0.85 + 0.8 - 1, 0\}, \min\{0.85, 0.8\}] = [0.65, 0.8]$.*

Remark 38. *We observe that, by Definition 19, it follows that $E^*_\varphi \wedge E^*_\varphi \neq E^*_\varphi$. Let E^*_φ be a fuzzy set. Then $E^*_\varphi \wedge E^*_\varphi = \{(E_\varphi|A_x) \wedge (E_\varphi|A_{x'}), \mu_{\varphi\wedge\varphi}(x,x')), (x,x') \in C_X \times C_X\}$, where $\mu_{\varphi\wedge\varphi}(x,x') = \mu_\varphi(x)$, if $x = x'$, and $\mu_{\varphi\wedge\varphi}(x,x') = \mu_\varphi(x)\mu_\varphi(x')$, if $x \neq x'$. Indeed, by Definition 3, when $x = x'$, it holds that $(E_\varphi|A_x) \wedge (E_\varphi|A_{x'}) = E_\varphi|A_x$ and hence by coherence $\mu_{\varphi\wedge\varphi}(x,x) = \mu_\varphi(x)$. Otherwise, when we consider pairs $(x,x') \in C_X \times C_X$ where $x \neq x'$, as $A_xA_{x'} = \varnothing$, by Remark 7, it holds that $(E_\varphi|A_x) \wedge (E_\varphi|A_{x'}) = (E_\varphi|A_x) \cdot$*

$(E_\varphi|A_{x'})$ and hence, by coherence, $\mu_{\varphi\wedge\varphi}(x,x') = \mu_\varphi(x)\mu_\varphi(x') \leqslant \min\{\mu_\varphi(x), \mu_\varphi(x')\}$. Therefore, in general, $E^*_\varphi \wedge E^*_\varphi \neq E^*_\varphi$.

It is well-known that, in fuzzy logic, the intersection between a fuzzy set and its complementary set in general does not coincide with the fuzzy empty set. We now observe that this feature still holds when applying Definitions 18 and 19.

Remark 39. *Let E^*_φ be a fuzzy set. Then, by Definition 18 and Definition 19, it holds that $E^*_\varphi \wedge \bar{E}^*_\varphi = \{((E_\varphi|A_x) \wedge (E_{\bar{\varphi}}|A_{x'}), \mu_{\varphi\wedge\bar{\varphi}}(x,x')) : (x,x') \in C_X \times C_X\}$, where $\mu_{\varphi\wedge\bar{\varphi}}(x,x') = 0$, if $x = x'$, and $\mu_{\varphi\wedge\bar{\varphi}}(x,x') = \mu_\varphi(x)\mu_{\bar{\varphi}(x')} = \mu_\varphi(x)\mu_\varphi(1-x')$, if $x \neq x'$. Indeed, we observe that, when $x = x'$, it holds that $(E_\varphi|A_x) \wedge (E_{\bar{\varphi}}|A_x) = (E_\varphi|A_x) \wedge (\bar{E}_\varphi|A_x) = 0$ and hence by coherence $\mu_{\varphi\wedge\bar{\varphi}}(x,x) = 0$. Otherwise, when we consider pairs $(x,x') \in C_X \times C_X$ with $x \neq x'$, by Remark 7 it holds that $(E_\varphi|A_x) \wedge (E_{\bar{\varphi}}|A_{x'}) = (E_\varphi|A_x) \cdot (E_{\bar{\varphi}}|A_{x'})$, which is not constant and can therefore not necessarily be equal to 0. Therefore, as $\mu_{\varphi\wedge\bar{\varphi}}(x,x')$ cannot necessarily be equal to 0, it follows that $E^*_\varphi \wedge E^*_{\bar{\varphi}} \neq \emptyset^*$.*

We recall that Definition 3 has been extended to the case of n conditional events in Definition 4. Then, Definition 19 can be extended to the case of n fuzzy sets as follows.

Definition 20. *Let $X_1, \dots, X_n$ be n random quantities and n associated properties $\varphi_1, \dots, \varphi_n$, respectively. Let $E^*_{\varphi_i} = \{(E_{\varphi_i}|A_{x_i}, \mu_{\varphi_i}(x_i)) : x_i \in C_{X_i}\}$ be a fuzzy set on $\mathcal{F}_{\varphi_i}$, $i = 1, \dots, n$. The fuzzy conjunction $E^*_{\varphi_1} \wedge \dots \wedge E^*_{\varphi_n}$ is defined as*

$$\begin{aligned} & E^*_{\varphi_1\wedge\dots\wedge\varphi_n} = E^*_{\varphi_1} \wedge \dots \wedge E^*_{\varphi_n} = \\ & = \{((E_{\varphi_1}|A_{x_1}) \wedge \dots \wedge (E_{\varphi_n}|A_{x_n}), \mu_{\varphi_1\wedge\dots\wedge\varphi_n}(x_1,\dots,x_n)) : (x_1,\dots,x_n) \in C_{X_1} \times \dots \times C_{X_n}\}, \end{aligned}$$

where $\mu_{\varphi_1\wedge\dots\wedge\varphi_n}(x_1,\dots,x_n) = \mathbb{P}[(E_{\varphi_1}|A_{x_1}) \wedge \dots \wedge (E_{\varphi_n}|A_{x_n})]$.

In the literature, different operators have been suggested to model the intersection and union of fuzzy sets, such as the max-operator, the min-operator ([99]), and the bold union ([44]). In particular, many authors proposed as aggregation operators for the intersection and the union of fuzzy sets the so called triangular norms (t-norms) and conorms (t-conorms), respectively ([72]). For what concerns the t-norm, with our approach not all of them are acceptable to describe coherent probability for the conjunction of conditional events. A class of t-norms that in cases of logical independence describe coherent prevision assessments for the conjunction of two conditional events is given by the Frank t-norms ([52]). They are t-norms T_λ that give a gradual transition between the Lukasiewicz t-norm T_L and the minimum t-norm T_M ([8]), that is

$$\max\{x+y-1, 0\} \leqslant T_{\lambda(x,y)} \leqslant \min\{x,y\}, \lambda \in [0, +\infty]. \tag{4.3}$$

In particular $T_0(x,y) = T_M(x,y) = \min\{x,y\}$, $T_1(x,y) = T_P(x,y) = xy$, and $T_{+\infty}(x,y) = T_L(x,y) = \max\{x+y-1, 0\}$. From (1.15) and (4.3) we have that, under logical independence, for every (coherent) pair of conditional probabilities $\mu_\varphi(x) = P(E_\varphi|A_x)$ and $\mu_\psi(y) = P(E_\psi|A_y)$ the assessment $\mu_{\varphi\wedge\psi}(x,y) = T_\lambda(\mu_\varphi(x), \mu_\psi(y))$ is a coherent extension on $(E_\varphi|A_x) \wedge (E_\psi|A_y)$, for every $\lambda \in [0, +\infty]$. Then, Frank t-norms can be used to construct the membership function of the intersection of two fuzzy sets. However, as shown in the example below, the construction of membership functions by means of Frank t-norms for all the intersections which involve three fuzzy sets could lead to case of incoherent conditional prevision [52, Example 7].

Example 7. *Given any logically independent events $E_1, E_2, E_3, H_1, H_2, H_3$, we observe that the assessment $(x_1, x_2, x_3) = (0.5, 0.6, 0.7)$ on $\{E_1|H_1, E_2|H_2, E_3|H_3\}$ is coherent. Moreover, as shown in [52, Example 7], the extension $(T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3)) = (0.1, 0.2, 0.3, 0)$ on the family $\{\mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$, where $\mathcal{C}_{ij} = (E_i|H_i) \wedge (E_j|H_j)$, and $\mathcal{C}_{123} = (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$, is not coherent. This counterexample shows that, when considering the fuzzy sets $E^*_{\varphi_1}, E^*_{\varphi_2}, E^*_{\varphi_1 \wedge \varphi_2}, E^*_{\varphi_1 \wedge \varphi_3}, E^*_{\varphi_2 \wedge \varphi_3}$, and $E^*_{\varphi_1 \wedge \varphi_2 \wedge \varphi_3}$, the t-norm of Lukasiewics cannot be always chosen as membership function in Definition 20 because it could lead to incoherent assessments.*

Chapter 5

Towards an algebraic and probabilistic setting for iterated Boolean conditionals

The results of this chapter are taken from [22].

In this chapter, we will follow the algebraic approach developed in [40] (see also [41, 42]) to deal with iterated (nested) conditionals. To represent iterated conditionals we employ an algebra that can be built from any Boolean algebra of plain events, by doubling the construction defined in [40] that defines Boolean algebras of conditionals. More precisely, given a Boolean algebra $\mathbf{A}$, its associated Boolean algebra of conditionals, denoted by $\mathcal{C}(\mathbf{A})$, contains all *basic* conditionals of the form $(a \mid b)$ for $a \in A$ and $b \in A \backslash \{\bot\}$ and their Boolean combinations. Since $\mathcal{C}(\mathbf{A})$ is itself a Boolean algebra, and it is finite if $\mathbf{A}$ is finite, we can consider its Boolean algebra of conditionals $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ that now contains *basic* iterated conditionals like $(a \mid b) \mid (c \mid d)$ for $a, c \in A$ and $b, d \in A \backslash \{\bot\}$ and their Boolean combinations as well. Thanks to the richer language available in the Boolean algebras of iterated conditionals it is possible to study algebraic and probabilistic versions of properties that we cannot express with basic conditionals. More precisely, in Section 5.1 we recall some preliminary notions on the construction of the algebra of conditionals $\mathcal{C}(\mathbf{A})$ and its atomic structure, the notion of canonical extension of a positive probability P on $\mathbf{A}$ and the "separability property" for a probability Q on $\mathcal{C}(\mathbf{A})$. In Section 5.2, we analyse the structure of the algebra of iterated conditionals $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ by also describing its atoms and the different ways of obtaining a probability on $\mathcal{C}(\mathcal{C}(\mathbf{A}))$. In Section 5.3, starting from the well-known Import-Export principle, we formulate a weaker version of it and we use the latter to characterize the separable probabilities on $\mathcal{C}(\mathbf{A})$. Then, in Section 5.4 we introduce the "conjunction rationality" principle for the canonical extension of a positive probability Q on $\mathcal{C}(\mathbf{A})$ and we prove it is related to the McGee decomposition formula for the conjunction of two basic conditional given for Q.

5.1 Basics on the Boolean algebra of conditionals

Boolean algebras will be considered in their usual language $(\wedge, \vee, ^{-}, \bot, \top)$ of type $(2, 2, 1, 0, 0)$ where $\wedge$ and $\vee$ denote the lattice operations of meet and join respectively, $^{-}$ indicates the operation of complementation (or negation), while $\bot$ and $\top$ are the constants for the bottom and the top elements respectively. For every Boolean algebra $\mathbf{B} = (B, \wedge, \vee, ^{-}, \bot, \top)$ we will denote by $\leqslant$ the lattice order of $\mathbf{B}$ so that for all $a, b \in B$, $a \leqslant b$ holds iff $a \wedge b = a$ or, equivalently, $a \vee b = b$. Moreover we will denote by B' the set $B \backslash \{\bot\}$.

Now, let $\mathbf{A}$ be a Boolean algebra. The *Boolean algebra of conditionals* of $\mathbf{A}$ ([40]), denoted by $\mathcal{C}(\mathbf{A})$, is defined by firstly considering the Boolean algebra $\text{Free}(A \times A')$ freely generated by the set $A \times A' = \{(a, b) : a \in A, b \in A \backslash \{\bot\}\}$, and then imposing on it some suitable properties of conditionals. This second step is done by taking the quotient of $\text{Free}(A \times A')$ by the minimal congruence $\equiv$ that extends the following set of relations:

- $(b \mid b) \equiv \top$, for all $b \in A \backslash \{\bot\}$;
- $(a_1 \mid b) \wedge (a_2 \mid b) \equiv (a_1 \wedge a_2 \mid b)$, for all $a_1, a_2 \in A$, $b \in A \backslash \{\bot\}$;
- $\overline{(a \mid b)} \equiv (\overline{a} \mid b)$, for all $a \in A$, $b \in A \backslash \{\bot\}$;
- $(a \mid b) \equiv (a \wedge b \mid b)$, for all $a \in A$, $b \in A \backslash \{\bot\}$;
- $(a \mid b) \wedge (b \mid c) \equiv (a \mid c)$, for all $a \in A$, $b, c \in A \backslash \{\bot\}$ such that $a \leqslant b \leqslant c$.

In the above we have adopted the notational convention of writing $a \mid b$ for the pair $(a, b) \in A \times A'$. Sometimes, to ease the reading, we will also write ab instead of $a \wedge b$ and we will use the abbreviation $a \to b$ for the material conditional $\overline{a} \vee b$.

Then, in symbols, $\mathcal{C}(\mathbf{A}) = \text{Free}(\mathbf{A} \times \mathbf{A}')/_{\equiv}$. If $\mathbf{A}$ is finite, then so is $\mathcal{C}(\mathbf{A})$ and we will henceforth assume to work with finite algebras only. In particular the algebras we will consider are atomic with atoms $\text{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$. In [40] the atoms of $\mathcal{C}(\mathbf{A})$ are characterized in terms of the atoms of $\mathbf{A}$ as follows: an element $\omega \in \mathcal{C}(\mathbf{A})$ is an atom iff there exists a permutation $\sigma = (i_1, \ldots, i_n)$ on the set of indices $\{1, \ldots, n\}$ in S_n,[1] and hence a linear sequence of pairwise different atoms of $\mathbf{A}$

$$(\alpha_{i_1}, \ldots, \alpha_{i_n}),$$

such that

$$\begin{aligned}
\omega_\sigma &= (\alpha_{i_1} \mid \top) \wedge (\alpha_{i_2} \mid \overline{\alpha_{i_1}}) \wedge \ldots \wedge (\alpha_{i_{n-1}} \mid \overline{\alpha_{i_1}} \wedge \ldots \wedge \overline{\alpha_{i_{n-2}}}) \wedge (\alpha_{i_n} \mid \overline{\alpha_{i_1}} \wedge \ldots \wedge \overline{\alpha_{i_{n-1}}}) \\
&= (\alpha_{i_1} \mid \top) \wedge (\alpha_{i_2} \mid \overline{\alpha_{i_1}}) \wedge \ldots \wedge (\alpha_{i_{n-1}} \mid \overline{\alpha_{i_1}} \wedge \ldots \wedge \overline{\alpha_{i_{n-2}}}),
\end{aligned} \tag{5.1}$$

since $(\alpha_{i_n} \mid \overline{\alpha_{i_1}} \wedge \ldots \wedge \overline{\alpha_{i_{n-1}}}) = (\alpha_{i_n} | \alpha_{i_n})$ and $(\alpha_{i_n} | \alpha_{i_n})$ is the top element of $\mathcal{C}(\mathbf{A})$. Because of this we can identify atoms of $\mathcal{C}(\mathbf{A})$ by shorter sequences $(\alpha_{i_1} \ldots, \alpha_{i_{n-1}})$ of $n-1$ pairwise different atoms of $\mathbf{A}$. To stress the identification of atoms of $\mathcal{C}(\mathbf{A})$ with sequences of atoms of $\mathbf{A}$, we will write ω_σ for the atoms of $\mathcal{C}(\mathbf{A})$. Moreover, for every atom $\omega_\sigma \in \text{at}(\mathcal{C}(\mathbf{A}))$ and every $j = 1, \ldots, n-1$, we write $\omega_\sigma[j]$ to denote the jth coordinate α_{i_j} of the sequence $(\alpha_{i_1}, \ldots, \alpha_{i_{n-1}})$ that uniquely determines ω_σ. Clearly, $\omega_\sigma[j] \in \text{at}(\mathbf{A})$.

Every positive probability $P : \mathbf{A} \to [0, 1]$ can be extended to a positive probability μ_P on $\mathcal{C}(\mathbf{A})$ by stipulating that for every atom $\omega_\sigma \in \text{at}(\mathcal{C}(\mathbf{A}))$ defined as above,

$$\mu_P(\omega_\sigma) = P(\alpha_{i_1}) \cdot \frac{P(\alpha_{i_2})}{P(\overline{\alpha_{i_1}})} \cdot \ldots \cdot \frac{P(\alpha_{i_{n-1}})}{P(\overline{\alpha_{i_1}} \wedge \ldots \wedge \overline{\alpha_{i_{n-2}}})}.$$

By [40, Lemma 6.8] the above map μ_P is a probability distribution on $\text{at}(\mathcal{C}(\mathbf{A}))$ and its associated probability on $\mathcal{C}(\mathbf{A})$, that we will still denote by the same symbol μ_P, is called the *canonical extension* of P to $\mathcal{C}(\mathbf{A})$.

Moreover, by [40, Theorem 6.13], for every $a, b \in A$, with $b \neq \bot$, $\mu_P(a \mid b) = \frac{P(a \wedge b)}{P(b)}$ and hence canonical extensions μ_P provide a *finitary* solution to what has been called the *strong conditional event problem* (see [67]) within the setting of Boolean algebras of conditionals. This reads as follows:

[1]Adhering to a standard notation we will henceforth denote by S_n the set of permutations over a set of n elements.

(SCEP) For every Boolean algebra $\mathbf{A}$, find another Boolean algebra $\mathbf{A}^*$ and a map $\rhd$: $\mathbf{A} \times \mathbf{A} \to \mathbf{A}^*$ such that $\mathbf{A}$ is subalgebra of $\mathbf{A}^*$ and, for every positive probability P on $\mathbf{A}$, there exists an extension P^* of P on $\mathbf{A}^*$ such that, for all $a, b \in A$ with $b \neq \bot$, $P^*(b \rhd a) = \frac{P(a \wedge b)}{P(b)}$.

The above (SCEP) can be further strengthened by requiring that *every* positive probability on the bigger algebra $\mathbf{A}^*$ behaves as a conditional probability on $\mathbf{A} \times \mathbf{A}'$. Call this stronger principle (SCEP)$^+$.

Although Boolean algebras of conditionals, and canonical extensions, provide a solution for the (SCEP), the stronger (SCEP)$^+$ fails in general in this setting. Indeed, [40, Example 6.3] shows that there are positive probability functions $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0}, \mathbf{1}]$ that fail to model the axioms of conditional probability (see for instance [69]) and therefore they do not satisfy $Q(a \mid b) = \frac{Q(a \wedge b|\top)}{Q(b|\top)}$. However, it is relevant to observe that, by [40, Corollary 6.14], $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0}, \mathbf{1}]$ is a conditional probability iff Q satisfies the next *separability* property.

(Sep) for all $a \leqslant b \leqslant c$ with $b \neq \bot$, $Q(a \mid c) = Q((a \mid b) \wedge (b \mid c)) = Q(a \mid b) \cdot Q(b \mid c)$.

We will call Q *separable* when it satisfies **(Sep)**. In other words, separable probabilities on $\mathcal{C}(\mathbf{A})$ behave as conditional probabilities on $\mathbf{A} \times \mathbf{A}'$.

5.2 Iterated conditionals in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$

In this section, we first analyse the algebraic structure $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ of iterated conditionals, describing in particular its atoms and the general semantic for iterated conditionals. Then, we consider different ways of deriving probabilities on $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ by also comparing the construction of the double canonical extension of a positive probability defined on $\mathbf{A}$, and of the canonical extension of a positive probability on $\mathcal{C}(\mathbf{A})$.

Mimicking the construction of $\mathcal{C}(\mathbf{A})$ from $\mathbf{A}$, being $\mathcal{C}(\mathbf{A})$ itself a Boolean algebra, it is possible to build its Boolean algebra of conditionals $\mathcal{C}(\mathcal{C}(\mathbf{A}))$, that is the Boolean algebra of iterated conditionals of $\mathbf{A}$. In $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ we allow *basic iterated conditionals*, i.e objects of the form $(a|b)|(c|d)$ with $a|b, c|d \in \mathcal{C}(\mathbf{A})$, $\mathbf{c}|\mathbf{d} \neq \bot_{\mathfrak{C}}$, which can be then freely combined using the usual Boolean operations.

Definition 21. *Given a Boolean algebra $\mathbf{A}$ and its Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$, we define the Boolean algebra of iterated conditionals of $\mathbf{A}$ as*

$$\mathcal{C}(\mathcal{C}(\mathbf{A})) = \mathrm{Free}(\mathcal{C}(\mathbf{A}) \times \mathcal{C}(\mathbf{A})')/_{\equiv}$$

where the congruence relation $\equiv$ is defined as in the case of $\mathcal{C}(\mathbf{A})$ except that in the equations the conditions $a, b, c \in \mathbf{A}$, $b, c \neq \bot$ are replaced by $a|b, c|d, e|f \in \mathcal{C}(\mathbf{A})$, $\mathbf{c}|\mathbf{d}, \mathbf{e}|\mathbf{f} \neq \bot$, respectively.

The following properties hold in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ by construction and by recalling how the congruence $\equiv$ is defined in the previous section.

Fact 1. *The following identities hold in every $\mathcal{C}(\mathcal{C}(\mathbf{A}))$.*

(i) $(a|b)|(a|b) = \top|\top$;

(ii) $((a|b)|(c|d)) \wedge ((a'|b')|(c|d)) = [(a|b) \wedge (a'|b')]|(c|d)$;

(iii) $\overline{(a|b)|(c|d)} = \overline{(a \mid b)} \mid (c \mid d) = (\overline{a}|b)|(c|d)$;

(iv) $[(a|b) \wedge (c|d)]|(c|d) = (a|b)|(c|d)$;

(v) *if* $(a|b) \leqslant (c|d) \leqslant (e|f)$, *then* $((a|b)|(c|d)) \wedge ((c|d)|(e|f)) = (a|b)|(e|f)$.

Now we want to give a characterization of the atoms of $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ in terms of the atoms in $\mathcal{C}(\mathbf{A})$. A basic observation is that if $\mathbf{A}$ is finite, then $\mathcal{C}(\mathbf{A})$ is also finite, and thus $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ is finite as well, and hence all of them are atomic.

We recall that, if $|\mathrm{at}(\mathbf{A})| = n$, the atoms of $\mathcal{C}(\mathbf{A})$ are determined by permutations in S_n and hence by sequences of length $n-1$ of pairwise different atoms of $\mathbf{A}$; as a consequence $|\mathrm{at}(\mathcal{C}(\mathbf{A}))| = \mathbf{n}!$.

We also recall that the lattice order relation $\leqslant$ in $\mathcal{C}(\mathbf{A})$ is defined as

$$t \leqslant s \text{ iff } t \wedge s = t \text{ iff } t \vee s = s, \text{ for every } s, t \in \mathcal{C}(\mathbf{A}). \tag{5.2}$$

Remark 40. *In [40, Proposition 4.7] it is shown that an atom* $\omega_\sigma = (\alpha_{\sigma(1)} \mid \top) \wedge (\alpha_{\sigma(2)} \mid \overline{\alpha_{\sigma(1)}}) \wedge \cdots \wedge (\alpha_{\sigma(n-1)} \mid \overline{\alpha_{\sigma(1)}} \cdots \overline{\alpha_{\sigma(n-2)}})$ *is below (i.e. it is a* model *for) a conditional* $(a \mid b)$ *w.r.t. the lattice order* $\leqslant$ *in* $\mathcal{C}(\mathbf{A})$*, i.e.* $\omega_\sigma \leqslant (a \mid b)$ *(also written as* $\omega_\sigma \models (a|b)$*), if and only if the following condition is satisfied:*

$$\begin{array}{l} \textit{either } \text{“}\alpha_{\sigma(1)} \leqslant ab\text{”}, \textit{ or } \text{“}\alpha_{\sigma(1)} \leqslant \overline{b} \textit{ and } \alpha_{\sigma(2)} \leqslant ab\text{”}, \textit{ or } \ldots \\ \textit{or } \text{“}\alpha_{\sigma(1)} \leqslant \overline{b} \textit{ and } \ldots \textit{and } \alpha_{\sigma(n-2)} \leqslant \overline{b} \textit{ and } \alpha_{\sigma(n-1)} \leqslant ab\text{”}. \end{array}$$

□

By repeating the same procedure as above, it follows that the atoms of $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ will be determined by sequences of length $|\mathrm{at}(\mathcal{C}(\mathbf{A}))| - \mathbf{1} = \mathbf{n}! - \mathbf{1}$ of pairwise different atoms of $\mathcal{C}(\mathbf{A})$, that is, by sequences of the form $(\omega_{\sigma_1}, \omega_{\sigma_2}, \ldots, \omega_{\sigma_{n!-1}})$, where $\sigma_j \in S_n$ is a permutation and $\omega_{\sigma_j} \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$ for all $j = 1, \ldots, n! - 1$.

Therefore, $|\mathrm{at}(\mathcal{C}(\mathcal{C}(\mathbf{A})))| = (\mathbf{n}!)!$. In terms of the atoms of $\mathbf{A}$, an atom of $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ can be represented by a sequence of sequences of atoms of $\mathbf{A}$, and it can be visualized as the matrix

$$M = \begin{pmatrix} \omega_{\sigma_1}, & \omega_{\sigma_2}, & \ldots, & \omega_{\sigma_{n!-1}} \end{pmatrix} = \begin{pmatrix} \omega_{\sigma_1}[1] & \omega_{\sigma_2}[1] & \ldots & \omega_{\sigma_{n!-1}}[1] \\ \omega_{\sigma_1}[2] & \omega_{\sigma_2}[2] & \ldots & \omega_{\sigma_{n!-1}}[2] \\ \vdots & \vdots & \ldots & \vdots \\ \omega_{\sigma_1}[n-1] & \omega_{\sigma_2}[n-1] & \ldots & \omega_{\sigma_{n!-1}}[n-1] \end{pmatrix}$$

Given a matrix M as the above, conforming to the previous notation, the corresponding atom ω_M of $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ is the following conjunction of iterated conditionals:

$$\omega_M = (\omega_{\sigma_1} \mid \top) \wedge (\omega_{\sigma_2} \mid \overline{\omega_{\sigma_1}}) \wedge \cdots \wedge (\omega_{\sigma_{n!-1}} \mid \overline{\omega_{\sigma_1}} \cdots \overline{\omega_{\sigma_{n!-2}}}).$$

The next example hints on what these formal expressions look like in the iterated case of $\mathcal{C}(\mathcal{C}(\mathbf{A}))$.

Example 8. *Let* $\mathbf{A}$ *be an algebra with* 3 *atoms* $\alpha_1, \alpha_2, \alpha_3$*. Then, the conditional algebra* $\mathcal{C}(\mathbf{A})$ *has* $3! = 6$ *atoms:*

$$\begin{array}{lll} \omega_1 = (\alpha_1|\top) \wedge (\alpha_2|\overline{\alpha}_1), & \omega_2 = (\alpha_1|\top) \wedge (\alpha_3|\overline{\alpha}_1), & \omega_3 = (\alpha_2|\top) \wedge (\alpha_1|\overline{\alpha}_2), \\ \omega_4 = (\alpha_2|\top) \wedge (\alpha_3|\overline{\alpha}_2), & \omega_5 = (\alpha_3|\top) \wedge (\alpha_1|\overline{\alpha}_3), & \omega_6 = (\alpha_3|\top) \wedge (\alpha_2|\overline{\alpha}_3). \end{array}$$

Conforming to the notation introduced above, the atoms of $\mathcal{C}(\mathbf{A})$ *are identified both with sequences* $(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})$ *of length* 3 *and with sequences* $(\alpha_{i_1}, \alpha_{i_2})$ *of length* 2 *(since the missing* *atom is univocally determined). For the sake of clarity, in this example we will consider sequences of length* 3. *For instance* ω_1 *will be identified with* $(\alpha_1, \alpha_2, \alpha_3)$. *Analogously, the conditional algebra* $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ *has* 6! *atoms whose associated sequences have length* 6 *(instead of* 5*) and they are hence of the form*

$$M = (\omega_{\sigma_1}, \omega_{\sigma_2}, \omega_{\sigma_3}, \omega_{\sigma_4}, \omega_{\sigma_5}, \omega_{\sigma_6})$$

where, for all $i = 1, \ldots, 6$, $\omega_{\sigma_i} \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$.
For example, let us consider the matrix

$$M = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_2 & \alpha_3 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_1 & \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_2 & \alpha_3 & \alpha_1 & \alpha_2 & \alpha_1 \end{pmatrix}$$

where the first column of M *corresponds to* ω_1, *the second to* ω_2 *and so forth. Its corresponding atom in* $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ *can be written as:*

$$\begin{aligned} \omega_M &= (\omega_1|(\top|\top)) \wedge (\omega_2|\overline{\omega}_1) \wedge (\omega_3|\overline{\omega}_1 \wedge \overline{\omega}_2) \wedge (\omega_4|\overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \overline{\omega}_3) \wedge \\ &\quad \wedge (\omega_5|\overline{\omega}_1 \wedge \overline{\omega}_2 \wedge \overline{\omega}_3 \wedge \overline{\omega}_4) = \\ &= (\omega_1|(\top|\top)) \wedge (\omega_2|\omega_2 \vee \omega_3 \vee \omega_4 \vee \omega_5 \vee \omega_6) \wedge \\ &\quad \wedge (\omega_3|\omega_3 \vee \omega_4 \vee \omega_5 \vee \omega_6) \wedge (\omega_4|\omega_4 \vee \omega_5 \vee \omega_6) \wedge (\omega_5|\omega_5 \vee \omega_6). \end{aligned}$$

To get more hints on the previous expression, let us observe that

$$\begin{aligned} \omega_5 \vee \omega_6 &= ((\alpha_3|\top) \wedge (\alpha_1|\overline{\alpha}_3)) \vee ((\alpha_3|\top) \wedge (\alpha_2|\overline{\alpha}_3)) = (\alpha_3|\top) \wedge (\alpha_1 \vee \alpha_2|\overline{\alpha}_3) = \\ &= (\alpha_3|\top) \wedge (\alpha_1 \vee \alpha_2|\alpha_1 \vee \alpha_2) = (\alpha_3|\top) \end{aligned}$$

and hence $(\omega_5|\omega_5 \vee \omega_6) = ((\alpha_3|\top) \wedge (\alpha_1|\overline{\alpha}_3))|(\alpha_3|\top) = (\alpha_1|\overline{\alpha}_3)|(\alpha_3|\top)$.
Moreover, recalling that for $a, b, c \in \mathbf{A}$

- *if* $a \leqslant b \leqslant d$ *then* $(a|b) \geqslant (a|d)$, *and hence* $\alpha_3 \leqslant \overline{\alpha}_2 = \alpha_1 \vee \alpha_3 \leqslant \top$ *and it implies that* $(\alpha_3|\overline{\alpha}_2) \geqslant (\alpha_3|\top)$,
- $(a \wedge b|\top) \leqslant (a|b) \leqslant (b \to a|\top)$ *and hence* $(\alpha_3|\overline{\alpha}_2) \leqslant (\overline{\alpha}_2 \to \alpha_3|\top)$

we have that

$$\begin{aligned} \omega_4 \vee \omega_5 \vee \omega_6 &= ((\alpha_2|\top) \wedge (\alpha_3|\overline{\alpha}_2)) \vee (\alpha_3|\top) \\ &= ((\alpha_2|\top) \vee (\alpha_3|\top)) \wedge ((\alpha_3|\overline{\alpha}_2) \vee (\alpha_3|\top)) \\ &= (\alpha_2 \vee \alpha_3|\top) \wedge (\alpha_3|\overline{\alpha}_2) \\ &= (\overline{\alpha}_2 \to \alpha_3|\top) \wedge (\alpha_3|\overline{\alpha}_2) = (\alpha_3|\overline{\alpha}_2). \end{aligned}$$

Then, $(\omega_4|\omega_4 \vee \omega_5 \vee \omega_6) = ((\alpha_2|\top) \wedge (\alpha_3|\overline{\alpha}_2))|(\alpha_3|\overline{\alpha}_2) = (\alpha_2|\top)|(\alpha_3|\overline{\alpha}_2)$.
Let us now consider

$$\omega_3 \vee \omega_4 \vee \omega_5 \vee \omega_6 = ((\alpha_2|\top) \wedge (\alpha_1|\overline{\alpha}_2)) \vee (\alpha_3|\overline{\alpha}_2) = (\alpha_2|\top) \vee (\alpha_3|\overline{\alpha}_2).$$

Therefore, $(\omega_3|\omega_3 \vee \omega_4 \vee \omega_5 \vee \omega_6) = ((\alpha_2|\top) \wedge (\alpha_1|\overline{\alpha}_2))|((\alpha_2|\top) \vee (\alpha_3|\overline{\alpha}_2))$.
We can also consider

$$\begin{aligned} \omega_2|\overline{\omega}_1 &= ((\alpha_1|\top) \wedge (\alpha_3|\overline{\alpha}_1))|((\alpha_2 \vee \alpha_3|\top) \vee (\alpha_1 \vee \alpha_3|\overline{\alpha}_1)) \\ &= ((\alpha_1|\top) \wedge (\alpha_3|\overline{\alpha}_1))|((\alpha_2 \vee \alpha_3|\top) \vee (\alpha_3|\overline{\alpha}_1)) \\ &= ((\alpha_1|\top) \wedge (\alpha_3|\overline{\alpha}_1))|((\alpha_2|\top) \vee (\alpha_3|\overline{\alpha}_1)). \end{aligned}$$

Finally, we come up with the following expression of the atom ω_M:

$$\begin{aligned}\omega_M = \quad & ((\alpha_1|\top) \wedge (\alpha_2|\overline{\alpha}_1))|(\top|\top) \quad \wedge \\ & ((\alpha_1|\top) \wedge (\alpha_3|\overline{\alpha}_1))|((\alpha_2|\top) \vee (\alpha_3|\overline{\alpha}_1)) \quad \wedge \\ & ((\alpha_2|\top) \wedge (\alpha_1|\overline{\alpha}_2))|((\alpha_2|\top) \vee (\alpha_3|\overline{\alpha}_2)) \quad \wedge \\ & (\alpha_2|\top)|(\alpha_3|\overline{\alpha}_2) \wedge (\alpha_1|\overline{\alpha}_3)|(\alpha_3|\top).\end{aligned}$$

□

Remark 41. *Given an iterated conditional $(a|b)|(c|d) \in \mathcal{C}(\mathcal{C}(\mathbf{A}))$ and a matrix $M = (\omega_1, \ldots, \omega_n)$, we say the atom ω_M* models *$(a|b)|(c|d)$, written $\omega_M \models (a|b)|(c|d)$, if and only if $\omega_M \leqslant (a|b)|(c|d)$ and hence, recalling Remark 40, if and only if, $\exists j$ such that $M[j] = \omega_j \models (a|b) \wedge (c|d)$ and, $\forall k < j, M[k] = \omega_k \not\models (c|d)$.*

Now, let us turn our attention on canonical extensions on $\mathcal{C}(\mathbf{A})$ and on $\mathcal{C}(\mathcal{C}(\mathbf{A}))$. Reproducing the results about the canonical extension of positive probabilities on $\mathbf{A}$ to probabilities on $\mathcal{C}(\mathbf{A})$, given a positive probability $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0}, \mathbf{1}]$ we can always define its canonical extension $\mu_Q : \mathcal{C}(\mathcal{C}(\mathbf{A})) \to [\mathbf{0}, \mathbf{1}]$ in such a way that, for every iterated conditional $(a|b) \mid (c|d) \in \mathcal{C}(\mathcal{C}(\mathbf{A}))$, we have

$$\mu_Q((a|b) \mid (c|d)) = \frac{Q((a|b) \wedge (c|d))}{Q(c|d)}.$$

In particular, if Q is the canonical extension μ_P of a positive probability P on $\mathbf{A}$, then μ_P on $\mathcal{C}(\mathbf{A})$ is again positive, and thus we can in turn canonically extend it to a probability μ_{μ_P} on $\mathcal{C}(\mathcal{C}(\mathbf{A}))$. In this case, [42, Theorem 8] (see also [41]) tells us that the expression for $\mu_P((a|b) \wedge (c|d))$ is

$$\mu_P((a|b) \wedge (c|d)) = P(abcd|b \vee d) + P(a|b)P(\bar{b}cd|b \vee d) + P(c|d)P(\bar{d}ab|b \vee d),$$

that coincides with the formula obtained by McGee [80] and Kaufmann [71]. Then, we get the following expression for the probability of an iterated conditional:

$$\mu_{\mu_P}((a|b) \mid (c|d)) = \frac{\mu_P((a|b) \wedge (c|d))}{\mu_P(c|d)} = \frac{P(abcd|b \vee d) + P(a|b)P(\bar{b}cd|b \vee d) + P(c|d)P(\bar{d}ab|b \vee d)}{P(c|d)}.$$

5.3 The weak Import-Export principle and the separability property

As already recalled in Section 2.1, a well-known and widely discussed principle for conditional logics is that of *Import-Export* (also known under the name of *exportation*). This principle needs a nested occurrence of the conditional operator to be written and hence, while it cannot be expressed in a conditional algebra like $\mathcal{C}(\mathbf{A})$, it can be formalized in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$. In the language of $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ it hence reads as the following equation: for all $a, b, c \in A$ such that $b, c \neq \perp$,

(IE) $(a \mid c) \mid (b \mid \top) = (a \mid c \wedge b) \mid (\top \mid \top)$.

In what follows we will be interested in the next equation that is in fact a weak version of **(IE)** obtained by taking $c = \top$ in **(IE)**:

(wIE) $(a \mid \top) \mid (b \mid \top) = (a \mid b) \mid (\top \mid \top)$.

Besides being a weakening of the Import-Export principle, the identity **(wIE)** states a condition that appears to be reasonable to ask for iterated conditionals. Indeed, keeping the discussion at an informal level, the identification of every plain event $a \in A$ with its conditional representation $(a \mid \top) \in \mathcal{C}(\mathbf{A})$ ensures the two sides of **(wIE)** read the same. More precisely, omitting the trivial reading of those antecedents "given that the sure event $\top$ holds", **(wIE)** reads tautologically: "a given b equals a given b".

Although reasonable, **(wIE)** does not always hold in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ as the next result shows.

Proposition 1. *For every algebra* $\mathbf{A}$ *with more than two atoms, there are* $a, b \in A$ *with* $b \neq \perp$ *for which* **(wIE)** *does not hold in* $\mathcal{C}(\mathcal{C}(\mathbf{A}))$.

Proof. Let us consider the case of an algebra $\mathbf{A}$ with three atoms $\{\alpha, \beta, \gamma\}$. We take $a = \alpha$ and $b = \alpha \vee \beta$. Consider a matrix like the following

$$M_1 = \begin{pmatrix} \gamma & \beta & \dots \\ \alpha & \gamma & \dots \end{pmatrix}.$$

By Remark 41, $\omega_{M_1} \models (a|b)|(\top|\top)$ but $\omega_{M_1} \not\models (a|\top)|(b|\top)$. Similarly, a matrix like

$$M_2 = \begin{pmatrix} \gamma & \alpha & \dots \\ \beta & \gamma & \dots \end{pmatrix}$$

is such that $\omega_{M_2} \not\models (a|b)|(\top|\top)$ but $\omega_{M_2} \models (a|\top)|(b|\top)$.

Therefore, $(a|b)|(\top|\top)$ and $(a|\top)|(b|\top)$ are different elements of $\mathcal{C}(\mathcal{C}(\mathbf{A}))$, and hence **(wIE)** does not hold in this $\mathcal{C}(\mathcal{C}(\mathbf{A}))$. □

Although the weak Import-Export principle **(wIE)** does not in general hold in the algebras $\mathcal{C}(\mathcal{C}(\mathbf{A}))$, however, it is interesting to observe that, if P is a positive probability on $\mathbf{A}$, then the double canonical extension $\mu_{\mu_P} : \mathcal{C}(\mathcal{C}(\mathbf{A})) \to [\mathbf{0}, \mathbf{1}]$ is a *probabilistic model* for **(wIE)**, that is to say, for any $a \in A$ and $b \in A'$,

$$\mu_{\mu_P}((a \mid \top) \mid (b \mid \top)) = \mu_{\mu_P}((a \mid b) \mid (\top \mid \top)).$$

Still, μ_{μ_P} does not provide a model for the Import-Export principle **(IE)**.

Proposition 2. *For every positive probability function* P *on* $\mathbf{A}$, μ_{μ_P} *is a probabilistic model of* **(wIE)**, *but, in general, it is not a probabilistic model of* **(IE)**.

Proof. Let us consider the quantities $\mu_P((a \mid c) \wedge (b \mid \top))$ and $\mu_{\mu_P}((a \mid c \wedge b) \mid (\top \mid \top))$. By [71] and [90, Section 8], and recalling the last equation of the above Section 5.2,

$$\mu_{\mu_P}((a \mid c) \mid (b \mid \top)) \quad = \frac{P(abc)+P(a|c)\cdot P(b\bar{c})}{P(b)}. \tag{5.3}$$

We distinguish two cases: (i) $bc = \perp$; (ii) $bc \neq \perp$.
Case (i). By (5.3), it follows that $\mu_{\mu_P}((a \mid c) \mid (b \mid \top)) = P(a|c)$, while $\mu_{\mu_P}((a \mid c \wedge b) \mid (\top \mid \top))$ is undefined, because $(a \mid c \wedge b) = (a|\perp) \notin \mathcal{C}(\mathbf{A})$.
Case (ii). By (5.3), it follows that

$$\begin{aligned} \mu_{\mu_P}((a \mid c) \mid (b \mid \top)) &= \frac{P(a|bc)P(c|b)P(b)+P(a|c)\cdot P(\bar{c}|b)P(b)}{P(b)} = \\ &= P(a|c \wedge b)P(c|b) + P(a|c)P(\bar{c}|b) \end{aligned} \tag{5.4}$$

while

$$\mu_{\mu_P}((a \mid c \wedge b) \mid (\top \mid \top)) = P(a \mid c \wedge b).$$

Therefore, μ_{μ_P} does not model **(IE)** in general. However, let us take $c = \top$. As $P(\top|b) = 1$, $P(\perp|b) = 0$ and $P(a|\top \wedge b) = P(a|b)$, it holds that

$$\mu_{\mu_P}((a \mid \top) \mid (b \mid \top)) = P(a|b) = \mu_{\mu_P}((a \mid b) \mid (\top \mid \top)) = P(a|b)$$

and hence **(wIE)** is satisfied by μ_{μ_P}.

□

Example 9. *In this example we show what kind of consequences can be obtained when a positive probability P on $\mathbf{A}$ is such that μ_{μ_P} is a probabilistic model for* (**IE**)*. By using the fact in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ we have $\mu_P(t \wedge s) = \mu_{\mu_P}(t \mid s)\mu_P(s)$ for every $t, s \in \mathcal{C}(\mathbf{A})$, the following equalities always hold for any two events $a \in A$ and $c \in A'$:*

$$\begin{aligned} P(a \mid c) = \mu_P(a \mid c) &= \mu_P((a \mid c) \wedge (a \mid \top)) + \mu_P((a \mid c) \wedge (\bar{a} \mid \top)) = \\ &= \mu_{\mu_P}((a \mid c) \mid (a|\top))\mu_P(a \mid \top) + \mu_{\mu_P}((a \mid c) \mid (\bar{a}|\top))\mu_P(\bar{a} \mid \top) = \\ &= \mu_{\mu_P}((a \mid c) \mid (a|\top))P(a) + \mu_{\mu_P}((a \mid c) \mid (\bar{a}|\top))P(\bar{a}). \end{aligned}$$

Now, if μ_{μ_P} is a probabilistic model for **(IE)** *then it follows that $\mu_{\mu_P}((a \mid c) \mid (a|\top)) = \mu_{\mu_P}((a \mid ac) \mid (\top|\top)) = P(a \mid ac) = 1$ and $\mu_{\mu_P}((a \mid c) \mid (\bar{a}|\top)) = \mu_{\mu_P}((a \mid \bar{a}c) \mid (\top|\top)) = P(a \mid \bar{a}c) = 0$, and thus it results that*

$$P(a|c) = 1 \cdot P(a) + 0 \cdot P(\bar{a}) = P(a),$$

which is in general not satisfactory. This derivation is actually another proof of one of the well-known Lewis' triviality results. However, as in general μ_{μ_P} is not a probabilistic model for (**IE**)*, Lewis' triviality result can be avoided by μ_{μ_P} in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$.*

Remark 42. *We observe that in the particular cases in which $b \leqslant c$ or $c \leqslant b$, we have that μ_P satisfies* **(IE)** *([54, Proposition 1]). If $b \leqslant c$ it holds that $\bar{c}b = \perp$ so that $P(\bar{c}|b) = 0$. Moreover, $a|(c \wedge b) = a|b$ and hence $P(a|c \wedge b) = P(a|b)$. Then, by (5.4), it holds that*

$$\mu_{\mu_P}((a \mid c) \mid (b \mid \top)) = P(a|b) \cdot 1 + P(a|c) \cdot 0 = P(a|b) = \mu_{\mu_P}((a \mid c \wedge b) \mid (\top \mid \top)).$$

Likewise, if $c \leqslant b$, $P(a|c \wedge b) = P(a|c)$ and hence

$$\mu_{\mu_P}((a \mid c) \mid (b \mid \top)) = P(a|c)[P(c|b) + P(\bar{c}|b)] = P(a|c) = \mu_{\mu_P}((a \mid c \wedge b) \mid (\top \mid \top)).$$

Now we turn our attention on the satisfiability of **(wIE)** by the canonical extensions μ_Q on $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ of a positive $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0}, \mathbf{1}]$.

As we recalled in Section 5.1, the separability property for a probability $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0}, \mathbf{1}]$ characterizes those functions that behave as conditional probabilities on $\mathbf{A}$. While separability seems to be non-characterizable in the language of $\mathcal{C}(\mathbf{A})$, the next result shows that, interestingly, it can be captured by requiring the canonical extension μ_Q of Q to model the equation **(wIE)** in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$.

Theorem 31. *Let $\mathbf{A}$ be any finite Boolean algebra and let $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0}, \mathbf{1}]$ be a positive probability function. Then, Q is separable iff $\mu_Q((a|b)|(\top|\top)) = \mu_Q((a|\top)|(b|\top))$ for every $a \in \mathbf{A}$ and $b \in \mathbf{A} \setminus \{\perp\}$.*

Proof. Let us start assuming that Q is separable. In such a case, we know that $Q(a \wedge b|\top) = Q((a|b) \wedge (b|\top)) = Q(a|b)Q(b|\top)$. Then, it follows that

$$\mu_Q((a|b)|(\top|\top)) = Q(a|b) = \frac{Q(a \wedge b|\top)}{Q(b|\top)} = \frac{Q((a|\top) \wedge (b|\top))}{Q(b|\top)} = \mu_Q((a|\top)|(b|\top)).$$

Conversely, let us suppose that $\mu_Q((a|\top)|(b|\top)) = \mu_Q((a|b)|(\top|\top))$ for every $a \in \mathbf{A}$ and $b \in \mathbf{A}\backslash\{\bot\}$. Now, let $P_Q : \mathbf{A} \to [0,1]$ be the probability on $\mathbf{A}$ defined by restricting Q to $\mathbf{A}$ (considered as a subalgebra of $\mathcal{C}(\mathbf{A})$), and consider $\mu_{P_Q} : \mathcal{C}(\mathbf{A}) \to [\mathbf{0},\mathbf{1}]$, the canonical extension of P_Q to $\mathcal{C}(\mathbf{A})$. Then, for $a, b \in \mathbf{A}$,

$$\begin{aligned}\mu_{P_Q}(a|b) &= \frac{P_Q(a \wedge b)}{P_Q(b)} = \frac{Q(a \wedge b|\top)}{Q(b|\top)} = \frac{Q((a|\top) \wedge (b|\top))}{Q(b|\top)} \\ &= \mu_Q((a|\top)|(b|\top)) = \mu_Q((a|b)|(\top|\top)) = Q(a|b).\end{aligned}$$

Therefore, Q and μ_{P_Q} coincide on the basic conditionals of the form $(x \mid y)$. Moreover, we know that μ_{P_Q} is a canonical extension and hence it is separable, i.e.

$$\mu_{P_Q}(a|c) = \mu_{P_Q}((a|b) \wedge (b|c)) = \mu_{P_Q}(a|b)\mu_{P_Q}(b|c)$$

for $a \leqslant b \leqslant c$, with $a, b, c \in \mathbf{A}$. But we found out that on basic conditionals, $\mu_{P_Q}(a|c) = Q(a|c)$, $\mu_{P_Q}(a|b) = Q(a|b)$, and $\mu_{P_Q}(b|c) = Q(b|c)$, and hence

$$Q(a|c) = Q(a|b)Q(b|c), \text{ for } a, b, c \in \mathbf{A} \text{ s.t. } a \leqslant b \leqslant c,$$

that is, Q is separable.

□

5.4 On the McGee formula and the conjunction rationality principle

In this final section we consider probabilistic properties related to the conjunction of basic conditionals. Any canonical extension μ_P of a (positive) probability P on $\mathbf{A}$ satisfies the McGee formula for conjunction of basic conditionals, but this is not the case for any probability Q on $\mathcal{C}(\mathbf{A})$.

Definition 22. *A (positive) probability $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0},\mathbf{1}]$ satisfies the* McGee decomposition *formula (McD) when, for every $(a|b), (c|d) \in \mathcal{C}(\mathbf{A})$,*

$$Q((a|b) \wedge (c|d)) = Q(abcd|b \vee d) + Q(a|b)Q(\bar{b}cd|b \vee d) + Q(c|d)Q(\bar{d}ab|b \vee d).$$

By the very definition of the canonical extension μ_Q on $C(\mathcal{C}(\mathbf{A}))$ the following properties hold.

Lemma 1. *For any probability $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0},\mathbf{1}]$ the following identities hold:*

(i) $\mu_Q([(a|b) \mid (\top|\top)] \wedge [(c|d) \mid (\top|\top)]) = \mu_Q((a|b) \wedge (c|d) \mid (\top|\top)) = Q((a|b) \wedge (c|d))$.

(ii) If Q is separable, then $\mu_Q([(a|\top) \mid (b|\top)] \wedge [(c|\top) \mid (d|\top)]) = Q(abcd|b \vee d) + Q(a|b)Q(\bar{b}cd|b \vee d) + Q(c|d)Q(\bar{d}ab|b \vee d)$.

Proof. (i) Since $Q(x) = \mu_Q(x \mid (\top|\top))$ for any $x \in C(\mathcal{C}(\mathbf{A}))$, we simply have that $\mu_Q([(a|b) \mid (\top|\top)] \wedge [(c|d) \mid (\top|\top)]) = \mu_Q((a|b) \wedge (c|d) \mid (\top|\top)) = Q((a|b) \wedge (c|d))$, where the first equality follows from Fact 1 (ii).

(ii) Since μ_Q is a canonical extension, it satisfies McGee decomposition formula for a conjunction of basic (iterated) conditionals, and we have:

$$\begin{aligned}&\mu_Q([(a|\top) \mid (b|\top)] \wedge [(c|\top) \mid (d|\top)]) = \\ &= \mu_Q((abcd|\top)|(b \vee d|\top)) + \mu_Q((a|\top)|(b|\top)) \cdot \mu_Q((\bar{b}cd|\top) \mid (b \vee d|\top)) + \\ &\quad + \mu_Q((c \mid \top) \mid (d|\top)) \cdot \mu_Q((\bar{d}ab|\top) \mid (b \vee d|\top)).\end{aligned}$$

Now, if Q is separable, by Theorem 31, $\mu_Q((a|\top) \mid (b|\top)) = \mu_Q((a|b) \mid (\top|\top)) = Q(a|b)$. Therefore,

$$\begin{aligned}&\mu_Q([(a|\top) \mid (b|\top)] \wedge [(c|\top) \mid (d|\top)]) = \\ &= Q(abcd|b \vee d) + Q(a|b)Q(\bar{b}cd|b \vee d) + Q(c|d)Q(\bar{d}ab|b \vee d).\end{aligned}$$

□

Let us consider now the following principle at the level of iterated conditionals.

Definition 23. *A probability $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0}, \mathbf{1}]$ satisfies the* conjunction rationality principle *(CRP) when, for all $a, b \in A$ and $c, d \in A'$:*

$$(CRP) \quad \mu_Q([(a|b) \mid (\top|\top)] \wedge [(c|d) \mid (\top|\top)]) = \mu_Q([(a|\top) \mid (b|\top)] \wedge [(c|\top) \mid (d|\top)])$$

Note that if Q satisfies (CRP) necessarily it must also be separable, since taking $c = d = \top$ above we get $\mu_Q((a|b) \mid (\top|\top)) = \mu_Q((a|\top) \mid (b|\top))$, and by Theorem 31, Q must be separable.

We can finally prove that (CRP) characterises in $\mathcal{C}(\mathcal{C}(\mathbf{A}))$ the satisfaction of McGee decomposition formula in $\mathcal{C}(\mathbf{A})$.

Theorem 32. *A probability $Q : \mathcal{C}(\mathbf{A}) \to [\mathbf{0}, \mathbf{1}]$ satisfies the* conjunction rationality principle *(CRP) iff Q satisfies the McGee decomposition (McD).*

Proof. If Q satisfies (CRP), Q is separable, and then by Lemma 1 above, $Q((a|b) \wedge (c|d)) = \mu_Q([(a|\top) \mid (b|\top)] \wedge [(c|\top) \mid (d|\top)]) = Q(abcd|b \vee d) + Q(a|b)Q(\bar{b}cd|b \vee d) + Q(c|d)Q(\bar{d}ab|b \vee d)$, hence Q satisfies (McD).

Conversely, since μ_Q is a canonical extension (and hence it satisfies McGee formula), we have:

$$\begin{aligned}&\mu_Q((a|\top) \mid (b|\top)) = \mu_Q([(a|\top) \mid (b|\top)] \wedge [(\top|\top) \mid (\top|\top)]) = \\ &\mu_Q((ab|\top)|(\top|\top)) + \mu_Q((a|\top)|(b|\top)) \cdot \mu_Q((\bar{b}|\top) \mid (\top|\top)) + 0 = \\ &Q(ab|\top) + \mu_Q((a|\top)|(b|\top)) \cdot Q((\bar{b}|\top))\end{aligned}$$

that is, $\mu_Q((a|\top)|(b|\top)) = Q(ab|\top)/Q(b|\top)$. On the other hand, if Q satisfies (McD), we have $Q(a|b) = Q((a|b) \wedge (\top|\top)) = Q(ab|\top) + Q(a|b) \cdot Q(\bar{b}|\top)$, hence $Q(a|b) = Q(ab|\top)/Q(b|\top)$, since we are assuming Q to be positive.

Therefore, it follows that $Q(a|b) = \mu_Q((a|\top) \mid (b|\top))$ and thus Q is separable because of (i) of Lemma 1. Then, we can finally conclude, by (ii) of the same lemma above, that:

$$\begin{aligned}&\mu_Q([(a|b) \mid (\top|\top)] \wedge [(c|d) \mid (\top|\top)]) = Q((a|b) \wedge (c|d)) \\ &= Q(abcd|b \vee d) + Q(a|b)Q(\bar{b}cd|b \vee d) + Q(c|d)Q(\bar{d}ab|b \vee d) \\ &= \mu_Q([(a|\top) \mid (b|\top)] \wedge [(c|\top) \mid (d|\top)]),\end{aligned}$$

that is, (CRP) is satisfied.

□

Chapter 6

Entropy and extropy for partial probability assessments on arbitrary families of events

The results of this chapter are taken from [19].

Among the many research topics covered by de Finetti during his career, he studied the theory of proper scoring rules in particular analysing the connections between the admissibility of a probability assessment and the concept of coherence by means of the penalty criterion and the Brier quadratic score ([38, 39]). In the following, by considering the asymmetric logarithmic scoring rule for the probability of an event, we suitably extend the definitions of entropy and extropy from a partition of events to the case of arbitrary families of events. Consequently, we also extend the well-known Kullback–Leibler divergence (and its dual) from a simplex of $\mathbb{R}^n$ to the larger domain $[0,1]^n$.

The chapter is organized as follows. In Section 6.1 we recall some basic notions on scoring rules, Bregman divergence, entropy, extropy, and the penalty criterion for defining coherence. In Section 6.2 we introduce the asymmetric logarithmic scoring rule, by showing that it is strictly proper. We also illustrate its relation to the 2-parameter Beta family of scoring rules. In Section 6.3 we extend the notion of entropy from partitions to an arbitrary finite family of events and introduce a Bregman divergence which generalises the Kullback –Leibler divergence. In Section 6.4, we introduce the complement of the asymmetric logarithmic score, which allows us to define the notion of extropy for an arbitrary family of events and the associated Bregman divergence. In Section 6.5, we propose a suitable symmetric strictly proper scoring rule, showing that the associated expected loss coincides, not only for the probability mass distribution of a random quantity but also for probability assessments on arbitrary families of events, with the sum of entropy and extropy.

6.1 Strictly proper scoring rules and coherence

We recall the notion of strictly proper scoring rule for an event E. In the literature, a score can be interpreted as a reward or as a penalty. In the following, we look at scores as penalties and hence improving the score means reducing it.

Definition 24. *[86] A function $s : \{0,1\} \times [0,1] \to [0,+\infty]$ is said to be a* strictly proper scoring rule *if the following conditions are satisfied:*
(a) for every $x, p \in [0,1]$, with $x \neq p$, it is

$$p\, s(1,p) + (1-p)\, s(0,p) < p\, s(1,x) + (1-p)\, s(0,x)\,; \tag{6.1}$$

(b) the functions $s(1,x)$ and $s(0,x)$ are continuous.

We observe that x is your announced probability for the event E while p is your degree of belief on E and hence the quantity $p\,s(1,x)+(1-p)\,s(0,x)$ represents your expected score. We recall that a scoring rule is said to be symmetric if $s(1,x) = s(0,1-x), \forall x \in [0,1]$.

We consider for any strictly proper scoring rule $s : \{0,1\} \times [0,1]$, the extension S on the set $[0,1] \times [0,1]$, defined as (see [51])

$$S(p,x) = ps(1,x) + (1-p)s(0,x).$$

We observe that if x is your announced probability for the event E, and p represents your degree of belief in E, then the quantity $S(p,x)$ represents your expected score. Using S, condition (a) in Definition 24 can be rewritten as

$$S(p,p) < S(p,x),\ \forall x,p \in [0,1],\ x \neq p.$$

That is, given a strictly proper scoring rule s, your expected score is minimum only when your announced probability x coincides with your degree of belief p. Furthermore, if $s(1,x)$ and $s(0,x)$ are continuous functions, then $S(p,x)$, for any given $p \in [0,1]$ is a continuous function in the variable x.

Remark 43. *Condition (b) in Definition 24 is consistent with s assuming value $+\infty$. As noticed in [86], the only cases where a strictly proper scoring rule s can assume value $+\infty$ are arguments $(1,0)$ and $(0,1)$, that is for categorically mistaken judgement. Therefore, when we consider the function S we have that $S(p,0) = ps(1,0)+(1-p)s(0,0) = +\infty$ for $p>0$, if $s(1,0) = +\infty$; $S(p,1) = ps(1,1)+(1-p)s(0,1) = +\infty$ for $p<1$, if $s(0,1) = +\infty$. In particular $S(p,p) < \infty$ for all $p \in [0,1]$.*

The notion of coherence is crucial in the subjective approach to probability and it can be defined through a penalty criterion. Given a strictly proper scoring rule s, with the assessment $\mathcal{P} = (p_1,\ldots,p_n)$ on $\mathcal{F} = \{E_1,\ldots,E_n\}$ we associate a random loss $\mathcal{L}(\mathcal{F},\mathcal{P}) = \sum_1^n s(E_i,p_i)$. Then, based on [39, 45], coherence can be characterized by the following definition ([61, 51])

Definition 25. *Given a strictly proper scoring rule s and an arbitrary family of events $\mathcal{K}$, a probability assessment P on the events in $\mathcal{K}$ is said to be* coherent *if and only if, for every integer n, for every finite subfamily $\mathcal{F} = \{E_1,\ldots,E_n\} \subseteq \mathcal{K}$, denoting by $\mathcal{P} = (p_1,\ldots,p_n)$ the restriction of P to $\mathcal{F}$, it does not exist another assessment $\mathcal{P}^* = (p_1^*,\ldots,p_n^*)$ on $\mathcal{F}$ such that $\mathcal{L}(\mathcal{F},\mathcal{P}^*) \leqslant \mathcal{L}(\mathcal{F},\mathcal{P})$, with $\mathcal{L}(\mathcal{F},\mathcal{P}^*) < \mathcal{L}(\mathcal{F},\mathcal{P})$ in at least one case.*

Definition 25 defines coherence in terms of the loss $\mathcal{L}$, which in turn depends on the chosen scoring rule s. However, as shown in [51, Theorem 4] (see also [86]), the set of all coherent assessments on $\mathcal{K}$ does not depend on the choice of the strictly proper scoring rule s. Then, Definition 25 coincides with the definition of coherence originally based on the Brier score, $s = (E-p)^2$.

Remark 44. *We observe (see the proof of [51, Theorem 3]) that if the assessment $\mathcal{P} = (p_1,\ldots,p_n)$ on $\mathcal{F} = \{E_1,\ldots,E_n\}$ is coherent, then for any coherent extension $(\lambda_1,\ldots,\lambda_m)$ (that is, with $\sum_{h:C_h\subseteq E_i}\lambda_h = p_i$, $i=1,\ldots,n$, $\sum_{h=1}^m \lambda_h = 1$ and $\lambda_h \geqslant 0$, $h=1,\ldots,m$) on the set of constituents $\{C_1,\ldots,C_m\}$ associated with $\mathcal{F}$, and for any $\mathcal{P}^* \neq \mathcal{P}$, it holds that*

$$\sum_{h=1}^m \lambda_h L_h = \sum_{i=1}^n S(p_i,p_i) < \sum_{i=1}^n S(p_i,p_i^*) = \sum_{h=1}^m \lambda_h L_h^*, \tag{6.2}$$

where L_h (resp., L_h^*) represents the value of $\mathcal{L}(\mathcal{F},\mathcal{P})$ (resp., of $\mathcal{L}(\mathcal{F},\mathcal{P}^*)$) when the constituent C_h is true. In other words, for any coherent extension $(\lambda_1,\dots,\lambda_m)$ of the coherent assessment $\mathcal{P}$, formula (6.2) implies that

$$\mathbb{E}(\mathcal{L}(\mathcal{F},\mathcal{P})) < \mathbb{E}(\mathcal{L}(\mathcal{F},\mathcal{P}^*)), \ \ \forall \mathcal{P}^* \neq \mathcal{P}.$$

6.1.1 Bregman divergence, entropy and extropy

Let C be a convex subset of $\mathbb{R}^n$ with non empty interior and let $\phi : C \to \mathbb{R}$ be a strictly convex function, differentiable in the interior of C whose gradient $\nabla\phi$ extends to a bounded, continuous function on C. In the definition below we recall the notion of Bregman divergence ([23]).

Definition 26. *For $x, y \in C$ the Bregman divergence $D_\phi : C \times C \to \mathbb{R}$ corresponding to ϕ is given by $D_\phi(x,y) = \phi(x) - \phi(y) - \nabla\phi(y)(x-y)$.*

We can construct a Bregman divergence starting from a given scoring rule s. We recall that it can be proved that $S(p,x) = S(x,x) + S'(x,x)(p-x)$, where $S'(x,x) = S(1,x) - S(0,x)$, and by setting $\phi(p) = -S(p,p)$ it holds that

$$D_\phi(p,x) = \phi(p) - \phi(x) - \nabla\phi(x)(p-x) = S(p,x) - S(p,p). \tag{6.3}$$

We introduce now two important objects strongly related to the notion of scoring rule. Given a finite random quantity X with possible values $x_1,\dots,x_n$, $n \geqslant 1$, we set $E_i = (X = x_i)$, $i = 1,\dots,n$. Then, the family $\mathcal{F} = \{E_1, E_2, \dots, E_n\}$ is a partition of Ω. Let $\mathcal{P} = (p_1,\dots,p_n)$, with $p_i = P(E_i)$, be a coherent probability assessment on $\mathcal{F}$. If $\mathcal{P}$ is coherent, then it is a probability mass function on X.

Definition 27. *The* entropy *associated with the pair $(X,\mathcal{P})$ is defined as*

$$H(X,\mathcal{P}) = -\sum_{i=1}^{n} p_i \log(p_i). \tag{6.4}$$

We recall that the entropy is related with the expected value of the logarithmic strictly proper scoring rule of a distribution ([74, pag. 343]), that is with the expected value of

$$s_{\log}(\mathcal{F},\mathcal{P}) = -\sum_{i=1}^{n} E_i \log(p_i). \tag{6.5}$$

Definition 28. *The* extropy *associated with the pair $(X,\mathcal{P})$ is defined as*

$$J(X,\mathcal{P}) = -\sum_{i=1}^{n} (1-p_i)\log(1-p_i). \tag{6.6}$$

The extropy can be seen as dual and complement of entropy ([78]). We also recall that the extropy can be seen as the expected value of the following strictly proper scoring rule of a distribution ([2, 77, 78])

$$s_{\log}^{c}(\mathcal{F},\mathcal{P}) = -\sum_{i=1}^{n} (1-E_i)\log(1-p_i). \tag{6.7}$$

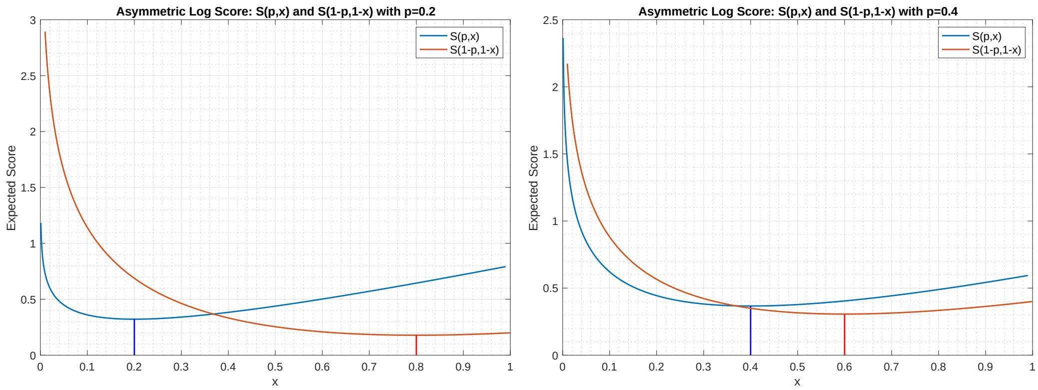

FIGURE 6.1: The functions $S(p,x)$ and $S(1-p,1-x)$ for the asymmetric log score with $p=0.2$ (left) and $p=0.4$ (right) illustrate the asymmetry of the score.

6.2 Asymmetric logarithmic score

In this section we analyse the asymmetric logarithmic score and we show that it is strictly proper. First, we recall the definition of symmetric score [97].

Definition 29. *A scoring rule s is symmetric if and only if*

$$s(i,x) = s(1-i,1-x)\,, i = 0,1, \textit{for all } x \in [0,1].$$

The well known Brier score $s(E,x) = (E-x)^2$ is an example of symmetric (strictly proper) scoring rule ([5]). Indeed $s(1-E,1-x) = (1-E-1+x)^2 = (E-x)^2$.

By Definition 29, a score s is said to be *asymmetric* if $s(i,x) \neq s(1-i,1-x)\,, i = 0,1$ for some $x \in [0,1]$. An example of asymmetric score is given by the following logarithmic score.

Definition 30. *Given an event E, the* asymmetric logarithmic *scoring rule is defined as*

$$s(E,x) = x - E\log(x) - E = \begin{cases} x - \log(x) - 1, & \textit{if } E = 1, \\ x, & \textit{if } E = 0\,, \end{cases}$$

where

$$s(1,0) = \lim_{x\to 0^+} s(1,x) = +\infty.$$

Of course $s(E,x)$ is not symmetric, because $s(1,x) = x - \log(x) - 1 \neq s(0,1-x) = 1-x$. We observe that the extension $S(p,x)$ is given by

$$S(p,x) = p\,(x - \log(x) - 1) + (1-p)\,x = -p\log(x) - p + x, \tag{6.8}$$

and in particular $S(1,x) = s(1,x) = x - \log(x) - 1$ and $S(0,x) = s(0,x) = x$. As a consequence of the asymmetry of the score, it follows that $S(p,x) \neq S(1-p,1-x)$ as illustrated in Figure 6.1. We show now that the previous score is a strictly proper scoring rule according to Definition 24.

Proposition 3. *The asymmetric scoring rule $s(E,x) = x - E\log(x) - E$ is a strictly proper score.*

Proof. We begin by observing that $s(1,x) = x - \log(x) - 1$ and $s(0,x) = x$ are continuous functions in $[0,1]$ and hence condition (b) of Definition 24 is satisfied. By Definition 24, to prove that $s(E,x)$ is strictly proper we have to show that $S(p,x) = ps(1,x) + (1-p)s(0,x)$ is uniquely minimized at $x = p$. We know that

$p = P(E)$ so that we can consider it fixed and looking at $S(p,x)$ as a function of x only. We distinguish two cases: (i) $p \in (0,1]$ and (ii) $p = 0$.
Case (i). Let $p \in (0,1]$ and $x \in [0,1]$. We first observe that for $x = 0$,

$$\lim_{x \to 0^+} S(p,x) = \lim_{x \to 0^+} (x - p\log(x) - 1) = +\infty$$

and hence $S(p,p) < S(p,0)$. Then, for $x \in (0,1]$, $x \neq p$, it holds that

$$\begin{aligned} S(p,p) < S(p,x) &\iff p - p\log(p) - p < x - p\log(x) - p \\ &\iff x - p - p\log(x) + p\log(p) > 0 \iff \tfrac{x}{p} - \log(\tfrac{x}{p}) - 1 > 0. \end{aligned}$$

For $x = p$ we have of course that $S(p,p) = S(p,x)$. Therefore for every $p \in (0,1]$ and for every $x \in [0,1]$ it holds that $S(p,p) < S(p,x)$ if $x \neq p$ and $S(p,p) = S(p,x)$ if and only if $x = p$, that is $S(p,x)$ is uniquely minimized at $x = p$.

Case (ii). It is immediate to see that $S(0,x) = s(0,x) = x$ takes its minimum in $x = p = 0$. □ □

From Figure 6.1 it can be also observed that the asymmetric logarithmic score is strictly proper, indeed for $p = 0.2$ (left) and $p = 0.4$ (right) the expected score $S(p,x)$ reaches its minimum only when $x = p$.

Remark 45 (Beta family)**.** *We note that this score belongs to the 2-parameter Beta family of scoring rules (see [7], see also [64, pag. 364]), that is the two-parameter family*

$$s(1,x) = \int_x^1 t^{\alpha-1}(1-t)^{\beta}dt,$$

$$s(0,x) = \int_0^x t^{\alpha}(1-t)^{\beta-1}dt,$$

with $\alpha, \beta > -1$. Indeed, we obtain the asymmetric logarithmic scoring rule by setting the parameters $\alpha = 0, \beta = 1$ and we have that

$$s(1,x) = x - \log(x) - 1 = \int_x^1 t^{\alpha-1}(1-t)^{\beta}dt, \ \ \alpha = 0, \beta = 1\,.$$

$$s(0,x) = x = \int_0^x t^{\alpha}(1-t)^{\beta-1}dt, \ \ \alpha = 0, \beta = 1\,.$$

We also recall that this family provides symmetric scoring rules when $\alpha = \beta$, indeed

$$\begin{aligned} s(1,x) &= \textstyle\int_x^1 t^{\alpha-1}(1-t)^{\beta}dt = \int_{1-x}^0 (1-c)^{\alpha-1}c^{\alpha}(-dc) = \\ &= \textstyle\int_0^{1-x} c^{\alpha}(1-c)^{\alpha-1}dc = s(0,1-x). \end{aligned}$$

Given the asymmetric logarithmic score $s(E,x) = x - E\log(x) - E$, using Equation (6.3), we can find its associated Bregman divergence. Therefore, by Equation (6.8), setting $\phi(p) = -S(p,p) = p\log(p)$, it holds that

$$\begin{aligned} D_{\phi}(x,y) &= S(p,x) - S(p,p) = \\ &= x - p\log(x) - p - p + p\log(p) + p = p\log(\tfrac{p}{x}) - (p-x). \end{aligned} \tag{6.9}$$

The divergence in Equation (6.9) will be used in Section 6.3 to generalise the Kullback–Leibler divergence (KLD).

6.3 Entropy for a probability assessment on an arbitrary family of events

Looking at Equation (6.4), we observe that the quantities $x_1, \ldots, x_n$, are not involved in the definition of the entropy and hence we can consider the entropy for any pair $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F}$ is a partition of Ω, even if $\mathcal{F}$ is not associated with a random quantity X. Then, given a partition $\mathcal{F}$ of Ω and a coherent probability assessment $\mathcal{P}$ on $\mathcal{F}$ we set the entropy associated with the pair $(\mathcal{F}, \mathcal{P})$ as

$$H(\mathcal{F}, \mathcal{P}) = -\sum_{i=1}^{n} p_i \log(p_i)\,.$$

Given a coherent probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on an arbitrary family of events $\mathcal{F} = \{E_1, \ldots, E_n\}$, we consider the penalty $\mathcal{L}$ w.r.t. the asymmetric strictly proper scoring rule $s(E, x) = x - E\log(x) - E$,

$$\begin{aligned} \mathcal{L}(\mathcal{F}, \mathcal{P}) &= \textstyle\sum_{i=1}^{n} s(E_i, p_i) = \sum_{i=1}^{n} (p_i - E_i \log(p_i) - E_i) = \\ &= \textstyle\sum_{i=1}^{n} p_i - \sum_{i=1}^{n} E_i \log(p_i) - \sum_{i=1}^{n} E_i\,. \end{aligned}$$

If $\mathcal{F}$ is a partition of Ω, as $\sum_{i=1}^{n} p_i = 1$, we have that the penalty

$$\mathcal{L}(\mathcal{F}, \mathcal{P}) = \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} E_i \log(p_i) - 1 = -\sum_{i=1}^{n} E_i \log(p_i) = s_{\log}(\mathcal{F}, \mathcal{P}).$$

In other words, the logarithmic score of a distribution coincides with the sum of the asymmetric logarithmic scoring rule applied to each event in the partition $\mathcal{F}$.

Given a coherent probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ and a probability assessment $\mathcal{Q} = (q_1, \ldots, q_n)$ on an arbitrary family of events $\mathcal{F} = \{E_1, \ldots, E_n\}$, we set

$$S(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^{n} S(p_i, q_i) = \sum_{i=1}^{n} q_i - \sum_{i=1}^{n} p_i \log(q_i) - \sum_{i=1}^{n} p_i,$$

which, by the linearity property of the expectation (see Remark 44), coincides with the expected value of your penalty $\mathbb{E}(\mathcal{L}(\mathcal{F}, \mathcal{Q}))$ when your announced probability assessment on $\mathcal{F}$ is $\mathcal{Q}$, while your degrees of belief on $\mathcal{F}$ are given by the assessment $\mathcal{P}$. In particular, $S(\mathcal{P}, \mathcal{P})$ is the expected value of $\mathcal{L}(\mathcal{F}, \mathcal{P})$, that is

$$S(\mathcal{P}, \mathcal{P}) = \mathbb{E}(\mathcal{L}(\mathcal{F}, \mathcal{P})) = -\sum_{i=1}^{n} p_i \log(p_i)\,. \tag{6.10}$$

By setting $\Phi(p_1, \ldots, p_n) = \sum_{i=1}^{n} \phi(p_i)$, where $\phi(p_i) = p_i \log(p_i)$, it follows from Equation (6.9) that the Bregman divergence $D_\Phi(\mathcal{P}, \mathcal{Q})$ is given by

$$D_\Phi(\mathcal{P}, \mathcal{Q}) = S(\mathcal{P}, \mathcal{Q}) - S(\mathcal{P}, \mathcal{P}) = \textstyle\sum_{i=1}^{n} [p_i \log(\frac{p_i}{q_i}) - (p_i - q_i)]. \tag{6.11}$$

The divergence in Equation (6.11) is the Generalised I-divergence ([3]) which generalises the Kullback–Leibler divergence (KLD) to the larger domain $[0,1]^n$.

If $\mathcal{F}$ is a partition of Ω, then Equation (6.10) shows that the expected value of the penalty $\mathcal{L}(\mathcal{F}, \mathcal{P})$, that is $S(\mathcal{P}, \mathcal{P})$, when your announced probability assessment coincides with your degree of belief, is the entropy associated with the pair $(\mathcal{F}, \mathcal{P})$.

Moreover, in this case, as $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$, it follows that

$$S(\mathcal{P}, \mathcal{Q}) = -\sum_{i=1}^{n} p_i \log(q_i),$$

which coincides with the cross entropy. The divergence $D_\Phi(\mathcal{P}, \mathcal{Q})$ given in Equation (6.11) reduces to KLD, that is

$$D_\Phi(\mathcal{P}, \mathcal{Q}) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

In the following definition, by exploiting the asymmetric logarithmic score, we extend the notion of entropy to an arbitrary finite family of events $\mathcal{F}$.

Definition 31. *Given a family of n events $\mathcal{F}$ and a coherent probability assessment $\mathcal{P}$ on the $\mathcal{F}$, let $s(E, x) = x - E\log(x) - E$ be the asymmetric logarithmic strictly proper scoring rule on an event E and $\mathcal{L}$ be the corresponding penalty associated with $(\mathcal{F}, \mathcal{P})$. We define the entropy associated with the pair $(\mathcal{F}, \mathcal{P})$, denoted by $H(\mathcal{F}, \mathcal{P})$, as the expected value w.r.t. $\mathcal{P}$ of the penalty $\mathcal{L}$, that is*

$$H(\mathcal{F}, \mathcal{P}) = \mathbb{E}(\mathcal{L}(\mathcal{F}, \mathcal{P})) = S(\mathcal{P}, \mathcal{P}) = -\sum_{i=1}^{n} p_i \log(p_i).$$

Remark 46. *In particular, if $n = 1$, that is $\mathcal{F} = \{E\}$, $\mathcal{P} = \{p\}$, we have*

$$H(E, p) = S(p, p) = -p\log(p),$$

which coincides with the well-known expected surprise for an event with probability p. Then, as the entropy of a random quantity X coincides with the expected surprise of a probability mass function on a partition of events, the entropy given in Definition 31 can be interpreted as the expected surprise of an assessment $\mathcal{P}$ on an arbitrary family of events $\mathcal{F}$.

6.4 Extropy for a probability assessment on an arbitrary family of events

Similarly to the previous section, we define the notion of extropy for an arbitrary family of events by considering the complement of the asymmetric logarithmic score $s(E, x) = x - E\log(x) - E$. We define the complement asymmetric logarithmic score score of $s(E, x)$ as

$$s^c(E, x) = s(1 - E, 1 - x) = E - x - (1 - E)\log(1 - x).$$

The score $s^c(E, x)$ is asymmetric (see Figure 6.1), strictly proper and its expected score is given by

$$S^c(p, x) = p - x - (1 - p)\log(1 - x).$$

Given a coherent probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on an arbitrary family of n event $\mathcal{F} = \{E_1, \ldots, E_n\}$, the penalty $\mathcal{L}^c$ w.r.t. $s^c(E, x)$ is given by

$$\begin{aligned}\mathcal{L}^c(\mathcal{F}, \mathcal{P}) &= \textstyle\sum_{i=1}^n s^c(E_i, p_i) = \\ &= \textstyle\sum_{i=1}^n E_i - \sum_{i=1}^n p_i - \sum_{i=1}^n (1 - E_i)\log(1 - p_i).\end{aligned}$$

If $\mathcal{F}$ is a partition of Ω, as $\sum_{i=1}^n p_i = 1$, we have that

$$\mathcal{L}^c(\mathcal{F},\mathcal{P}) = 1 - 1 - \sum_{i=1}^{n}(1-E_i)\log(1-p_i) = -\sum_{i=1}^{n}(1-E_i)\log(1-p_i) = s^c_{\log}(\mathcal{F},\mathcal{P}).$$

By recalling that $Q = (q_1,\ldots,q_n)$ is your announced probability assessment on $\mathcal{F}$ is Q, and $\mathcal{P} = (p_1,\ldots,p_n)$ the vector of your degrees of belief on $\mathcal{F}$, we set

$$S^c(\mathcal{P},Q) = \sum_{i=1}^{n} S^c(p_i,q_i) = \sum_{i=1}^{n} p_i - \sum_{i=1}^{n}(1-p_i)\log(1-q_i) - \sum_{i=1}^{n} q_i,$$

which coincides with $\mathbb{E}(\mathcal{L}^c(\mathcal{F},Q))$. In particular, $S^c(\mathcal{P},\mathcal{P})$, that is the expected value of the penalty when $Q = \mathcal{P}$, is

$$S^c(\mathcal{P},\mathcal{P}) = \mathbb{E}(\mathcal{L}^c(\mathcal{F},\mathcal{P})) = -\sum_{i=1}^{n}(1-p_i)\log(1-p_i)\,. \tag{6.12}$$

As done in Section 6.3, we set $\Phi^c(p_1,\ldots,p_n) = \sum_{i=1}^{n}\phi^c(p_i)$, where $\phi^c(p_i) = (1-p_i)\log(1-p_i)$. Then, if $\mathcal{P}$ is your degree of belief on $\mathcal{F}$ and Q your announced probability, it follows from Equation (6.3) that the Bregman divergence $D_{\Phi^c}(\mathcal{P},Q)$ is given by

$$D_{\Phi^c}(\mathcal{P},Q) \quad = S^c(\mathcal{P},Q) - S^c(P,P) = \textstyle\sum_{i=1}^{n}[(1-p_i)\log(\frac{1-p_i}{1-q_i}) - (q_i - p_i)]. \tag{6.13}$$

If $\mathcal{F}$ is a partition of Ω, then Equation (6.12) shows that the expected value of the penalty $\mathcal{L}^c(\mathcal{F},\mathcal{P})$, that is $S^c(\mathcal{P},\mathcal{P})$, when your announced probability assessment coincides with your degree of belief, is the extropy associated with the pair $(\mathcal{F},\mathcal{P})$. Moreover, in this case it holds that

$$S^c(\mathcal{P},Q) = \sum_{i=1}^{n} S^c(p_i,q_i) = -\sum_{i=1}^{n}(1-p_i)\log(1-q_i),$$

coincides with the cross extropy ([76]).

Now, by exploiting the complement asymmetric logarithmic score, we extend the notion of extropy to an arbitrary finite family of events $\mathcal{F}$.

Definition 32. *Given a family of n events $\mathcal{F}$ and a coherent probability assessment $\mathcal{P} = (p_1,\ldots,p_n)$ on $\mathcal{F} = \{E_1,\ldots,E_n\}$, let $s^c(E,x)$ be the complement asymmetric logarithmic score on an event E and $\mathcal{L}^c$ be the corresponding penalty associated with $(\mathcal{F},\mathcal{P})$. We define the extropy associated with the pair $(\mathcal{F},\mathcal{P})$, denoted by $J(\mathcal{F},\mathcal{P})$, as the expected value w.r.t. $\mathcal{P}$ of the penalty $\mathcal{L}^c$, that is*

$$J(\mathcal{F},\mathcal{P}) = \mathbb{E}(\mathcal{L}^c(\mathcal{F},\mathcal{P})) = S^c(\mathcal{P},\mathcal{P}) = -\sum_{i=1}^{n}(1-p_i)\log(1-p_i).$$

6.5 Symmetrized logarithmic score of the probability of an event

The notions of entropy and extropy for a random quantity, as well as our extensions to a probability assessment on an arbitrary family of events, can be interpreted as the expected value of the penalties associated with asymmetric scores of the probability

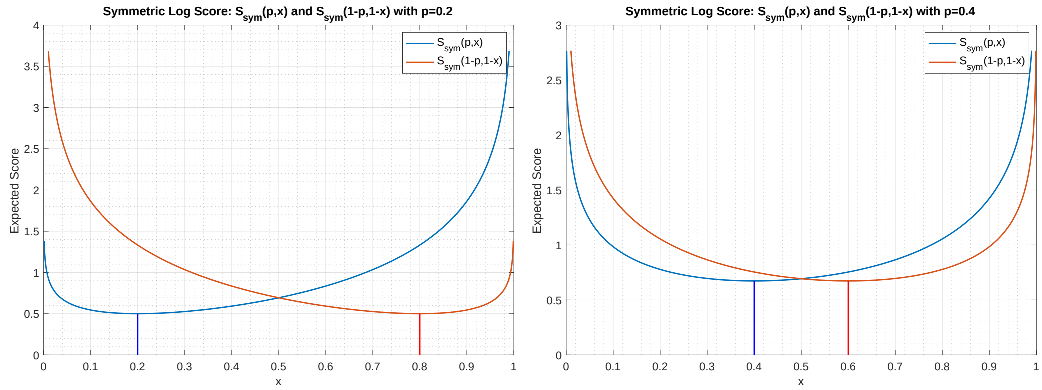

FIGURE 6.2: The functions $S_{\text{sym}}(p, x)$ and $S_{\text{sym}}(1-p, 1-x)$ for the symmetric log score with $p = 0.2$ (left) and $p = 0.4$ (right) illustrate the symmetry of the score.

of an event. We consider now the symmetrized logarithmic score obtained as the sum of $s(E, x)$ and $s^c(E, x)$, that is

$$\begin{aligned} s_{\text{sym}}(E, p) &= s(E, p) + s^c(E, p) = s(E, p) + s(1-E, 1-p) = \\ &= (p - E\log(p) - E) + (1 - p - (1-E)\log(1-p) - 1 + E) = \\ &= -E\log(p) - (1-E)\log(1-p). \end{aligned} \tag{6.14}$$

Of course, the symmetric score s_{sym} is strictly proper because it is the sum of two strictly proper scoring rules. Denoting by p you degree of belief on E, and by x your announced probability, the expected value of s_{sym} is

$$S_{\text{sym}}(p, x) = -p\log(x) - (1-p)\log(1-x),$$

with

$$S_{\text{sym}}(p, p) = -p\log(p) - (1-p)\log(1-p) < S_{\text{sym}}(p, x), \quad \text{if } x \neq p.$$

As a consequence of the symmetry of the score, it follows that $S_{\text{sym}}(p, x) = S_{\text{sym}}(1-p, 1-x)$ as illustrated in Figure 6.2.

Given a coherent probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on an arbitrary family of n events $\mathcal{F} = \{E_1, \ldots, E_n\}$, the penalty $\mathcal{L}_{\text{sym}}$ w.r.t. $s_{\text{sym}}(E, x)$ is given by

$$\begin{aligned} \mathcal{L}_{\text{sym}}(\mathcal{F}, \mathcal{P}) &= \textstyle\sum_{i=1}^n s_{\text{sym}}(E_i, p_i) = \sum_{i=1}^n (s(E_i, p_i) + s^c(E_i, p_i)) = \\ &= -\textstyle\sum_{i=1}^n E_i \log(p_i) - \sum_{i=1}^n (1-E_i)\log(1-p_i)\,. \end{aligned}$$

Still denoting by $Q = (q_1, \ldots, q_n)$ your announced probability assessment on $\mathcal{F}$, and by $\mathcal{P} = (p_1, \ldots, p_n)$ the vector of your degrees of belief on $\mathcal{F}$, we set

$$S_{\text{sym}}(\mathcal{P}, Q) = S(\mathcal{P}, Q) + S^c(\mathcal{P}, Q) = -\sum_{i=1}^n p_i \log(q_i) - \sum_{i=1}^n (1-p_i)\log(1-q_i). \tag{6.15}$$

Then,

$$S_{\text{sym}}(\mathcal{P}, Q) = \mathbb{E}(\mathcal{L}_{\text{sym}}(\mathcal{F}, Q)) = \mathbb{E}(\mathcal{L}(\mathcal{F}, Q)) + \mathbb{E}(\mathcal{L}^c(\mathcal{F}, Q)).$$

Considering the Definitions 31 and 32 of Entropy and Extropy on an arbitrary family of events $\mathcal{F}$, given a coherent assessment $\mathcal{P}$ on $\mathcal{F}$, the expected value of the penalty associated w.r.t. s_{sym} given in Equation (6.15) coincides with the sum of entropy and extropy, that is

$$S_{\text{sym}}(\mathcal{P}, \mathcal{P}) = H(\mathcal{F}, \mathcal{P}) + J(\mathcal{F}, \mathcal{P}). \tag{6.16}$$

Since both entropy and extropy represent the expected value of the penalty with respect to asymmetric scores, a useful measure of information can be obtained by considering the expected value of the corresponding symmetrized score. This measure is given by Equation (6.16), that is by $H(\mathcal{F}, \mathcal{P}) + J(\mathcal{F}, \mathcal{P})$, which take into account both entropy and extropy for arbitrary family of events.

To look at the Bregman divergence that we can obtain using the expected score in Equation (6.16), from Equations (6.11) and (6.13), by setting $\Phi_{\text{sym}}(p_2, \ldots, p_n) = \sum_{i=1}^{n} \phi_{\text{sym}}(p_i)$, where $\phi_{\text{sym}}(p_i) = \phi(p_i) + \phi^c(p_i) = p_i \log(p_i) + (1 - p_i) \log(1 - p_i)$, it holds that

$$\begin{aligned} &D_{\Phi_{\text{sym}}}(\mathcal{P}, \mathcal{Q}) = D_{\Phi}(\mathcal{P}, \mathcal{Q}) + D_{\Phi^c}(\mathcal{P}, \mathcal{Q}) = \\ &= \textstyle\sum_{i=1}^{n} [p_i \log(\frac{p_i}{q_i}) - (p_i - q_i)] + \sum_{i=1}^{n} [(1 - p_i) \log(\frac{1-p_i}{1-q_i}) - (q_i - p_i)] = \\ &= \textstyle\sum_{i=1}^{n} \left[p_i \log \left(\frac{p_i}{q_i} \right) + (1 - p_i) \log \left(\frac{1-p_i}{1-q_i} \right) \right]. \end{aligned} \tag{6.17}$$

Then, the Bregman divergence of the symmetrized score is nothing but the sum of the divergence obtained in Equations (6.11) and (6.13) using the expected value of the asymmetric logarithmic score (Equation (6.5)) and its "complementary" asymmetric score (Equation (6.7)). Notice that, since we are considering that $\mathcal{P}$ and $\mathcal{Q}$ are probability assessments on an arbitrary family of events $\mathcal{F}$, unlike the KLD, in Equation (6.17) it is not required that $\mathcal{P}$ and $\mathcal{Q}$ belong to the simplex $\mathbb{R}^n$. In the particular case where $\mathcal{P}$ and $\mathcal{Q}$ belong to the simplex, that is they are coherent probability assessment on partitions, Equation (6.17) coincides with the expected value of the total score associated with the following score for a distribution ([78])

$$s_{\text{totallog}}(\mathcal{F}, \mathcal{P}) = s_{\text{log}}(\mathcal{F}, \mathcal{P}) + s_{\text{log}}^{c}(\mathcal{F}, \mathcal{P}) = \sum_{i=1}^{n} [-E_i \log(p_i) - (1 - E_i) \log(1 - p_i)],$$

which in our case is obtained as the penalty $\mathcal{L}_{\text{sym}}$ w.r.t. the symmetric logarithmic score s_{sym} for the probability of an event.

Conclusions

This thesis focused on the study of conditional events, compound and iterated conditionals in a coherent subjective probability setting.

In Chapter 2, different definitions of iterated conditionals have been analysed. First, in the framework of trivalent logics, we studied the notions of iterated conditionals in the logics of Cooper-Calabrese ($|_C$), de Finetti ($|_{dF}$) and Farrell ($|_F$). Then, by exploiting the structure $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\overline{\bigcirc}$ used in order to define $|_{gs}$ from the conjunction $\wedge_{gs}$, for each $i \in \{K, L, B, S\}$, we defined the iterated conditioning $|_i$ from the conjunction $\wedge_i$. We observed that the iterated conditionals $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$, are all conditional random quantities (not always in $[0,1]$) which satisfy the compound prevision theorem. For all these objects we analysed the validity of the Import-Export principle and of the properties (see Section 2.1)

- P1. $(B|K)|(A|H) = [(A|H) \wedge (B|K)]|(A|H)$
- P2. $(A|H) \wedge (B|K) \leqslant (B|K)|(A|H)$
- P3. $\mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(B|K)|(A|H)]P(A|H)$
- P4. Lower and upper bounds for $(B|K)|(A|H)$.

A brief summary of the results is given in Table 6.1. We observed that all the basic logical and probabilistic properties are satisfied only by the iterated conditioning $|_{gs}$. We also showed that a generalised version of the Bayes' Rule $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$ for

	$\|_C$	$\|_{dF}$	$\|_F$	$\|_K$	$\|_L$	$\|_B$	$\|_S$	$\|_{gs}$
No IE				✓	✓	✓	✓	✓
P1		✓	✓	✓	✓	✓	✓	✓
P2		✓	✓	✓	✓	✓	✓	✓
P3				✓	✓	✓	✓	✓
P4								✓

TABLE 6.1: Properties P1-P4, the non validity of Import-Export Principle (No IE), and iterated conditionals in their own logic. The symbol ✓ means that the property is satisfied. The blank space means that the property is not satisfied.

iterated conditionals is satisfied by $|_i$ with $i \in \{K, L, B, S, gs\}$. We discussed the implications of the obtained results on the probability propagation rules from $\{A|H, B|K\}$ to the iterated conditional $(B|K)|_i(A|H)$, by studying the p-validity of the generalised versions of Modus Ponens and two-premise centering, which involve conditional events and iterated conditionals. We observed that for $|_B$ and $|_S$, as the iterated conditionals $(B|K)|_B(A|H)$ and $(B|K)|_S(A|H)$ can take value outside the interval [0,1], the study of p-validity is meaningless. For the remaining ones $|_C, |_{dF}, |_F, |_K, |_L$, and $|_{gs}$, we showed that only for the iterated conditioning $|_{gs}$ turns out that both generalised versions of the inference rules are p-valid. We also examined two examples of the generalised inference rules in natural language by observing that, when we adopt the notion of iterated conditioning $|_i$, with $|_i \in \{|_C, |_{dF}, |_F, |_K, |_L\}$, some results which are counterintuitive in commonsense reasoning can be obtained. Future work on these structures could concern the (possible) characterization of the p-entailment of Adams in the setting of coherence by means of the iterated conditioning $|_K, |_L, |_B$, and $|_S$, and the study of the different notions of iterated conditioning in the framework of nonmonotonic reasoning in System P and in other non-classical logics, like connexive logic ([85]). In Chapter 3 we proposed a generalisation of the definitions of conjunction and disjunction of two conditional events defined as conditional random quantities. These new structures take arbitrary values $a, b \in [0, 1]$ not necessarily equal to the probabilities $x = P(A|H)$ and $y = P(B|K)$, in the cases where one conditional event is void and the other is true (for the conjunction) or false (for the disjunction). For these objects, even though a generalised De Morgan's relation holds, the Sum rule, written for conditional events, is not satisfied, except in the case $a = x = P(A|H)$ and $b = y = P(B|K)$. Then, we proved that the Fréchet-Hoeffding bounds, both for conjunction and disjunction, are satisfied for every coherent assessment (x, y) on $\{A|H, B|K\}$ only when we take $a = x$ and $b = y$. Moreover, we analysed some particulare cases, such as incompatibility between the antecedents and for particular values of a and b. In particular, we studied the case $(a, b) = (0, 0)$ for which the generalised conjunction reduces to one of the four associative conjunction proposed by Walker ([94]), and the generalised disjunction reduces to the quasi disjunction. Similarly, we analysed the case $(a, b) = (1, 1)$ for which the generalised conjunction reduces to quasi conjunction and the generalised disjunction reduces to the fourth Walker's disjunction. Finally, we considered the case $(a, b) = (p, p)$ in which the generalised five-valued structures coincide with the three-valued conjunction and disjunction in the trivalent logic of Kleene-de Finetti. Furthermore, we analysed the coherent assessment on the conjunction and the disjunction in the case of interval-valued assessments and we gave an interpretation for the generalised conjunction in the setting of subjective probability theory. Further work could concern the investigation of these generalizations within the theory of the Boolean algebra of conditionals ([40, 42]), of Markov graphs ([98]) and the study of related generalized notions of iterated conditionals which could give further insights on the generalized structures and their applicability (see, e.g., [63]).

In Chapter 4 following the approach introduced by Coletti and Scozzafava in [26], we proposed a framework for fuzzy set theory based on conditional events and coherent conditional probability theory. Then considering the operations of conditional events as random quantities, we defined the complement and the intersection of fuzzy sets, interpreting the membership functions as the prevision of the corresponding operations on conditional events. Future work will concern the analysis of other set operations and relations, such as the union, the inclusion of fuzzy sets [29], De Morgan's Laws and the formalization and interpretation of more complex structures such as the "conditional" of two fuzzy sets, $E^*_\varphi | E^*_\psi$ defined through the notion of iterated conditional (see, e.g. [90]).

In Chapter 5, we took a step further in the study of the Boolean algebra of conditional started by [40] analysing the Boolean algebra of iterated conditionals. We studied how some properties of a probability on these Boolean algebras can be characterized in terms of satisfiability of known principles of its "canonical extensions" to the algebra of iterated conditionals. In particular, we proved that a weaker version of the Import-Export principle is satisfied by the canonical extension μ_Q of a probability Q defined on the Boolean algebra of conditionals if and only if Q is a separable probability, and that if μ_Q satisfies what we called "conjunction rationality principle" then Q satisfies the McGee's decomposition formula for the conjunction. As for future work we plan to further extend our analysis when, instead of starting from a positive probabilities on $\mathbf{A}$ and on $\mathcal{C}(\mathbf{A})$, we directly start from a conditional probability (in the axiomatic sense) on $\mathbf{A} \times \mathbf{A}'$ and on $\mathcal{C}(\mathbf{A}) \times \mathcal{C}(\mathbf{A})'$. Also we plan to explore the relationship, within $\mathcal{C}(\mathcal{C}(\mathbf{A}))$, between the general import-export principle and the p-validity of inference rules in nonmonotonic reasoning, as done in [63] within the framework of conditional random quantities.

Finally in Chapter 6, we studied the relation of the study of coherence in terms of penalty criterion and the theory of the proper scoring rules. In particular, we considered the asymmetric logarithmic scoring rules for the probability of an event and thanks to it we suitably extended the definitions of entropy and extropy, usually given for a partition of events, to the case of arbitrary families of events, and we find the associated Bregman divergences. Further developments could concern the study of the properties of entropy, extropy, cross-entropy, and cross-extropy in the case of an arbitrary family of events (or conditional events), by also exploiting the connections between scoring rules and Bregman divergence. Moreover, the structure free approach used in this paper may also be of interest when considering partial assessments, as it allows for a connection to work on scoring rules for imprecise models ([73, 83, 92]).

Bibliography

[1] E. W. Adams. *The logic of conditionals*. Dordrecht: Reidel, 1975.

[2] G. Agró, F. Lad, and G. Sanfilippo. "Sequentially forecasting economic indices using mixture linear combinations of EP distributions." In: *Journal of Data Science* 8.1 (2010). ISSN 1680-743X, pp. 101–126.

[3] A. Banerjee et al. "Clustering with Bregman Divergences". In: *Journal of Machine Learning Research* 6.58 (2005), pp. 1705–1749.

[4] V. Biazzo, A. Gilio, and G. Sanfilippo. "Generalized coherence and connection property of imprecise conditional previsions". In: *Proc. IPMU 2008, Malaga, Spain, June 22 - 27*. 2008, pp. 907–914.

[5] G. W. Brier. "Verification of Forecasts expressed in terms of probability". In: *Monthly Weather Review* 78.1 (1950), pp. 1–3.

[6] G. Bruno and A. Gilio. "Confronto fra eventi condizionati di probabilità nulla nell'inferenza statistica Bayesiana". In: *Rivista di matematica per le scienze economiche e sociali* 8.2 (1985), pp. 141–152.

[7] A. Buja, W. Stuetzle, and Y. Shen. *Loss functions for binary class probability estimation and classification: Structure and applications.* http://www-stat.wharton.upenn.edu/ buja/.

[8] D. Butnariu, E. Klement, and S. Zafrany. "On triangular norm-based propositional fuzzy logics." In: *Fuzzy Sets and Systems* **69**(3) (1995), pp. 241 –255.

[9] P. G. Calabrese. "An algebraic synthesis of the foundations of logic and probability". In: *Information Sciences* 42.3 (1987), pp. 187 –237.

[10] P. G. Calabrese. "Deduction with uncertain conditionals (revised & simplified, with examples)". In: *Heliyon* 7.11 (2021), e08328. ISSN: 2405-8440.

[11] P. G. Calabrese. *Logic and Conditional Probability: A Synthesis*. College Publications, 2017.

[12] P. G. Calabrese. "Reasoning with uncertainty using conditional logic and probability". In: *Proceedings. First International Symposium on Uncertainty Modeling and Analysis*. 1990, pp. 682–688.

[13] J. Cantwell. "Revisiting McGee's Probabilistic Analysis of Conditionals". In: *Journal of Philosophical Logic* 51.5 (2022), pp. 973–1017. DOI: 10.1007/s10992-022-09657-5.

[14] J. Cantwell. "The Logic of Conditional Negation". In: *Notre Dame Journal of Formal Logic* 49.3 (2008), pp. 245 –260. DOI: 10.1215/00294527-2008-010.

[15] G. Casini and U. Straccia. "A General Framework for Modelling Conditional Reasoning - Preliminary Report". In: *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning*. Aug. 2022, pp. 112–121. DOI: 10.24963/kr.2022/12.

[16] L. Castronovo and G. Sanfilippo. "A Generalized Notion of Conjunction for Two Conditional Events". In: *13th International Symposium on Imprecise Probabilities: Theories and Applications (ISIPTA 2023), Oviedo July 11-14, Oviedo, Spain*. Ed. by E. Miranda et al. Vol. 215. Proceedings of Machine Learning Research. ML Research Press, 2023, pp. 96–108.

[17] L. Castronovo and G. Sanfilippo. "A probabilistic analysis of selected notions of iterated conditioning under coherence". In: *International Journal of Approximate Reasoning* 165.109088 (2024). DOI: 10.1016/j.ijar.2023.109088.

[18] L. Castronovo and G. Sanfilippo. "Compound Conditionals and Fuzzy Sets". In: *Combining, Modelling and Analyzing Imprecision, Randomness and Dependence*. Ed. by Jonathan Ansari et al. Cham: Springer Nature Switzerland, 2024, pp. 67–75. ISBN: 978-3-031-65993-5. DOI: 10.1007/978-3-031-65993-5_8.

[19] L. Castronovo and G. Sanfilippo. "Entropy and Extropy for Partial Probability Assessments on Arbitrary Families of Events". In: *Scalable Uncertainty Management*. Ed. by Sébastien Destercke, Maria Vanina Martinez, and Giuseppe Sanfilippo. Cham: Springer Nature Switzerland, 2025, pp. 85–99. ISBN: 978-3-031-76235-2. DOI: 10.1007/978-3-031-76235-2_7.

[20] L. Castronovo and G. Sanfilippo. "Generalized conjunction and disjunction of two conditional events in the setting of conditional random quantities". In: *International Journal of Approximate Reasoning* 187 (2025), p. 109533. ISSN: 0888-613X. DOI: https://doi.org/10.1016/j.ijar.2025.109533.

[21] L. Castronovo and G. Sanfilippo. "Iterated conditionals, trivalent logics, and conditional random quantities". In: *15th International Conference on Scalable Uncertainty Management (SUM 2022), October 17-19, 2022, Paris, France*. Ed. by M. Dupin, M. Öztürk, and N. Potyka. Vol. 13562. LNCS. Springer, 2022, pp. 47–63.

[22] L. Castronovo et al. "Towards an algebraic and probabilistic setting for iterated Boolean conditionals". In: *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2025)*. Ed. by K. Sauerwald and M. Thimm. Springer Lecture Notes in Artificial Intelligence (LNAI). Springer, To appear.

[23] Y. Al Censor and S. A. Zenios. *Parallel Optimization: Theory, Algorithms and Applications*. Oxford University Press, 1997. ISBN: 019510062X.

[24] D. Ciucci and D. Dubois. "A map of dependencies among three-valued logics". In: *Information Sciences* 250 (2013), pp. 162 –177. ISSN: 0020-0255.

[25] D. Ciucci and D. Dubois. "Relationships between Connectives in Three-Valued Logics". In: *Advances on Computational Intelligence*. Vol. 297. CCIS. Springer, 2012, pp. 633–642.

[26] G. Coletti and R. Scozzafava. "Conditional probability, fuzzy sets, and possibility: a unifying view." In: *Fuzzy Sets and Systems* **144**(1) (2004), pp. 227–249.

[27] G. Coletti and R. Scozzafava. "Conditioning and inference in intelligent systems". In: *Soft Computing* 3.3 (1999), pp. 118–130.

[28] G. Coletti and R. Scozzafava. *Probabilistic logic in a coherent setting*. Dordrecht: Kluwer, 2002.

[29] G. Coletti, R. Scozzafava, and B. Vantaggi. "Coherent Conditional Probability, Fuzzy Inclusion and Default Rules". In: *Soft Computing: State of the Art Theory and Novel Applications*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 193–208.

[30] W. S. Cooper. “The propositional logic of ordinary discourse1”. In: *Inquiry* 11.1-4 (1968), pp. 295–320. DOI: 10.1080/00201746808601531.

[31] Á. Császár. “Sur la structure des espaces de probabilité conditionnelle”. In: *Acta Mathematica Academiae Scientiarum Hungarica* 6.3-4 (1955), pp. 337–361. ISSN: 0001-5954.

[32] B. de Finetti. “La Logique de la Probabilité”. In: *Actes du Congrès International de Philosophie Scientifique, Paris, 1935*. Hermann et C.ie, Paris, 1936, pp. IV 1–IV 9.

[33] B. de Finetti. “Sul significato soggettivo della probabilità”. In: *Fundamenta Mathematicae* 17 (1931), pp. 298–329.

[34] T. Denoeux, D. Dubois, and H. Prade. “Representations of Uncertainty in Artificial Intelligence: Probability and Possibility”. In: *A Guided Tour of Artificial Intelligence Research Volume I: Knowledge Representation, Reasoning and Learning*. Springer International Publishing, May 2020, pp. 69–117. DOI: 10.1007/978-3-030-06164-7_3.

[35] L. E. Dubins. “Finitely additive conditional probabilities, conglomerability and disintegrations”. In: *The Annals of Probability* 3 (1975), pp. 89–99.

[36] P. Egré, L. Rossi, and J. Sprenger. “De Finettian Logics of Indicative Conditionals Part I: Trivalent Semantics and Validity”. In: *J Philos Logic* (2020).

[37] R. J. Farrell. “Material implication, confirmation, and counterfactuals.” In: *Notre Dame Journal of Formal Logic* 20.2 (1979), pp. 383 –394. DOI: 10.1305/ndjfl/1093882546.

[38] B. de Finetti. “Does it make sense to speak of ‘good probability appraisers’?” In: *The scientist speculates: an anthology of partly-baked ideas*. Ed. by I J Good. Heinemann, London, 1962, pp. 357–364.

[39] B. de Finetti. *Teoria delle probabilitá*. English version: Theory of Probability 1 (2), Chichester, Wiley, 1974 (1975). Torino: Ed. Einaudi, 2 voll., 1970.

[40] T. Flaminio, L. Godo, and H. Hosni. “Boolean algebras of conditionals, probability and logic”. In: *Artificial Intelligence* 286 (2020), p. 103347.

[41] T. Flaminio et al. “Compound Conditionals as Random Quantities and Boolean Algebras”. In: *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning, KR 2022, Haifa, Israel. July 31 - August 5, 2022*. Ed. by Gabriele Kern-Isberner, Gerhard Lakemeyer, and Thomas Meyer. 2022, pp. 141–151.

[42] T. Flaminio et al. “On conditional probabilities and their canonical extensions to Boolean algebras of compound conditionals”. In: *Int. J. Approx. Reason.* 159 (2023), p. 108943. DOI: 10.1016/j.ijar.2023.108943.

[43] A. Gibbard. “Two recent theories of conditionals”. In: *Ifs*. Ed. by W. L. Harper, R. Stalnaker, and G. Pearce. Dordrecht: Reidel, 1981, pp. 221–247.

[44] R. Giles. “‘Ąukasiewicz logic and fuzzy set theory.” In: *International Journal of Man-Machine Studies* **8**(3) (1976), pp. 313–327.

[45] A. Gilio. “Criterio di penalizzazione e condizioni di coerenza nella valutazione soggettiva della probabilità”. In: *Bollettino dell’Unione Matematica Italiana* 4–B (1990), pp. 645–660.

[46] A. Gilio. “Probabilistic reasoning under coherence in System P”. In: *Annals of Mathematics and Artificial Intelligence* 34 (2002), pp. 5–34.

[47] A. Gilio and S. Ingrassia. "Totally coherent set-valued probability assessments." In: *Kybernetika* **34**(1) (1998), pp. 3–15.

[48] A. Gilio, N. Pfeifer, and G. Sanfilippo. "Probabilistic entailment and iterated conditionals". In: *Logic and Uncertainty in the Human Mind: A Tribute to David E. Over*. Oxon: Routledge, 2020, pp. 71–101.

[49] A. Gilio, N. Pfeifer, and G. Sanfilippo. "Transitivity in coherence-based probability logic." In: *Journal of Applied Logic* 14 (2016), pp. 46–64.

[50] A. Gilio and G. Sanfilippo. "Algebraic aspects and coherence conditions for conjoined and disjoined conditionals". In: *International Journal of Approximate Reasoning* 126 (2020), pp. 98 –123.

[51] A. Gilio and G. Sanfilippo. "Coherent conditional probabilities and proper scoring rules". In: *ISIPTA'11: Proceedings of the Seventh International Symposium on Imprecise Probability: Theories and Applications*. Ed. by F Coolen et al. Innsbruck: SIPTA, 2011, pp. 189–198.

[52] A. Gilio and G. Sanfilippo. "Compound conditionals, Fréchet-Hoeffding bounds, and Frank t-norms". In: *International Journal of Approximate Reasoning* 136 (2021), pp. 168–200. ISSN: 0888-613X.

[53] A. Gilio and G. Sanfilippo. "Conditional Random Quantities and Compounds of Conditionals". In: *Studia Logica* 102.4 (2014), pp. 709–729.

[54] A. Gilio and G. Sanfilippo. "Conditional random quantities and iterated conditioning in the setting of coherence". In: *ECSQARU 2013*. Ed. by L. C. van der Gaag. Vol. 7958. LNCS. Berlin, Heidelberg: Springer, 2013, pp. 218–229.

[55] A. Gilio and G. Sanfilippo. "Conjunction, disjunction and iterated conditioning of conditional events". In: *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*. Vol. 190. AISC. Berlin: Springer, 2013, pp. 399–407.

[56] A. Gilio and G. Sanfilippo. "Generalized Logical Operations among Conditional Events". In: *Applied Intelligence* 49.1 (2019), pp. 79–102.

[57] A. Gilio and G. Sanfilippo. "Iterated Conditionals and Characterization of P-Entailment". In: *Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU 2021*. Vol. 12897. LNCS. Springer, 2021, pp. 629–643.

[58] A. Gilio and G. Sanfilippo. "On compound and iterated conditionals". In: *Argumenta* 6.2 (2021), pp. 241–266.

[59] A. Gilio and G. Sanfilippo. "Quasi conjunction, quasi disjunction, t-norms and t-conorms: Probabilistic aspects". In: *Information Sciences* 245 (2013), pp. 146–167.

[60] A. Gilio and G. Sanfilippo. "Subjective probability, trivalent logics and compound conditionals". In: *Handbook of Three-Valued Logics*. Ed. by P. Égré and L. Rossi. Cambridge: MIT Press, (forthcoming). URL: http://arxiv.org/abs/2301.09327.

[61] A. Gilio and G. Sanfilippo. "Subjective probability, trivalent logics and compound conditionals". In: *Handbook of Trivalent Logics*. Ed. by P. Égré and L. Rossi. Cambridge: MIT Press, (forthcoming).

[62] A. Gilio et al. "On Independence and Compound and Iterated Conditionals". In: Springer Nature Switzerland, Mar. 2024, pp. 159–177. ISBN: 978-3-031-55244-1. DOI: 10.1007/978-3-031-55245-8_11.

[63] A. Gilio et al. “On trivalent logics, probabilistic weak deduction theorems, and a general import-export principle”. In: *Artificial Intelligence* 337 (2024), p. 104229. ISSN: 0004-3702. DOI: https://doi.org/10.1016/j.artint.2024.104229. URL: https://www.sciencedirect.com/science/article/pii/S0004370224001656.

[64] T. Gneiting and A. E. Raftery. “Strictly Proper Scoring Rules, Prediction, and Estimation”. In: *Journal of the American Statistical Association* 102 (2007), pp. 359–378. URL: http://EconPapers.repec.org/RePEc:bes:jnlasa:v:102:y:2007:p:359-378.

[65] I. R. Goodman and H. T. Nguyen. “Conditional Objects and the Modeling of Uncertainties”. In: *Fuzzy Computing*. Ed. by M. M. Gupta and T. Yamakawa. North-Holland, 1988, pp. 119–138.

[66] I. R. Goodman, H. T. Nguyen, and Elbert A. Walker. *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*. North-Holland, 1991, pp. I–VIII, 1–288. ISBN: 978-0-444-88685-9.

[67] I.R. Goodman and H. T. Nguyen. “A theory of conditional information for probabilistic inference in intelligent systems: II. Product space approach”. In: *Information Sciences* 76.1 (1994), pp. 13–42. ISSN: 0020-0255. DOI: https://doi.org/10.1016/0020-0255(94)90065-5. URL: https://www.sciencedirect.com/science/article/pii/0020025594900655.

[68] T. Hailperin. *Sentential probability logic. Origins, development, current status, and technical applications*. Bethlehem: Lehigh University Press, 1996.

[69] J. Y. Halpern. *Reasoning About Uncertainty*. MIT Press, 2003.

[70] S. Holzer. “On Coherence and Conditional Prevision”. In: *Bollettino dell’Unione Matematica Italiana* 4.6 (1985), pp. 441–460.

[71] S. Kaufmann. “Conditionals Right and Left: Probabilities for the Whole Family”. English. In: *Journal of Philosophical Logic* 38 (2009), pp. 1–53. ISSN: 0022-3611.

[72] E.P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*. Springer, 1 edn. (2000).

[73] J. Konek. “Evaluating imprecise forecasts”. In: *Proceedings of the Thirteenth International Symposium on Imprecise Probability: Theories and Applications*. Ed. by Enrique Miranda et al. Vol. 215. Proceedings of Machine Learning Research. PMLR, 2023, pp. 270–279. URL: https://proceedings.mlr.press/v215/konek23a.html.

[74] F. Lad. “Coherent prevision as a linear functional without an underlying measure space: the purely arithmetic structure of conditional quantities”. In: *Mathematical Models for Handling Partial Knowledge in Artificial Intelligence*. Ed. by G Coletti et al. New York: Plenum Press, 1995, pp. 101–112.

[75] F. Lad. *Operational subjective statistical methods: A mathematical, philosophical, and historical introduction*. New York: Wiley, 1996.

[76] F. Lad and G. Sanfilippo. “Scoring alternative forecast distributions: completing the Kullback symmetric distance complex”. In: *Global and Local Economic Review* 22.02 (2018), pp. 63–90. ISSN: 1722-4241.

[77] F. Lad, G. Sanfilippo, and G. Agró. "Completing the logarithmic scoring rule for assessing probability distributions". In: *AIP Conference Proceedings* 1490.1 (2012). Ed. by Julio Michael Stern et al. ISBN: 978-0-7354-1102-9, ISSN 0094-243X, pp. 13–30. ISSN: 0094-243X. DOI: 10.1063/1.4759585. URL: http://link.aip.org/link/?APC/1490/13/1.

[78] F. Lad, G. Sanfilippo, and G. Agró. "Extropy: complementary dual of entropy". In: *Statistical Science (Journal of the Institute of Mathematical Statistics)* 30.1 (2015), pp. 40–58. ISSN: 0883-4237. DOI: 10.1214/14-STS430.

[79] D. Lewis. "Probabilities of Conditionals and Conditional Probabilities". In: *The Philosophical Review* 85.3 (1976), pp. 297–315.

[80] V. McGee. "Conditional Probabilities and Compounds of Conditionals". In: *Philosophical Review* 98.4 (1989), pp. 485–541.

[81] D. E. Over et al. "The probability of causal conditionals". In: *Cognitive Psychology* 54.1 (2007), pp. 62–97. DOI: https://doi.org/10.1016/j.cogpsych.2006.05.002.

[82] R. Pelessoni and P. Vicig. "The Goodman-Nguyen relation within imprecise probability theory". In: *International Journal of Approximate Reasoning* 55.8 (2014), pp. 1694–1707. DOI: 10.1016/j.ijar.2014.06.002.

[83] D. Petturiti and B. Vantaggi. "How to Assess Coherent Beliefs: A Comparison of Different Notions of Coherence in Dempster-Shafer Theory of Evidence". In: *Reflections on the Foundations of Probability and Statistics: Essays in Honor of Teddy Seidenfeld*. Cham: Springer International Publishing, 2022, pp. 161–185. ISBN: 978-3-031-15436-2. DOI: 10.1007/978-3-031-15436-2_8. URL: https://doi.org/10.1007/978-3-031-15436-2_8.

[84] D. Petturiti and B. Vantaggi. "Modeling agent's conditional preferences under objective ambiguity in Dempster-Shafer theory". In: *International Journal of Approximate Reasoning* 119 (2020), pp. 151–176. ISSN: 0888-613X.

[85] N. Pfeifer and G. Sanfilippo. "Connexive Logic, Probabilistic Default Reasoning, and Compound Conditionals". In: *Studia Logica* (2023). DOI: 10.1007/s11225-023-10054-5.

[86] J.B. Predd et al. "Probabilistic Coherence and Proper Scoring Rules". In: *IEEE Transactions on Information Theory* 55.10 (2009), pp. 4786–4792. ISSN: 0018-9448. DOI: 10.1109/TIT.2009.2027573.

[87] E. Regazzini. "Finitely Additive Conditional Probabilities". In: *Rendiconti del Seminario Matematico e Fisico di Milano* 55 (1985), pp. 69–89.

[88] G. Sanfilippo. "Lower and upper probability bounds for some conjunctions of two conditional events". In: *SUM 2018*. Vol. 11142. LNCS. Cham: Springer, 2018, pp. 260–275. ISBN: 978-3-030-00460-6.

[89] G. Sanfilippo, N. Pfeifer, and A. Gilio. "Generalized probabilistic modus ponens". In: *ECSQARU 2017*. Ed. by A. Antonucci, L. Cholvy, and O. Papini. Vol. 10369. LNCS. Springer, 2017, pp. 480–490.

[90] G. Sanfilippo et al. "Probabilistic inferences from conjoined to iterated conditionals". In: *International Journal of Approximate Reasoning* 93.Supplement C (2018), pp. 103 –118. ISSN: 0888-613X.

[91] G. Sanfilippo et al. "Probabilities of conditionals and previsions of iterated conditionals". In: *International Journal of Approximate Reasoning* 121 (2020), pp. 150 –173. ISSN: 0888-613X.

[92] T. Seidenfeld, M.J. Schervish, and J.B. Kadane. “Forecasting with imprecise probabilities”. In: *International Journal of Approximate Reasoning* 53.8 (2012), pp. 1248–1261. DOI: https://doi.org/10.1016/j.ijar.2012.06.018.

[93] C. G. Wagner. “Modus Tollens Probabilized”. In: *The British Journal for the Philosophy of Science* 55.4 (2004), pp. 747–753. (Visited on 06/30/2023).

[94] E. A. Walker. “Stone algebras, conditional events, and three-valued logic”. In: *IEEE Transaction of Systems, Man, and Cybernetics* 24 (12) (1994), 1699‘Ă‘Ş17.

[95] P. Walley. *Coherent upper and lower previsions*. The Imprecise Probabilities Project. 1997.

[96] P. M. Williams. “Notes on conditional previsions”. In: *International Journal of Approximate Reasoning* 44.3 (2007), pp. 366–383.

[97] R. L. Winkler. “Evaluating Probabilities: Asymmetric Scoring Rules”. In: *Management Science* 40.11 (1994), pp. 1395–1405. ISSN: 00251909, 15265501.

[98] K. Wójtowicz and A. Wójtowicz. “Probabilities of Conditionals via Labeled Markov Graphs”. In: *Erkenntnis* (2025). DOI: 10.1007/s10670-024-00896-3.

[99] L. Zadeh. “Fuzzy sets.” In: *Information and Control* **8**(3) ((1965)), pp. 338–353.