

Thermodynamics of ferromagnetic crystals with a non-Euclidean structure as internal variable

Marina Dolfin¹, Mauro Francaviglia², Liliana Restuccia¹

¹*Dipartimento di Matematica, Facoltà di Scienze MM.FF.NN.
Università degli Studi di Messina, Italy
mdolfin@unime.it
lrest@dipmat.unime.it*

²*Dip. di Matematica, Facoltà di Scienze MM.FF.NN.
Università degli Studi di Torino, Italy
mauro.francaviglia@unito.it*

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Abstract

In this paper we construct a geometric model for deformable magnetizable bodies in the framework of thermodynamics of simple materials, taking into account a Maugin's approach for ferromagnetic crystals, developed within the irreversible thermodynamics with vectorial and tensorial internal variables. We explicitly consider an internal (non-Euclidean) metric as a thermodynamical non-equilibrium variable obtaining the dynamical system on the fibre bundle of processes for simple material elements of the media under consideration. The derivation of this system is the first step to apply the qualitative theory of dynamical systems. Furthermore, we work out the entropy function and the entropy 1-form, which represents the starting point to introduce an extended thermodynamical phase space. Finally, from Clausius-Duhem inequality we give the extra-entropy flux and the state laws.

1. Introduction

In a previous paper [1] a geometrization technique was applied to the thermodynamics of magnetizable simple materials with internal variables [2]- [8], using a model given by Maugin in [9]- [11], within the irreversible thermodynamics. The aim of this paper is to develop, in the same geometric framework, a thermodynamical model for these media, where an extra internal variable is introduced in terms of a non-Euclidean intrinsic material metric \mathbf{g} mixing up with the deformation gradient \mathbf{F} and accounting for internal disarrangement. In Sections 2 and 3 we give a continuum description of deformable ferromagnetic crystals and we introduce the concept of physical metric in thermodynamics due to K.C. Valanis [12]. In Section 4

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we recall, following G.A. Maugin, the fundamental laws that govern the processes occurring inside the considered media. In Section 5 we construct a geometric model for magnetizable crystals with a non-Euclidean metric working out the dynamical system for simple material elements of these media and the expressions of the entropy function and of the entropy 1-form. This last result is the starting point to introduce an extended thermodynamical phase space [13]. Furthermore, the obtained dynamical system is a set of ordinary differential equations with respect to time on the fibre bundle of processes, while the thermodynamical models developed in the framework of irreversible thermodynamics are described by partial differential equations [9]- [11]. Finally, using Maugin's technique, we derive the extra-entropy flux and the state laws, with the aid of Clausius-Duhem inequality. Concrete materials covered by these models include ferrites, ferro-magnets and anti-ferromagnets. In [14] a thermodynamic geometrical model was developed by the authors for deformable dielectrics with a non-Euclidean structure by the same geometrization technique.

2. Basic assumptions of a model for deformable magnetizable bodies

We recall here briefly the continuous description of deformable ferro-magnetic bodies due to Maugin [9] - [11].

We work in a suitable Galilean quasi-static approximation considering an Eulerian configuration K_t of the body, where a rectangular Cartesian coordinate system is assumed. A deformable magnetically ordered crystal below its magnetic transition temperature T_{cr} is assumed to be composed of n molecular (ionic) species, each one of them giving rise to a *field of magnetization* per unit of mass denoted by μ_α ($\alpha = 1, \dots, n$) (magnetic sublattices interactions). A spin density \mathbf{s}_α per unit mass is associated with each μ_α via the gyromagnetic relation

$$(1) \quad \mathbf{s}_\alpha(\mathbf{x}, t) = \gamma_\alpha^{-1} \boldsymbol{\mu}_\alpha(\mathbf{x}, t)$$

where $\gamma_\alpha = g_\alpha e / 2m_0 c$ (e : electronic charge; m_0 = rest mass of the electron; c : light velocity in vacuo) is the gyromagnetic ratio and g_α is the α -th corresponding spectroscopic splitting factor associated to the ionic species. The total spin intrinsic momentum per unit mass is thus given by

$$(2) \quad \mathbf{s} = \sum_{\alpha} \mathbf{s}_\alpha = \sum_{\alpha} \gamma_\alpha^{-1} \boldsymbol{\mu}_\alpha.$$

Furthermore, it is

$$(3) \quad \boldsymbol{\mu}(\mathbf{x}, t) = \sum_{\alpha} \boldsymbol{\mu}_{\alpha}(\mathbf{x}, t).$$

If all magnetic moments of the ions arise only from spin, it is $g_{\alpha} = g_e = e$. In [9] it is argued that $\boldsymbol{\mu}_{\alpha}$ has necessarily at \mathbf{x} and instantaneously at time t the following time evolution equation

$$(4) \quad \dot{\boldsymbol{\mu}}_{\alpha} = \boldsymbol{\omega}_{\alpha} \times \boldsymbol{\mu}_{\alpha} = -\gamma_{\alpha} \mathbf{B}^{eff} \times \boldsymbol{\mu}_{\alpha}$$

where $\boldsymbol{\omega}_{\alpha}(\mathbf{x}, t) = -\gamma_{\alpha} \mathbf{B}^{eff}$ is the instantaneous and local precessional velocity of $\boldsymbol{\mu}_{\alpha}$ and the *effective magnetic field* \mathbf{B}^{eff} is assumed to be

$$(5) \quad \mathbf{B}^{eff} = \mathbf{B} + {}^L\mathbf{B} + m^{-1}(\nabla \cdot {}^L\mathbb{B}),$$

where: \mathbf{B} is the external magnetic induction; ${}^L\mathbf{B}$ is a vector which accounts for the intermagnetic-sublattice interactions coming from the different ionic species; ${}^L\mathbb{B}$ is a rank two tensor which accounts for the *spin-lattice interactions* arising between each magnetic sublattice and the material continuum (i.e. the crystal lattice) that are responsible for the typical ferromagnetic ordering. It is also obvious that it is

$$(6) \quad \boldsymbol{\omega}_{\alpha} \cdot \dot{\boldsymbol{\mu}}_{\alpha} = 0$$

or

$$(7) \quad \gamma_{\alpha}^{-1} \dot{\boldsymbol{\mu}}_{\alpha} \cdot \boldsymbol{\omega}_{\alpha} = 0$$

taking into account equation (1). This last equation means that \mathbf{s}_{α} is a *d'Alembertian inertia couple* (i.e. a gyroscopic couple). In the following we will consider only one species of molecules.

3. Intrinsic Material Metric or "physical metric".

The concept of physical metric in thermodynamics has been introduced by K.C Valanis [12] in an effort to determine a conceptual connection between the micromotion and the global material behavior. In classical continuum theories the material metric is always Euclidean, or can be reduced to Euclidean by a transformation, but in [12] a concrete example is given where the physical metric evolves from Euclidean to non-Euclidean when

the deformation of a neighborhood is non-affine. Moreover, cases are shown where the intrinsic material metric is an internal variable, which is function of the internal material structure and its evolution. The physical interpretation is that although the initial material metric in the reference frame may be Euclidean, an internal disarrangement of the atomic structure of a material neighborhood will cause its metric to become "intrinsically" non-Euclidean, so that

$$(8) \quad ds_0^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

where ds_0^2 is the square of the initial distance of two neighboring particles at $t = 0$ and $g_{\alpha\beta}$ is the intrinsic material metric. Its effect is to change the expression for the Finger (material Cauchy-Green tensor) and Left Cauchy-Green tensors. In a previous paper [15] it was already considered the possibility of introducing as extra internal variable an "inner metric" \mathbf{g} which mixes up with the deformation gradient \mathbf{F} through an algebraic relation of the type

$$(9) \quad \mathbf{C} = \mathbf{F}^T \mathbf{g} \mathbf{F},$$

where \mathbf{C} is the material Cauchy-Green tensor and the symbol " T " denotes transposition. The internal variable \mathbf{g} is then a non-Euclidean metric tensor that has to do with an average tensor characterizing the local deviations from the Euclidean structure associated with effects on the mesoscopic scale [12]. What we mean is that if the mesoscopic subdomains have characteristic dimension too small to be detected by technical instruments of measure then one may only observe the motion of certain aggregates of *mesoscopic subdomains*. Along a thermodynamic process the individual mesoscopic subdomain migrates and diffuses, so that a domain's neighborhood is constantly changing. This micromotion may influence the topology of the body, resulting thus in a non-affine deformation superposed on the deformation of a mesodomain. This additional effect gives rise to a non-Euclidean (local) structure. Moreover, it can produce an additional dissipation of energy inside the body which is not due to macroscopic phenomena. We take into account this type of situation by regarding the physical metric as an internal variable.

4. Fundamental laws

The object of our study is a non linear thermoelastic magnetizable body (see [1]), in which the following fields interact with each other: the *elastic*

field described in the case of finite deformations by stress tensor t_{ij} (in general non symmetric) and the rate of deformation $\mathbf{L} = \nabla \mathbf{v}$ (where \mathbf{v} is the velocity field); the *thermal field* described by the internal energy e and the gradient of temperature $\nabla\theta$; the *magnetization* $\boldsymbol{\mu}$ and its gradient $\nabla\boldsymbol{\mu}$. If we split \mathbf{C} by using eq.(9), the independent variables are represented by the set

$$(10) \quad (\mathbf{F}, \mathbf{g}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu}).$$

Alternatively, if we prefer not to split \mathbf{C} thence we assume as variables the following

$$(11) \quad (\mathbf{C}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu}).$$

No dependence whatsoever on $\nabla\mathbf{g}$ will be considered, since covariant effects associated with the metric would in fact pass through higher order derivatives of \mathbf{g} involved in curvature [12].

All the process occurring in the body considered are governed, in the quasi-static case, by the following balance equations [9] - [11]

- Maxwell's equations in the quasi-magnetostatic case

$$(12) \quad \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \wedge \mathbf{B} = \nabla \wedge (m\boldsymbol{\mu}), \quad \mathbf{H} = \mathbf{B} - m\boldsymbol{\mu},$$

where \mathbf{H} is the magnetic field (Lorentz-Heaviside EM units are used throughout);

- the balance equation of mass density

$$(13) \quad \dot{m} + m\nabla \cdot \mathbf{v} = 0,$$

where m denotes the mass density and \mathbf{v} is the velocity of body particles;

- the balance equation of momentum density

$$(14) \quad m\dot{\mathbf{v}} = \nabla \cdot \mathbf{t} + \mathbf{f}^{em},$$

in which \mathbf{f}^{em} is the *electromotive force* and \mathbf{t} is the non-symmetric *total stress tensor*, given in components, respectively, by:

$$(15) \quad \mathbf{f}_i^{em} = m\boldsymbol{\mu}_j B_{/i}^j,$$

$$(16) \quad t^{ij} = T^{ij} + m\boldsymbol{\mu}^{[jL}B^i] - {}^L\mathbb{B}_{[i|k|\mu_{j],k}} \neq t^{ji},$$

where the brackets [] denote skew-symmetrization and \mathbf{T}^{ij} are the components of the standard symmetric *elastic stress tensor* of Cauchy.

- the balance equation of moment of momentum

$$(17) \quad \epsilon_{ijk}t_{jk} = f_i,$$

where f_i is a couple of forces.

- the balance equation of internal energy (1^o law of thermodynamics)

$$(18) \quad m\dot{e} = p_{(i)} - \nabla \cdot \mathbf{q} + mr,$$

where e and r are the internal energy and the heat radiation per unit of mass respectively; \mathbf{q} is the *heat flux* and $p_{(i)}$ is the *virtual power* of internal forces that in the model developed by Maugin [9]- [11] is explicitly given by

$$(19) \quad p_{(i)} = \mathbf{T} \cdot \mathbf{L} - m{}^L\mathbf{B} \cdot \dot{\boldsymbol{\mu}} + {}^L\mathbb{B} \cdot \nabla \dot{\boldsymbol{\mu}},$$

where \mathbf{L} is the velocity gradient. Here and in the sequel $\mathbf{A} \cdot \mathbf{B}$ denotes the full contraction of vectors and tensors, i.e. $\mathbf{A} \cdot \mathbf{B} = \text{tr}(\mathbf{A}^T \mathbf{B})$; moreover in the following the heat radiation r will be neglected. The thermodynamical model regarding the media under consideration presented in this Section is based on the assumption that elasto-mechanical phenomena should couple, in concrete materials both to the magnetization $\boldsymbol{\mu}$ and to its gradient.

We are now ready to work on the problem of finding expressions for \mathbf{L} , and the quantity $\mathbf{T} \cdot \mathbf{L}$ which enters eqn. (19) for $p_{(i)}$. Notice that we are facing an implicit function problem, hidden in the fact that among the three variables \mathbf{F} , \mathbf{g} and \mathbf{C} only two are independent because of equation (9). Specific calculations suggest us to choose the two independent variables in different ways depending upon the formula envisaged.

Through derivation the relation

$$(20) \quad \mathbf{C} = \mathbf{F}^T \mathbf{g} \mathbf{F}$$

takes the form

$$(21) \quad \dot{\mathbf{C}} = \dot{\mathbf{F}}^T \mathbf{g} \mathbf{F} + \mathbf{F}^T \dot{\mathbf{g}} \mathbf{F} + \mathbf{F}^T \mathbf{g} \dot{\mathbf{F}}$$

From (21), using the notation $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ we obtain

$$(22) \quad \dot{\mathbf{C}} = \dot{\mathbf{F}}^T \mathbf{g}\mathbf{F} + \mathbf{F}^T \dot{\mathbf{g}}\mathbf{F} + \mathbf{F}^T \mathbf{g}(\mathbf{L}\mathbf{F}).$$

Multiplying eq.(22) by \mathbf{F}^{-1} one has

$$(23) \quad \dot{\mathbf{C}}\mathbf{F}^{-1} = (\dot{\mathbf{F}}^T \mathbf{g} + \mathbf{F}^T \dot{\mathbf{g}}) + \mathbf{F}^T \mathbf{g}\mathbf{L}.$$

Finally, from eq.(23), one obtains

$$(24) \quad \mathbf{L} = \mathbf{M}^{-1}\dot{\mathbf{C}}\mathbf{F}^{-1} - \mathbf{M}^{-1}\dot{\mathbf{M}}$$

with the positions $\mathbf{M} = \mathbf{F}^T \mathbf{g} = \mathbf{C}\mathbf{F}^{-1}$.

Using eq.(24), one obtains the following explicit expression for the power of internal forces (19) in our model

$$(25) \quad p_{(i)} = -\mathbf{Z} \cdot \dot{\mathbf{M}} + (\mathbf{Z}\mathbf{F}^{-T}) \cdot \dot{\mathbf{C}} - m^L \mathbf{B} \cdot \dot{\boldsymbol{\mu}} + {}^L \mathbb{B} \cdot \nabla \dot{\boldsymbol{\mu}},$$

where we set for simplicity $\mathbf{Z} \equiv \mathbf{M}^{-T}\mathbf{T} = (\mathbf{T}\mathbf{M}^{-1})^T$; the notation $\mathbf{X}^{-T} = (\mathbf{X}^{-1})^T$ is used for all matrices \mathbf{X} . Using equation (25), and the first law of thermodynamics (18), one obtains the following expression

$$(26) \quad -\nabla \cdot \mathbf{q} = m\dot{e} + \mathbf{Z} \cdot \dot{\mathbf{M}} - (\mathbf{Z}\mathbf{F}^{-T}) \cdot \dot{\mathbf{C}} + m^L \mathbf{B} \cdot \dot{\boldsymbol{\mu}} - {}^L \mathbb{B} \cdot \nabla \dot{\boldsymbol{\mu}}.$$

To be sure that the physical processes occurring in the body considered are real, all the admissible solutions of the proposed evolution equations should be restricted by the following entropy inequality (2° law of thermodynamics):

$$(27) \quad m\dot{S} + \nabla \cdot \mathbf{J}_S \geq 0$$

where S denotes the entropy per unit of mass and the entropy flux \mathbf{J}_S is given by

$$(28) \quad \mathbf{J}_S = \frac{1}{\theta} \mathbf{q} + \mathbf{k},$$

with the additional term \mathbf{k} called *extra entropy flux density* which includes contributions due to the internal variables [25]; θ is the thermodynamic temperature (here supposed to be such that $0 < \theta \leq \theta_c$, since the range of temperature considered is much below the Curie ferromagnetic phase-transition temperature θ_c).

5. A geometrical model for a magnetizable crystal

We recall briefly a geometrical model for the thermodynamics of material point model developed in [2]- [8]. The first leading idea consists in assuming from the beginning that time resides on an equal footing with all other state variables, so that terms in dt will directly enter the so-called entropy 1-form. Moreover, a second fundamental ingredient is the introduction of internal variables which, together with their space gradients, enter directly the definition of the state spaces and of state functions. An instantaneous state space B_t is assumed to contain all state variables which fit the configuration of the element at time t and B_t is assumed to be a manifold. The *total state space* is then given by the disjoint union

$$(29) \quad \mathcal{B} = \bigcup_{t \in \mathbb{R}} B_t,$$

with a given natural structure of fibre bundle over the real line \mathbb{R} where time flows [6,7]. If the instantaneous state space B_t does not vary in time, the state space \mathcal{B} reduces to a Cartesian product $\mathbb{R} \times B$. Moreover, following Noll [3] we consider an abstract space of *processes* which consists of a set Π of functions

$$(30) \quad P_t^i : [0, t] \rightarrow \mathcal{G},$$

where $[0, t]$ is any time interval, the space \mathcal{G} is a suitable target space suggested by the model, i is a label ranging in an unspecified index set for all allowed processes and $t \in \mathbb{R}$ is the so called *duration* of the process. Then, a continuous function is defined

$$(31) \quad \rho : \mathbb{R} \times \Pi \rightarrow C^0(B_0, B_t),$$

so that for any instant of time t and for any process $P_t^i \in \Pi$ a continuous mapping called *transformation* (induced by the process) is generated.

In some cases, however, the description of the evolution of a thermodynamical system requires an extension of the state space through the introduction of further dynamical variables like, for instance, internal variables. Internal variables represent micro-local or mesoscopic phenomena which one cannot control in full detail and can be described as averaged quantities described by a set of parameters (of mechanical, thermodynamical or other physical origin suggested by the particular model chosen for the continuum (see [20]-[24] for the general theory and applications).

Following standard ideas (see [17]- [19]), it is assumed that the body \mathcal{B} , having a regular boundary $\partial\mathcal{B}$, is regularly embedded into the Euclidean

space \mathbb{R}^3 by a regular family of instantaneous time-dependent configurations \mathcal{B}_t . The properties of \mathcal{B}_t are encoded by the state variables (11). The total state space is then:

$$(32) \quad B = Lin(W) \oplus \mathbb{R} \oplus W \oplus W \oplus Lin(W),$$

where W is any vector space accounting for the variable $\boldsymbol{\mu}$ and $Lin(W)$ accounts for its gradients $\nabla\boldsymbol{\mu}$.

By assuming $\boldsymbol{\mu}$ and $\nabla\boldsymbol{\mu}$ as state variables, we specify then the space W of (32) to be $W \simeq \mathbb{R}^3$, where the time derivative of the polarization vector $\boldsymbol{\mu}$ lives; in other words we shall assume as state variables $(\mathbf{C}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu})$. We assume now that the following dynamical system holds

$$(33) \quad \begin{cases} \dot{\mathbf{C}} = \mathbf{F}^T \mathbf{g} [\mathbf{g}^{-1} \mathbf{L}^T \mathbf{g} + \mathbf{g}^{-1} \dot{\mathbf{g}} + \mathbf{L}] \mathbf{F} \\ m\dot{e} = p_{(i)} - \nabla \cdot \mathbf{q} \\ \dot{\nabla}\theta = \boldsymbol{\Lambda} \\ \dot{\boldsymbol{\mu}} = -\gamma \mathbf{B}^{eff} \times \boldsymbol{\mu} \\ (\dot{\nabla}\boldsymbol{\mu}) = \boldsymbol{\Gamma} \\ \dot{\mathbf{g}} = \mathbf{G} \end{cases}$$

where $p_{(i)}$ and $-\nabla \cdot \mathbf{q}$ are given by eqs.(25) and (26), respectively and we take into account eq.(9) and eq.(5). Moreover the following constitutive functions are defined

$$(34) \quad \begin{aligned} \theta &: \mathbb{R} \times B \rightarrow \mathbb{R}^{++}, \\ \mathbf{T} &: \mathbb{R} \times B \rightarrow Sym(W), \\ \boldsymbol{\Lambda} &: \mathbb{R} \times B \rightarrow W, \\ \boldsymbol{\Gamma} &: \mathbb{R} \times B \rightarrow Lin(W), \\ \mathbf{G} &: \mathbb{R} \times B \rightarrow Sym(W). \end{aligned}$$

The set $(B, \Pi, \theta, \mathbf{T}, \boldsymbol{\Lambda}, \boldsymbol{\Gamma})$ defines the simple material element of ferromagnetic crystals with a non-Euclidean internal metric. This dynamical system (33) on the fibre bundle of the processes has the advantage to be a system of ordinary differential equations with respect to time, while Maugin's model, within the context of irreversible thermodynamics, is described by partial differential equations.

5.1. Entropy action

In this geometrical framework we are able to introduce an “entropy action”, which is related to a reversible transformation between the initial

and the final states b and $\rho_t^i(b) = b_t$, respectively, by setting:

$$(35) \quad s(\rho_t^i, b, t) = - \int_0^t \frac{1}{m} \nabla \cdot \mathbf{J}_S d\tau,$$

where \mathbf{J}_S is defined according to equation (28). Using eq. (26), we get the following form for the entropy action

$$(36) \quad s = \int_0^t \left[-\frac{1}{m\theta} \nabla \cdot \mathbf{q} + \frac{1}{m\theta^2} \mathbf{q} \cdot \nabla \theta - \frac{1}{m} \nabla \cdot \mathbf{k} \right] d\tau.$$

Then, we can write s in the following form

$$(37) \quad s = \int_0^t \left[\frac{1}{\theta} \dot{e} + \frac{1}{m\theta} \mathbf{Z} \cdot \dot{\mathbf{M}} - \frac{1}{m\theta} (\mathbf{Z} \cdot \mathbf{F}^{-T}) \dot{\mathbf{C}} + \frac{1}{\theta} {}^L\mathbf{B} \cdot \dot{\boldsymbol{\mu}} - \frac{1}{m\theta} {}^L\mathbb{B} \cdot \nabla \boldsymbol{\mu} + \frac{1}{m\theta^2} \mathbf{q} \cdot \nabla \theta - \frac{1}{m} \nabla \cdot \mathbf{k} \right] d\tau.$$

In the next Section, by using a technique by Maugin [25], we derive the explicit expression for the extra entropy flux \mathbf{k} .

5.2. Exploitation of the dissipation inequality

According to our hypothesis above, the free energy is a constitutive function of the type

$$(38) \quad \Psi = \Psi(\mathbf{C}, \theta, \nabla \theta, \boldsymbol{\mu}, \nabla \boldsymbol{\mu}).$$

By derivation with respect to time one obtains

$$(39) \quad \dot{\Psi} = \frac{\partial \Psi}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} + \frac{\partial \Psi}{\partial \theta} \dot{\theta} + \frac{\partial \Psi}{\partial \nabla \theta} \cdot \dot{\nabla} \theta + \frac{\partial \Psi}{\partial \boldsymbol{\mu}} \cdot \dot{\boldsymbol{\mu}} + \frac{\partial \Psi}{\partial (\nabla \boldsymbol{\mu})} \cdot (\dot{\nabla} \boldsymbol{\mu}).$$

We make the following positions

$$(40) \quad \tilde{\mathbf{T}} = \frac{\partial \Psi}{\partial \mathbf{C}}; \quad \mathbf{H} = \frac{\partial \Psi}{\partial \boldsymbol{\mu}}; \quad \mathbf{Y} = \frac{\partial \Psi}{\partial (\nabla \boldsymbol{\mu})}; \quad \tilde{s} = -\frac{\partial \Psi}{\partial \theta}; \quad \mathbf{R} = \frac{\partial \Psi}{\partial (\nabla \theta)}$$

and after some simple manipulation, the following form of equation (39) is obtained

$$(41) \quad \dot{\Psi} = \tilde{\mathbf{T}} \cdot \dot{\mathbf{C}} + \mathcal{A} \cdot \dot{\boldsymbol{\mu}} - \tilde{s} \dot{\theta} + \nabla \cdot (\mathbf{Y}^T \cdot \dot{\boldsymbol{\mu}}) + \mathbf{R} \cdot (\dot{\nabla} \theta),$$

where

$$(42) \quad \mathcal{A} = \frac{\partial \Psi}{\partial \boldsymbol{\mu}} - \nabla \cdot \left[\frac{\partial \Psi}{\partial (\nabla \boldsymbol{\mu})} \right] = \frac{\delta \Psi}{\delta \boldsymbol{\mu}}$$

is the variational derivatives of Ψ with respect to the (state) variable $\boldsymbol{\mu}$.

From the first law of thermodynamics (18) (neglecting heat radiation) and the second principle of thermodynamics (27), using the Legendre transformation $\Psi = e - \theta S$ one obtains the general form for the Clausius-Duhem inequality in the form

$$(43) \quad -m(\dot{\Psi} + S\dot{\theta}) + p_{(i)} + \nabla \cdot (\theta \mathbf{k}) - \mathbf{J}_S \cdot \nabla \theta \geq 0$$

where the extra entropy flux \mathbf{k} of equation (28) is considered and the *virtual power* of internal forces $p_{(i)}$ has the explicit expression (19).

By substituting equation (25) into the Clausius-Duhem inequality, the following relation is obtained

$$(44) \quad -m(\dot{\Psi} + S\dot{\theta}) - \mathbf{Z} \cdot \dot{\mathbf{M}} + (\mathbf{Z}\mathbf{F}^{-T}) \cdot \dot{\mathbf{C}} - m {}^L\mathbf{B} \cdot \dot{\boldsymbol{\mu}} + {}^L\mathbb{B} \cdot \nabla \dot{\boldsymbol{\mu}} + \nabla \cdot (\theta \mathbf{k}) - \mathbf{J}_S \cdot \nabla \theta \geq 0.$$

We now replace equation (41) into equation (44) and obtain

$$(45) \quad (\mathbf{Z}\mathbf{F}^{-T} - m\tilde{\mathbf{T}}) \cdot \dot{\mathbf{C}} - m(S - \tilde{s})\dot{\theta} - m(\mathcal{A} + {}^L\mathbf{B}) \cdot \dot{\boldsymbol{\mu}} + {}^L\mathbb{B} \cdot \nabla \dot{\boldsymbol{\mu}} + \mathbf{R} \cdot \nabla \dot{\theta} + \frac{1}{m} \theta \mathbf{k} \cdot \nabla m - \mathbf{J}_S \cdot \nabla \theta \geq 0.$$

From inequality (45) it is possible to obtain the following state laws

$$(46) \quad \tilde{\mathbf{T}} = \frac{\partial \Psi}{\partial \mathbf{C}} = m^{-1} \mathbf{Z}\mathbf{F}^{-T},$$

$$(47) \quad \tilde{s} = \frac{\partial \Psi}{\partial \theta} = S, \quad \mathbf{R} = \frac{\partial \Psi}{\partial (\nabla \theta)} = 0,$$

$$(48) \quad \mathcal{A} = \frac{\delta \Psi}{\delta \boldsymbol{\mu}} = -{}^L\mathbf{B},$$

$$(49) \quad \mathbf{k} = \frac{m}{\theta} \left[(\mathbf{Y})^T \dot{\boldsymbol{\mu}} \right] = \frac{m}{\theta} \left[\left(\frac{\partial \Psi}{\partial \nabla \boldsymbol{\mu}} \right)^T \dot{\boldsymbol{\mu}} \right].$$

Then the following dissipation inequality remains

$$(50) \quad -\mathbf{Z} \cdot \dot{\mathbf{M}} + {}^L\mathbb{B} \cdot \nabla \dot{\boldsymbol{\mu}} + \frac{1}{m} \theta \mathbf{k} \cdot \nabla m - \mathbf{J}_S \cdot \nabla \theta \geq 0$$

5.3. Exploitation of the entropy action

We use now the results of Section 5.2 to exploit the expression for the entropy action (37) for the model of ferromagnetic crystals.

By substituting the last relation into (35) the explicit expression for the entropy action along the transformation for the system is obtained in the following form

$$(51) \quad s = \int_{\rho} -\frac{\mathbf{ZF}^{-T}}{m\theta} \cdot d\mathbf{C} + \frac{1}{\theta} d\mathbf{e} + \left[\frac{L\mathbf{B}}{\theta} - \frac{1}{\mathbf{m}} \nabla \cdot \left(\frac{\mathbf{m}[\mathbf{Y}^T]}{\theta} \right) \right] \cdot d\boldsymbol{\mu} +$$

$$(52) \quad - \frac{1}{\theta} \left(\frac{L\mathbb{B}}{m} + \mathbf{Y} \right) \cdot d(\nabla\boldsymbol{\mu}) + \frac{1}{m\theta} \left[\mathbf{Z} \cdot \dot{\mathbf{M}} + \frac{1}{\theta} \mathbf{q} \cdot \nabla\theta \right] dt =$$

$$(53) \quad = \int_{\rho} \Omega,$$

so that the entropy function is now calculated as an integral of the 1-form Ω along a path ρ into the appropriate space $\mathbb{R} \times B$ of all thermodynamical variables, together with the independent time variable. Here the space of state variables amounts to choose $(\mathbf{C}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu})$ as thermodynamical variables.

In components the *entropy 1-form* Ω becomes:

$$(54) \quad \Omega = \omega_{\mu} dq^{\mu} + \omega_0 dt = \omega_A dq^A, \quad (A = 1, 2, \dots, 6)$$

where

$$(55) \quad q^A = (\mathbf{C}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu}, t)$$

and

$$(56) \quad \omega_A = \left[-\frac{(\mathbf{ZF})^{-T}}{m\theta}, \frac{1}{\theta}, 0, \frac{L\mathbf{B}}{\theta} - \frac{1}{m} \nabla \cdot \left(\frac{m\mathbf{Y}^T}{\theta} \right), \frac{1}{\theta} \left(\frac{L\mathbb{B}}{m} + \mathbf{Y} \right), \frac{1}{m\theta} (\mathbf{Z} \cdot \dot{\mathbf{M}} + \frac{1}{\theta} \mathbf{q} \cdot \nabla\theta) \right].$$

Starting from the entropy 1-form, it's possible to investigate and to introduce an extended thermodynamical phase space in a suitable way [13, 26].

REFERENCES

1. M. Francaviglia, L. Restuccia and P. Rogolino, *A geometric model for magnetizable bodies with internal variables*, Atti Accademia Peloritana dei Pericolanti di Messina, **LXXXIII** (2005) 1-8 .

2. B.D. Coleman and M.E. Gurtin, *Thermodynamics with internal state variables*, J. Chem. Phys., **47** (1967) 597-613 .
3. W. Noll, *A new mathematical theory of simple materials*, Arch. Rat. Mech. Anal., **48** (1972) 1-50.
4. B. D. Coleman and D. R. Owen, *A mathematical foundation for thermodynamics*, Arch. Rat. Mech. Anal., **54** (1974) 1-104 .
5. D. R. Owen, *A first course in the Mathematical Foundations of Thermodynamics*, Springer-Verlag, New-York (1984).
6. M. Dolfín, M. Francaviglia and P. Rogolino, *A geometric perspective on irreversible thermodynamics with internal variables*, J. Non-Equilib. Thermodyn., **23(3)** (1998) 250-263 .
7. M. Dolfín, M. Francaviglia and P. Rogolino, *A geometric model on the thermodynamics of simple materials*, Periodica Polytechnica Serie Mech. Eng., **43** (1999) 29-36.
8. M. Dolfín, M. Francaviglia and P. Rogolino, *Thermodynamic transformations in magnetically polarizable undeformable media*, J. Non-Equilib. Thermodyn., **26** (2001) 255-267.
9. G.A. Maugin, *A continuum theory of deformable ferrimagnetic bodies. I. Field equations*, Journal of Mathematical Physics, **17** (1976) 1727-1738.
10. G.A. Maugin, *A continuum theory of deformable ferrimagnetic bodies. II. Thermodynamics, constitutive theory*, Journal of Mathematical Physics, **17**(1976) 1739-1751.
11. G.A. Maugin, *The Method of Virtual Power in Continuum Mechanics: Application to Coupled Fields*, Acta Mechanica, **35** (1980) 1-70.
12. K.C. Valanis, *The concept of physical metric in thermodynamics*, Acta Mechanica, **113** (1995) 169-184.
13. S. Preston and J. Vargo, *Indefinite metric of R. Mrugala and the geometry of thermodynamical phase state*, Atti Accademia Peloritana dei Pericolanti, **LXXXVI** Suppl. 1 (2008) 11-12.
14. M. Dolfín, M. Francaviglia and L. Restuccia, *Thermodynamics of deformable dielectrics with a non-Euclidean structure as internal variable*, Technische Mechanik, **24(2)** (2004) 137-145.
15. V. Ciancio, M. Francaviglia and M. Dolfín, *Localization of deformations in finite elastoplasticity*, Technische Mechanik, **22** (2002) 111-117.
16. M. Dolfín, M. Francaviglia and P. Rogolino, *Thermodynamic transformations in magnetically polarizable undeformable media*, J. Non-Equilib. Thermodyn., **26** (2001) 255-267.

17. C. A. Truesdell , *Termodinamica Razionale*, Accademia Nazionale dei Lincei, Roma (1976).
18. C. Truesdell and C. C. Wang, *Introduction to Rational Elasticity*, Noordhoff Int. Pub., Leyden (1973).
19. C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, Handbuch der Physik, Vol. **III/3**, Springer, Berlin (1965).
20. S.R. De Groot, P. Mazur, *Non-Equilibrium Thermodynamics*, Amsterdam: North Holland Publ. Co. (1962).
21. G.A. Maugin and W. Muschik, *Thermodynamics with internal variables. Part I. General Concepts*, J. Non-Equilib. Thermodyn., **19** (1994) 217-249.
22. G.A. Maugin and W. Muschik, *Thermodynamics with internal variables. Part II. Applications*, J. Non-Equilib. Thermodyn., **19** (1994) 250-289.
23. W. Muschik, *Non-equilibrium thermodynamics with applications to solids*, CISM Courses and lectures, **336**, Springer-Verlag (Berlin 1994).
24. J. Verhas, *Thermodynamics and Rheology*, Kluwer Academic Publisher, Dordrecht (1997).
25. G.A. Maugin, *Internal Variables and dissipative structures*, J Non-Equilib. Thermodyn., **15** (1990) 173-192 .
26. M. Dolfín, M. Francaviglia and L. Restuccia, *Material point model and the geometry of entropy form*, International Journal of geometric Methods in Modern Physics, **6(1)** (2010) 1021-1042.